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Totally Umbilical Semi-Invariant Submanifolds in Locally Decomposable Metallic Riemannian Manifolds

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Abstract. In this paper, we obtain some classification theorems for totally umbilical semi-invariant submanifolds in locally decomposable metallic Riemannian manifolds. We also prove that there exist no totally umbilical proper semi-invariant submanifolds in a positively or negatively curved locally decomposable metallic Riemannian manifold.

1. Introduction

CR submanifolds arose from the idea of generalizing of both complex and totally real submanifolds in Kaehlerian manifolds [4]. Later, in this sense, the notion of a semi-invariant submanifold was put forward in almost contact metric manifolds [7]. Actually, the semi-invariant submanifold is the extension of the concept of a CR submanifold in Kaehlerian manifolds to submanifolds of almost contact metric manifolds. After that, this notion was applied to the other ambient manifolds and important results were obtained, such as Kenmotsu manifolds [30], locally product Riemannian manifolds [3, 5], Sasakian space forms [8], cosymplectic manifolds [2, 13], almost contact manifolds [25, 26], nearly Sasakian manifolds [32], Lorentzian para-Sasakian manifolds [31], nearly trans-Sasakian manifolds [24], Lorentzian Sasakian manifolds [1], golden Riemannian manifolds [17]. To sum up, semi-invariant submanifolds in different kinds of ambient manifolds have still continued to be a rich research field.

For different values of two positive integer numbers p and q, the positive solutions of quadratic equations of type $x^2 = px + q$ form the metallic means family, each of whose members is called a (p, q)-metallic number and denoted by $\sigma_{p,q}$. By carrying the metallic numbers $\sigma_{p,q}$'s into tensor fields of type $(1, 1)$ on C^∞-differentiable real manifolds in order to apply them to differential geometry, the concept of a metallic structure on C^∞-differentiable real manifolds was introduced by M. C. Crâşmăreanu and C. E. Hreţcanu as a polynomial structure with the structure polynomial $Q(x) = x^2 = px + q$ in [23].

There exist a lot of studies regarding the differential geometry of metallic structures on C^∞-differentiable real manifolds, in particular on Riemannian manifolds. Some certain kinds of submanifolds of metallic Riemannian manifolds, such as invariant, anti-invariant, slant, semi-slant, hemi-slant, bi-slant submanifolds were defined according to the behaviour of their tangent bundles with respect to the action of the metallic structure of the ambient manifold and investigated their fundamental properties by C. E. Hreţcanu and

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A. M. Blaga in [10, 18–20]. On the other hand, metallic structures have appeared in different contexts of differential geometry. A. M. Blaga and C. E. Hreţcanu examined metallic conjugate connections with respect to the structural and virtual tensors of the metallic structure and their behaviour on invariant distributions in [11]. Also, the authors showed that the metallic structure of the product of two metallic Riemannian manifolds could be expressed from the point of metallic maps, obtained an equivalent condition to the locally decomposability of the warped product of two locally decomposable metallic Riemannian manifolds and found a necessary and sufficient condition for the warped product of two metallic Riemannian manifolds to have the invariant Ricci tensor with respect to the metallic structure in [12]. C. Özgür and N. Y. Özgür gave the full classification of metallic shaped hypersurfaces in real space forms [28] and Lorentzian space forms [29]. A. Gezer and Ç. Karaman made an examination of metallic structures with the help of a special operator introduced by them in [15]. In addition, golden manifolds, which are a particular class of metallic manifolds, were analyzed many geometers in [9, 14, 16, 21, 22].

The main purpose of this paper is to classify totally umbilical semi-invariant submanifolds and to give a theorem on the absence of totally umbilical proper semi-invariant submanifolds in the case that the ambient manifold is a locally decomposable metallic Riemannian manifold.

The paper has three sections. The first section is introduction and the rest is prepared as follows: Section 2 is concerned with preliminaries regarding some fundamental facts from metallic Riemannian manifolds and their submanifolds. Section 3 deals with the classification and absence of totally umbilical semi-invariant submanifolds in locally decomposable metallic Riemannian manifolds. To begin with, we give a main classification theorem for totally umbilical semi-invariant submanifolds. Secondly, we find a necessary condition for any totally umbilical semi-invariant submanifold to be totally geodesic. Next, we demonstrate that if the dimension of the invariant distribution in the tangent bundle of any totally umbilical semi-invariant submanifold is greater than or equal to 2, the totally umbilical semi-invariant submanifold is an extrinsic sphere. Lastly, we show that there are no totally umbilical proper semi-invariant submanifolds in a posivitely or negatively curved locally decomposable metallic Riemannian manifold.

2. Preliminaries

This section provides a brief summary concerning metallic manifolds and their submanifolds.

Let $\overline{M}$ be a C^{∞}-differentiable real manifold with a tensor field $\overline{J}$ of type $(1,1)$ such that

$$\overline{J}^2 = p\overline{J} + qI \tag{1}$$

for $p,q \in \mathbb{N} - \{0\}$, where I is the identity $(1,1)$-tensor field on the Lie algebra $\Gamma\left(T\overline{M}\right)