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On Quasi-Nested Wandering Domains

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Abstract. In this paper, the nature of the singularity of a meromorphic functions of the form $f(z) = \frac{1}{h(z)} + a$ for $a \in \mathbb{C}$ and h is an entire function having a Baker wandering domain, lying over the Baker omitted value is discussed. Various dynamical issues relating to the singular values of f have been studied. Also following are shown in this paper. If a be the Baker omitted value of f then f has a Quasi-nested wandering domain U if and only if there exists $\{n_k\}_{k>0}$ such that each U_{n_k} surrounds a and $U_{n_k} \to a$ as $k \to \infty$. If f is a function having Quasi-nested wandering domain then all the Fatou components of f are bounded. In particular, f has no Baker domain. Also existence of Quasi-nested wandering domain ensures that the Julia component containing ∞ i.e., J_∞ is a singleton buried component. At the end of the paper a result about the non existence of Quasi-nested wandering domain is given.

1. Introduction.

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function. For a given meromorphic function we are interested in the convergence of the *orbit*: z, $f(z)$, $f^2(z)$, ..., $f^n(z)$ .... of a point $z \in \mathbb{C}$. The nth iterates of f, denoted by f^n, generate a dynamical system. The set of points $z \in \mathbb{C}$ in a neighbourhood of which the sequence of iterates $\{f^n\}_{n>0}$ is defined and forms a *normal family* is called the *Fatou set* of f and is denoted by $\mathcal{F}(f)$. The *Julia set*, denoted by $\mathcal{J}(f)$, is the complement of $\mathcal{F}(f)$ in $\widehat{\mathbb{C}}$. The Fatou set is open and the Julia set is perfect. One can get some preliminary ideas of these sets from [2] and [9]. The dynamics of transcendental functions can be seen in [8] and [10]. A *Fatou component* is a maximally connected subset of the Fatou set. For a Fatou component U, U_k denotes the Fatou component containing $f^k(U)$. A Fatou component U is called *wandering* if $U_n \neq U_m$ for all $n \neq m$. A maximally connected subset of the Julia set is called a Julia component. A component of the Julia set is called a *Buried component* if it is not contained in the boundary of any Fatou component.

Let $a \in \widehat{\mathbb{C}}$. If for every open neighborhood U containing a, there exists a component V of $f^{-1}(U)$ such that $f : V \to U$ is not injective then a is called a singular value of f. The singular values of f are very much essential in studying the dynamics of the function. Denote the set of singular values of f by $S(f)$.

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These are the closure of critical values and asymptotic values of f. The image of a critical point, that is, $f(z_0)$ where $f'(z_0) = 0$ is called a critical value. A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value of f if there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ with $\lim_{t\to\infty} |\gamma(t)| = \infty$ such that $a = \lim_{t\to\infty} f(\gamma(t))$. A more general definition of singular values is given below [3].

For $a \in \widehat{\mathbb{C}}$ and $r > 0$, let $B_r(a)$ be a disk (in the spherical metric) and choose a component U_r of $f^{-1}(B_r(a))$ in such a way that $U_{r_1} \subset U_{r_2}$ for $0 < r_1 < r_2$. There are two possibilities.

1. $\bigcap_{r>0} U_r = \{z\}$ for $z \in \mathbb{C}$. Then $f(z) = a$. The point z is called an ordinary point if (i) $f'(z) \neq 0$ and $a \in \mathbb{C}$, or (ii) z is a simple pole. The point z is called a critical point if $f'(z) = 0$ and $a \in \mathbb{C}$, or z is a multiple pole. In this case, a is called a critical value and we say that a critical point or an algebraic singularity lies over a.
2. $\bigcap_{r>0} U_r = \phi$. The choice $r \mapsto U_r$ defines a transcendental singularity of f^{-1}. We say a transcendental singularity lies over a. The singularity lying over a is called direct if there exists $r > 0$ such that $f(z) \neq a$ for all $z \in U_r$. The singularity lying over a is called logarithmic if $f : U(r) \to B_r(a) \setminus \{a\}$ is a universal covering for some $r > 0$. A singularity is called indirect if it is not direct.

A value $a_0 \in \widehat{\mathbb{C}}$ is said to be an *omitted value* for a function f if a_0 is never taken by f. It is easy to note that each singularity lying over an omitted value is direct [3] .

Let M denote the class of meromorphic functions with at least two poles or one pole that is not an omitted value. Let M_o and ${M_o}^1$ be the subset of M having at least one omitted value and exactly one omitted value respectively. The dynamics of the functions in classes M_o and ${M_o}^1$ have been already studied in [12]. We know from [12] that if a function has only one omitted value and the Julia component containing it, is non-empty and non-singleton then the iterated forward image of each multiply connected periodic Fatou component must be a Herman ring. On the other hand, if the Julia component containing the omitted value is singleton then every multiply connected Fatou component is either wandering or eventually becomes a Herman ring or an infinitely connected Baker domain of period greater than 1. If all the omitted values are in a Fatou component then any multiply connected Fatou component whenever it exists must ultimately land on a Herman ring or on the Fatou component containing all the omitted values.

Let $\mathcal{F}$ denote the class consisting of meromorphic functions of the form $f(z) = \frac{1}{h(z)} + a$ for $a \in \mathbb{C}$ and h is an entire function having a Baker wandering domain. The class $\mathcal{F}$ is a subclass of ${M_o}^1$ where the singularity lying over the omitted value is of a particular type of non-logarithmic singularity. In [4], we call an omitted value $a \in \widehat{\mathbb{C}}$ a *Baker omitted value* of f if for all $r > 0$, $f^{-1}(B_r(a)) = \mathbb{C} \setminus \bigcup_{i=1}^{\infty} D_i$ where $D_i \cap D_j = \phi$ for $i \neq j$ and each D_i is a bounded simply connected domain. A Baker wandering domain is a wandering component U of $\mathcal{F}(f)$ such that, for n large enough, U_n is bounded, multiply connected and surrounds 0, and $U_n \to \infty$ as $n \to \infty$. A wandering domain U is said to be Quasi-nested if there exists a sequence $\{n_k\}$ in $\mathbb{N}$ such that each U_n is bounded, U_{n_k} surrounds 0 for all $k > 0$ and $U_{n_k} \to \infty$ as $k \to \infty$. Here $U_{n_k} \to \infty$ as $k \to \infty$ means in any neighbourhood of ∞ contains infinitely many elements of the sequence $\{U_{n_k}\}$. Any Baker wandering domain is a Quasi-nested wandering domain. But the converse is not true in general. A Quasi-nested wandering domain is a Baker wandering domain when $n_k = k$ for all k. For transcendental entire function both the concepts are same. Rippon and Stallard in [15] gave an example of a meromorphic function having a Quasi-nested wandering domain.

The existence of Quasi-nested wandering domains and how such components control the dynamics of a function is discussed in the current paper. In Section 2, we discuss the nature of the singularities of $f \in \mathcal{F}$ lying over a where a is the Baker omitted value of f. Also we give some implications of Baker omitted value on the singular values of f. In Section 3, it is mainly shown that for a given meromorphic function with the Baker omitted value a, f has a Quasi-nested wandering domain U if and only if there exists $\{n_k\}_{k>0}$ such that each U_{n_k} surrounds a and $U_{n_k} \to a$ as $k \to \infty$. Finally, we have shown that if f has a Quasi-nested wandering domain then all the Fatou components are bounded. Then Baker domain does not exist for f as a particlar case. Also, it follows that the Julia component containing the point ∞ i.e., J_∞ is a singleton buried component. Also it is shown that when the omitted value is contained in a Fatou component, then Quasi-nested wandering domains do not exist.

2. The Baker omitted value.

In this section, the nature of singularity of $f \in \mathcal{F}$ lying over the Baker omitted value a has been described. A result similar to below result was already proved in ([4], Lemma 2.1). Here we give the analysis with a different approach.

Lemma 2.1. *Let h be an entire function. Suppose that h has a Baker wandering domain. Let $D_r = \mathbb{C} \setminus B_r(0)$. Then for all $r > 0$, $h^{-1}(D_r)$ is connected and each of its boundary component is bounded. In particular, $h^{-1}(D_r)$ is infinitely connected and is the complement of infinitely many simply connected domains.*

Proof. First, we show that every component of $h^{-1}(D_r)$ is unbounded. If possible let U_r be a bounded component of $h^{-1}(D_r)$. Then $h(U_r) = D_r$ is bounded. This is a contradiction.

If U_r is any arbitrary component of $h^{-1}(D_r)$, we assert that U_r has no unbounded boundary component. On the contrary, let γ be an unbounded boundary component of U_r. Then γ is a component of the pre-image of $\{z : |z| = r\}$. If γ intersects $\{z : |z| = r\}$, then γ' be the part of γ such that $\{z : |z| = r\} \cap \gamma' = \phi$. Let W be the Baker wandering domain of h and $h^k(W)$ is contained in the Fatou component W_k for some $k \in \mathbb{N}$. Then $\exists\ n_0 \in \mathbb{N}$ such that $W_n \cap \gamma' \neq \phi$ for all natural number $n \geq n_0$. Suppose that $\gamma'_{n_0} = \gamma' \cap W_{n_0}$. As $h(\gamma') \subset \{z : |z| = r\}$, $h(\gamma'_{n_0}) \subset \{z : |z| = r\}$. Again as $\gamma'_{n_0} \subset W_{n_0}$ and $h(W_{n_0}) \subset W_{n_0+1}$ we get that $h(\gamma'_{n_0}) \subset W_{n_0+1}$. So $dist(h(\gamma'_{n_0}), 0) > r$ which is not true. This shows that U_r contains no unbounded boundary component. Next to show that U_r is infinitely connected and this is the whole of $h^{-1}(D_r)$.

If $c(U_r) = 1$, since U_r is unbounded and has no unbounded boundary component then $U_r = \mathbb{C}$. Since h is entire, by Picard's Theorem it can omit at most one point in $\mathbb{C}$, but $h(U_r) = D_r$. Hence $c(U_r) > 1$. Suppose that U_r is finitely connected. Since $h^{-1}(D_r) = U_r$, $h^{-1}(B_r(0))$ is equal to the union of the finite number of bounded complementary components of U_r. Then the points of $B_r(0)$ have only finite number of pre-images which contradicts Picard's Theorem. This shows that U_r is infinitely connected. Next to show that $h^{-1}(D_r)$ is connected. If possible there is another component U_s of $h^{-1}(D_r)$. Then U_s is infinitely connected and has no unbounded boundary component, which is not possible. So we conclude that U_r is equal to $h^{-1}(D_r)$ and hence $h^{-1}(D_r)$ is connected. □

Remark 2.2. *Lemma 2.1 is true for all $r > 0$. If it does not hold for some $r_0 > 0$, then $h^{-1}(D_{r_0})$ is finitely connected. Then there are infinitely many points in $B_{r_0}(0)$ which are having finite number of pre-images. This is impossible because of Picard's Theorem. Hence it concludes that Lemma 2.1 is true for any neighborhood of ∞.*

The above result can be extended to the class $\mathcal{F}$ of meromorphic functions of the form $f = \frac{1}{h} + a$ where $a \in \mathbb{C}$ and h is an entire function having a Baker wandering domain.

Lemma 2.3. *Let $f \in \mathcal{F}$. Then $f^{-1}(B_r(a))$ is unbounded and infinitely connected. In particular, there is only one transcendental singularity lying over a and that is not logarithmic.*

Proof. Since $f^{-1}(w) = h^{-1}(\frac{1}{w-a})$ for $w \in B_r(a)$, from Lemma 2.1, we have $f^{-1}(B_r(a))$ is an unbounded, connected and infinitely connected subset of $\mathbb{C}$ and none of the boundary components is unbounded. Hence the singularity lying over a is transcendental and not logarithmic. □

Now the next remark is immediate from the definition of Baker omitted value.

Remark 2.4. *If $a \in \mathbb{C}$ is a Baker omitted value of f then for all $r > 0$, $f^{-1}(B_r(a)) = \mathbb{C} - \bigcup_{i=1}^{\infty} D_i$ where each D_i is a simply connected bounded domain and $D_i \cap D_j = \phi$ for all $i \neq j$.*

If f is a transcendental meromorphic function and a is a Baker omitted value of f then each D_i contains at least one pole. So when we increase the radius of the ball $B_r(a)$ then the diameter (the supremum of distances of any pair of points of D_i) of each D_i decreases but they cannot vanish. If one D_i is vanished completely then a pole will be mapped to a finite number which is not possible.

If f is a transcendental entire function then ∞ can be the only Baker omitted value for f. For this case, each D_i contains at least one zero. So when we increase the radius of the ball $B_r(a)$ then the diameter of each D_i decreases but they also cannot vanish. If one D_i vanishes completely then each point of D_i must be mapped to a nonzero finite number. This is a contradiction to the fact that, for all $r > 0$ and for the Baker omitted value a of f, $f^{-1}(B_r(a)) = \mathbb{C} - \bigcup_{i=1}^{\infty} D_i$ where each D_i is a simply connected bounded domain and $D_i \cap D_j = \phi$ for all $i \neq j$.

Lemma 2.3 says that the omitted value a of f is actually a Baker omitted value. Let E denote the class of transcendental entire functions. If $f \in E \cup M$ has two asymptotic values in $\widehat{\mathbb{C}}$, then f can not have a Baker omitted value. To see it, let a and b be two asymptotic values of f. If possible let a be a Baker omitted value. Then for some $r > 0$, $f^{-1}(B_r(a)) = \mathbb{C} \setminus \bigcup_{i=1}^{\infty} D_i$ where $D_i \cap D_j = \phi$ for $i \neq j$ and each D_i is a bounded simply connected domain. Then for any neighborhood $N_\delta(b)$ $(\delta > 0)$ of b each component of $f^{-1}(N_\delta(b))$ will be contained in some D_i. Since each D_i's is bounded, each component of $f^{-1}(N_\delta(b))$ is also bounded. This is a contradiction to b is an asymptotic value. Thus f can not have any Baker omitted value.

From above it follows that, if $f \in M$ has a Baker omitted value then $f \in M_o^1$. We now give certain characterizations of functions in M_o^1. Let $E' = \{h \in E : h \text{ has no finite omitted value}\}$. Let $f \in M_o^1$ omit a then $h(z) = \frac{1}{f(z)-a}$ is entire and does not omit any finite value. If $h(z)$ omits some finite value b, then f omits $a + \frac{1}{b}$ and then f omits two values a and $a + \frac{1}{b}$ which is a contradiction. So $h(z) \in E'$. For any $g \in E'$ the function $\frac{1}{g}$ is meromorphic and omits 0. Two things are combined to say that functions in M_o^1 are in one-to-one correspondence with functions in E'. That is $M_o^1 \cong E'$. If $f \in E$ has a Baker omitted value then $Tf \in M_o^1$ has a Baker omitted value for a suitable Mobius transformation T.

In [1], Baker showed that if a transcendental entire function is bounded on a curve tending to infinity then all of its Fatou components are necessarily simply connected. If an entire function h has a Baker wandering domain, then it has multiply connected Fatou components. This implies that h can not be bounded on a curve tending to ∞. This in turn shows that h has no finite asymptotic value. If h takes a finite value only finitely often then that value is an asymptotic value for h. Hence we have the following.

Lemma 2.5. *The function h takes every finite value infinitely often.*

Lemma 2.5 tells that h has infinitely many zeros. Thus f is a transcendental meromorphic function with infinite number of poles. The function f can be written as the composition of two functions, that is, $f(z) = g(h(z))$, where $g(z) = \frac{1}{z} + a$. Any asymptotic value of f is either an asymptotic value of g or an image of some asymptotic value of $h(z)$ under g. Since there is no asymptotic value of $h(z)$, and a is the asymptotic value of $g(z)$ it shows that a is the only asymptotic value of $f(z)$.

Lemma 2.6. *If the critical values of h never accumulate at the origin, then $S(f)$ is bounded. Further, the function f has a single asymptotic value a and infinitely many critical values converging to a.*

Proof. The critical points of f are the same as that of $h(z)$. As h is having a wandering domain it has infinite number of singular values. But, since h has no finite asymptotic value, there are infinite number of critical values and hence critical points. Let z_n, $n = 1, 2, ...$, be the critical points of h. Now by our assumption, ∞ is not a limit point of the set $\{f(z_n) : z_n \text{ is a critical point of h}, \ n = 1, 2, 3, ...\}$. So $S(f)$ is bounded. Again as a is an omitted value of f, it is an asymptotic value.

By Bolzano-Weierstrass Theorem every infinite sequence will have a convergent subsequence. Let z_{n_k} be the convergent subsequence. But as $k \to \infty$, $z_{n_k} \nrightarrow b$ for b in $\mathbb{C}$ because if $z_{n_k} \to b$ as $k \to \infty$ then $h'(z) = 0$ on a set that contains a limit point in it and hence by Uniqueness Theorem $h'(z) \equiv 0$ and h becomes a constant function. This shows that $z_{n_k} \to \infty$ as $k \to \infty$. By our assumption, since the singular values of h never accumulate at the origin, therefore either $h(z_{n_k}) \to \infty$ or $h(z_{n_k})$ tends to some finite number say b which may not be an asymptotic value. If $h(z_{n_k}) \to \infty$ then $f(z_{n_k}) \to a$ as $k \to \infty$. If $h(z_{n_k}) \to b$ then $f(z_{n_k}) \to a + \frac{1}{b}$ as $k \to \infty$. Hence in both the cases, the set of all the singular values of f is bounded. □

It has been proved in [12] that the functions in the class M_0 do not have Baker wandering domains. So this is true for functions in $\mathcal{F}$. A periodic Fatou component U of period p is called a *Baker domain* if there exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined. In [13], Rippon and Stallard showed that transcendental meromorphic functions with $S(f)$ bounded have no Baker domain of period 1. Hence the Fatou set of f has no Baker domain of period 1 for $f \in \mathcal{F}$. Here we give a result about the existence of multiply connected Fatou component of $f \in \mathcal{F}$.

Lemma 2.7. *If J_a is singleton and buried for $f \in \mathcal{F}$ then there is a multiply connected Fatou component of f.*

Proof. Given that J_a is singleton and buried of f. It is shown in [7] that if f is a transcendental meromorphic function with at least one not omitted pole, if there exists a buried component of $\mathcal{J}(f)$, then $\mathcal{J}(f)$ is disconnected. Since any $f \in \mathcal{F}$ has infinitely many poles and by Picard's theorem for meromorphic functions there exists at least one pole which is not omitted, we have $\mathcal{J}(f)$ is disconnected, for any $f \in \mathcal{F}$. Then the disconnected Julia set implies that there exists at least one multiply connected Fatou component. □

3. Existence of Quasi-nested wandering domains.

Let γ be any closed curve, and let $B(\gamma)$ denote the union of all the bounded complementary components of γ. We denote U_{n_k-1} as the component of $f^{-1}(U_{n_k})$ such that $f^{n_k-1}(U) \subset U_{n_k-1}$.

Lemma 3.1. *Let f be a meromorphic function with the Baker omitted value a. Let there exists $\{n_k\}_{k>0}$ and a Fatou component U such that each U_{n_k} surrounds a. Then $\widehat{\mathbb{C}} \setminus U_{n_k-1}$ has a bounded component which surrounds a pole of f.*

Proof. Take $\gamma \subset U_{n_k}$ such that $a \in B(\gamma)$. Then $f^{-1}(\gamma) \cap U_{n_k-1}$ is a disjoint union of closed curves. Let γ' be any such closed curve. Consider a component C of $B(\gamma')$. Let $x \in \partial f(C)$. If $f(z) = x$ then z does not belong to the interior of C by the Open Mapping Theorem. Take a sequence $\{x_n\}_{n>0}$ in $f(C)$ such that $x_n \to x$ as $n \to \infty$. Then there is $z_n \in C$ such that $f(z_n) = x_n$. Each limit point of $\{z_n\}_{n>0}$ is in $\overline{C}$. Let z be one limit point such that $z \in \partial C$ then by the continuity of f at z, $z_n \to z$ implies $f(z_n) \to f(z)$ as $n \to \infty$. i.e. as $n \to \infty$, $x_n \to f(z)$. Since limit of a sequence is unique, $f(z) = x$. Now take a sequence $\{a_n\}_{n>0}$ in $B(\gamma')$ such that $a_n \to z$ as $n \to \infty$. Then by continuity of f at z, $f(a_n) \to f(z)$ as $n \to \infty$. But $f(a_n) \in f(B(\gamma')) = A$ gives that $f(z) \in \partial A \subseteq \gamma$. Thus $f(z) = x \in \gamma$. This proves that $\partial f(C) \subseteq \gamma$. To ensure this, one can see [14]. If $\partial f(C) \subsetneq \gamma$ then $f(C)$ is unbounded. Then $a \in f(C)$ which is not possible. Therefore $\partial f(C) = \gamma$. This implies that $f(C) = B(\gamma)$ or $\widehat{\mathbb{C}} \setminus B(\gamma)$. But $f(C) = B(\gamma)$ implies that $a \in f(C)$ which is not true. So $f(C) = \widehat{\mathbb{C}} \setminus B(\gamma)$ and thus the pole is in a bounded component of $\widehat{\mathbb{C}} \setminus U_{n_k-1}$. □

For a multiply connected Fatou component U, let $B(U)$ denote the component of $\widehat{\mathbb{C}} \setminus U$ containing the Baker omitted value a. Now we prove the following theorem.

Lemma 3.2. *Let a be the Baker omitted value for f. Let there exists $\{n_k\}_{k>0}$ and a Fatou component U such that each $U_{n_k} \to a$ as $k \to \infty$ and $a \in B(U_{n_k})$. Suppose that the Fatou components U_{n_k} and $U_{n_{k'}}$ surrounds a. Let N_a be a choosen neighborhood of a which does not intersect U_{n_k} and $U_{n_{k'}}$. Then there exists a Fatou component U_l surrounding a and is contained in N_a such that it has a pre-image U^{**} (which is not necessarily the U image under f^n) surrounding both U_{n_k-1} and $U_{n_{k'}-1}$.*

Proof. Firstly, by Lemma 3.1, the pre-images U_{n_k-1} and $U_{n_{k'}-1}$ of Fatou components U_{n_k} and $U_{n_{k'}}$ respectively are bounded. Since $U_{n_k} \to a$ as $k \to \infty$, then any neighbourhood of a contains infinitely many U_{n_k}'s surrounding a. Then N_a must contain at least one Fatou component which surrounds a. Let U_l be that Fatou component. Since a is the Baker omitted value of f, pre-image component of $B(U_l)$ under f is contained in a smaller neighborhood of infinity (say) N that does not contain U_{n_k-1} and $U_{n_{k'}-1}$. So there exists one bounded component of $\widehat{\mathbb{C}} \setminus N$ containing U_{n_k-1} and $U_{n_{k'}-1}$ and hence the result.
□

Theorem 3.3. *Let f be a meromorphic function with the Baker omitted value a. Then f has a Quasi-nested wandering domain if and only if there exists $\{n_k\}_{k>0}$ and a Fatou component U such that each U_{n_k} surrounds a, $U_{n_k} \to a$ as $k \to \infty$.*

Proof. Let U be a Quasi-nested wandering domain of f. Then there is a subsequence $\{n_k\}_{k>0} \subset \{n\}_{n>0}$ such that U_{n_k} surrounds poles of f and $f(U_{n_k}) = U_{n_k+1}$ surrounds the omitted value a [12]. Since each limit function of $\{f^n\}_{n>0}$ on a wandering domain is constant, $f^{n_k+1}|_U \to a$ or ∞ as $k \to \infty$. Again since a is an asymptotic value there is an asymptotic path γ such that $f(\gamma(t)) \to a$ when $\gamma(t) \to \infty$ as $(t \to \infty)$. There exists $R > 0$ large enough, such that

$f(\gamma \cap (B_R(0))^c) \subset B_\delta(a)$ for $\delta > 0$. Let for $k \geq k_0$, $U_{n_k} \cap (\gamma \cap (B_R(0))^c) \neq \phi$ and denote this set by γ_{n_k}. For $k > k_0$, $f(\gamma_{n_k}) \subset B(U_{n_{k_0}})$. So $f(U_{n_k}) \subset B(U_{n_{k_0}}) \subset B_\delta(a)$. As $\delta \to 0$, $f^{n_k+1}|_U \to a$ as $k \to \infty$ and thus we have proved the first part of the theorem.

Let U be any Fatou component and there exists $\{n_k\}_{k>0}$ such that U_{n_k} surrounds a with $U_{n_k} \to a$ as $k \to \infty$. Then, to prove that U is a Quasi-nested wandering domain, we first show that U_{n_k-1} is bounded, where U_{n_k-1} is such that $f^{n_k-1}(U) \subset U_{n_k-1}$ and $f(U_{n_k-1}) = U_{n_k}$. Since a is the Baker omitted value of f, $f^{-1}(B(U_{n_k})) = \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^{\infty} D_i$, where $D_i \cap D_j = \phi$ for $i \neq j$ and each D_i is a simply connected bounded domain. Again, since $U_{n_k} \subset \widehat{\mathbb{C}} \setminus B(U_{n_k})$, any component U_{n_k-1} of $f^{-1}(U_{n_k})$ lie in some D_i and hence is bounded.

We know by Lemma 3.1, that for each $k > 0$, U_{n_k-1} surrounds a pole. Now suppose that there are infinitely many k such that U_{n_k-1} surrounds a single pole (say) w_0. Then $f^{n_k-1}|_U \to w_0$ or ∞ as $k \to \infty$. If $f^{n_k-1}|_U \to \infty$ as $k \to \infty$ then U satisfies all the conditions of Quasi-nested wandering domain and we are done. If $f^{n_k-1}|_U \to w_0$ as $k \to \infty$ then $f^{n_k}|_U \to f(w_0) = \infty$ as $k \to \infty$ which is a contradiction to the fact that $U_{n_k} \to a$ as $k \to \infty$. Now assuming that there are only finitely many k for which U_{n_k-1} surrounds a single pole. There exists Fatou components U_{n_k-1} and $U_{n_{k'}-1}$ surrounding w_k and $w_{k'}$ respectively where $w_k \neq w_{k'}$ and none of them surrounds the other. It is described in Figure 1. By the definition of Baker omitted value a and mapping patern of f, we find another Fatou component U^* surrounding U_{n_k-1} which is the pre-image of some Fatou component $U_{n_{k''}}$ surrounded by $U_{n_{k'}}$. Then by Lemma 3.2, we can choose $U_{n_{k''}}$ with pre-images U^* and U^{**} such that U^{**} (which may not be the image of U under f^n) surrounds both U_{n_k-1} and $U_{n_{k'}-1}$. (see Figure 1)

Let A be the annulus bounded by γ_{k-1} and γ^{**} where γ_{k-1} is the inner boundary of U_{n_k-1} surrounding w_k and γ^{**} is the outer boundary of U^{**} surrounding both w_k and $w_{k'}$. Then $f(A)$ is connected as A is connected, and $f(A)$ is unbounded as $w_{k'} \in A$. We can write

$$\partial f(A) \subset f(\partial A) \subset \partial U_{n_k} \cup \partial U_{n_{k''}}.$$

Now $U^* \subset A$ implies that $f(U^*) \subset f(A)$, that is, $U_{n_{k''}} \subset f(A)$. Thus

$$\partial f(A) \subset f(\partial A) \subset \partial U_{n_k}.$$

This shows that $f(A)$ contains everything in $B(U_{n_k})$ and in particular the omitted value a, which is a contradiction. □

Corollary 3.4. *From the proof of the first part of the theorem it follows that, we have a sequence of shrinking Fatou components U_{n_k} surrounding a. Again a is not in the boundary of any Fatou component, and hence there exists a sequence $z_{n_k} \subset \partial U_{n_k}$ such that $z_{n_k} \to a$. Thus J_a is a singleton buried component.*

Remark 3.5. *The above theorem shows that if there is a Quasi-nested wandering domain then by* ([12], Theorem 5) *all the multiply connected Fatou components not landing on any Herman ring are wandering and a is a limit point of $\{f^n\}_{n>0}$ on each of these wandering domains. Further, in this case the forward orbit of a is an infinite set and singleton buried components are dense in $\mathcal{J}(f)$.*

Example 3.6. *Let $f(z) = \frac{1}{e^z+z} + a$ for $a \in \mathbb{C}$. Here a is the Baker omitted value for $f(z)$. If $a = -0.567 \in \mathcal{J}(f)$ and $|J_a| = 1$, then we have the following cases. If J_a is not buried, then by* ([12], Theorem 4) *there exists an infinitely connected Baker domain B with period at least* 2 *such that for any multiply connected Fatou component U, there is a non-negative integer n depending on U with $U_n = B$, and hence U is a Quasi-nested wandering domain. If J_a is buried, then by Remark 3.5, all the multiply connected Fatou components are wandering.*

4. Implications of Quasi-nested wandering domains.

Theorem 4.1. *If f has a Quasi-nested wandering domain, then the following are true.*

1. *If $f \in M_o$, then f has at most one asymptotic value and that is the omitted value of f.*
2. *All the Fatou components are bounded. In particular, f has no Baker domain.*
3. *J_∞ is a singleton buried component.*

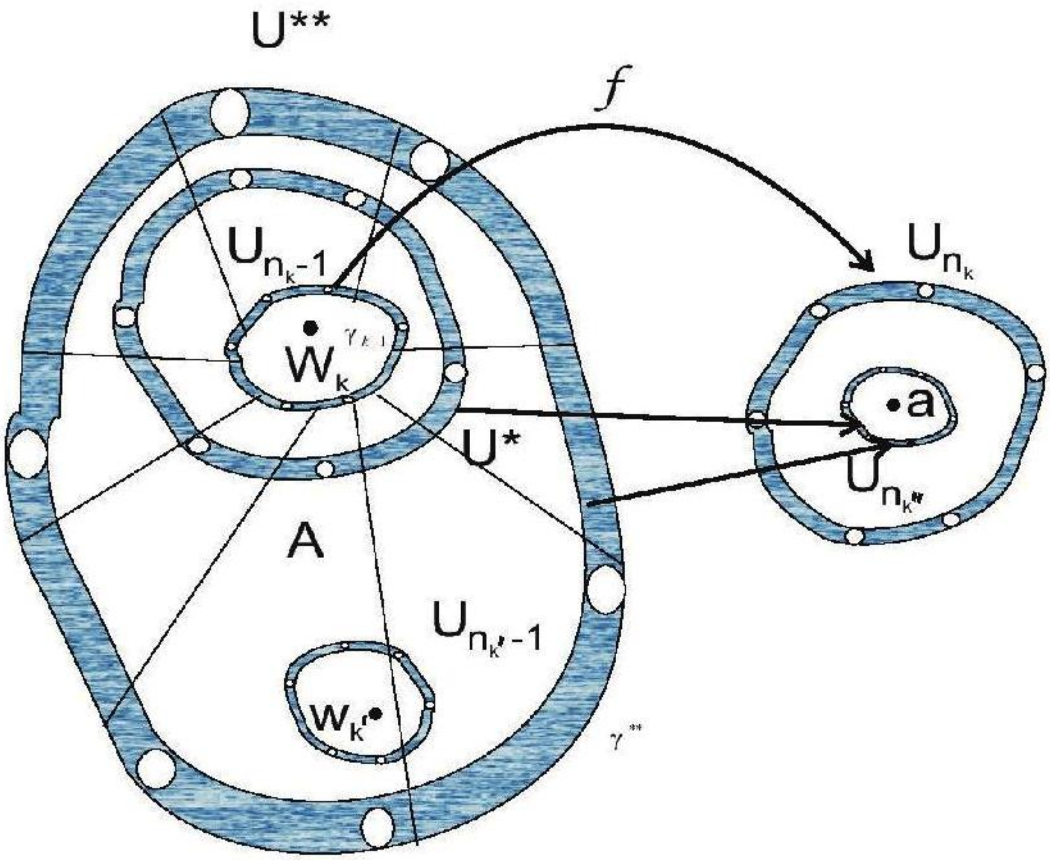

Figure 1: Here a is the Baker omitted value. The above figure describes the Theorem 3.1.

Proof. 1. On the contrary if f has two asymptotic values say a and b, then there are two different asymptotic paths γ_a and γ_b corresponding to a and b respectively. Consider neighborhoods $N_\delta(a)$ and $N_\delta(b)$ of a and b respectively for some $\delta > 0$ such that $N_\delta(a) \cap N_\delta(b) = \phi$. Let $\widetilde{\gamma_a}$ be the part of γ_a such that $f(\widetilde{\gamma_a}) \subset N_\delta(a)$. Also $\widetilde{\gamma_b}$ be the part of γ_b such that $f(\widetilde{\gamma_b}) \subset N_\delta(b)$. By the definition of Quasi-nested wandering domain, if it has such a component U, then there exists n_k such that $U_{n_k} \to \infty$ and take k_0 large such that $U_{n_{k_0}} \cap \widetilde{\gamma_a}$ as well as $U_{n_{k_0}} \cap \widetilde{\gamma_b}$ are nonempty and denote them by γ'_a and γ'_b respectively. Now $f(\gamma'_a) \subset N_\delta(a)$ and $f(\gamma'_b) \subset N_\delta(b)$. This is a contradiction.

2. If U is a Quasi-nested wandering domain, then there exists n_k such that $U_{n_k} \to \infty$ as $k \to \infty$ where all the U_{n_k}'s are multiply connected, bounded and surrounds the origin. If V is a Fatou component of f then it must be in the complement of $\bigcup_{k\geq 0} U_{n_k}$ and as all the complementary components of $\bigcup_{k\geq 0} U_{n_k}$ are bounded, any such Fatou component is also bounded. So, ∞ is not in the boundary of any Fatou component.

   Suppose that it has a Baker domain B_i of period p (say). Then $f^{np} \to \infty$ on B_i. This gives that B_i is unbounded. This is a contradiction.

3. Suppose U is a Quasi-nested wandering domain. Then from the definition, given any neighborhood $N(\infty)$ of ∞, there exists n_{k_0} for $k_0 > 0$ such that $U_{n_{k_0}} \subset N(\infty)$ and $U_{n_{k_i}} \subset \widehat{\mathbb{C}} \setminus B(U_{n_{k_0}})$ for infinitely many i. Each $U_{n_{k_i}}$ is a multiply connected domain and $U_{n_{k_i}} \to \infty$ uniformly and hence $|J_\infty| = 1$. If ∞ is in the boundary of some Fatou component then that Fatou component is unbounded. Then by (2) Quasi-nested wandering domain does not exist. Thus ∞ does not belong to the boundary of any Fatou component. Hence J_∞ is a singleton buried component.
□

Corollary 4.2. *Let $f \in M_0$ have Baker omitted value a and $a \in \mathcal{F}(f)$. Then f has no Quasi-nested wandering domain.*

Proof. If $f \in M_0$ having Baker omitted value a and $a \in \mathcal{F}(f)$ then there exist a unbounded Fatou component. By Theorem 4.1(2), f has no quasi-nested wandering domain. □

5. Future prospects.

The sigularity of f^{-1} lying over the Baker omitted value is non-logarithmic and this plays an important role throughout this paper. This is in contrast with earlier works on dynamics of transcendental meromorphic functions with logarithmic singularities. The existence of Quasi nested wandering domain of a meromorphic function f ensures that there exist no unbounded Fatou component of f. So for this case, unbounded Herman ring does not exist for f. In [5, 6, 11], many results regarding the existence of Herman ring of meromorphic functions are proved. So we incline to make the following conjecture. Meromorphic functions with Quasi nested wandering domain have no Herman ring. In [4], we have seen that a function with Baker omitted value has no asymptotic value other than the Baker omitted value. This restriction seems to simplify the investigation of the dynamics of the function. Thus investigation of the dynamics of meromorphic function with Quasi nested wandering domain is worth-doing.

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References

[1] I. N. Baker, *Wandering domains in the iteration of entire functions*, Proc. London Math. Soc., (3) 49, (1984), 563-576.
[2] W. Bergweiler, *Iterations of meromorphic functions*, Bull. Amer. Math. Soc., no.2, 29, (1993), 151-188.
[3] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana, no.2, 11, (1995), 355–373.
[4] T. Chakra , G. Chakraborty and T. Nayak, *Baker omitted value*, Complex Variables and Elliptic Equations, no. 10, 61 (2016), 1353-1361.
[5] T. Chakra, G. Chakraborty and T. Nayak, *Herman rings with small periods and omitted values*, Acta Math. Sci., 38(6) (2018), 1951–1965.
[6] G. Chakraborty, S. K. Datta and S. Sahoo, *Confgurations of Herman rings in the complex plane*, Indian J. Math., 63(3) (2021), 375–391.
[7] P. Dominguez and N. Fagella, *Residual Julia sets of rational and transcendental functions*, London Mathematical Society Lecture Note Series (No.348) (2008), Cambridge University Press.
[8] X.H. Hua and C.C. Yang, *Dynamics of Transcendental Functions*, Gordon and Breach Sci. Pub., Amsterdam (1998).
[9] G.P. Kapoor, M. Sajid and M.G.P. Prasad, *Dynamics of entire functions: a review*, Mathematical Analysis and Applications, A.P. Dwivedi (Ed) (2008), Narosa Publishing House, New Delhi, India.
[10] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic Dynamics*, Cambridge University Press, Cambridge (2000).
[11] T. Nayak, *Herman rings of meromorphic maps with an omitted value*, Proc. Amer. Math. Soc., 144(2), (2016), 587-597.
[12] T. Nayak and J. H. Zheng, *Omitted values and dynamics of meromorphic functions*, J. London Math. Soc., 83(2), (2011), 121–136.
[13] P.J. Rippon and G.M. Stallard, *Iteration of a class of hyperbolic meromorphic functions*, Proc. Amer. Math. Soc., 127, no. 11, (1999), 3251-3258.
[14] P.J. Rippon and G.M. Stallard, *On multiply connected wandering domains of meromorphic functions*, J. London Math. Soc., 77 (2), (2008), 405–423.
[15] P.J. Rippon and G.M. Stallard, *Slow escaping points of meromorphic functions*, Trans. Amer. Math. Soc., no. 8, 363 (2011), 4171-4201.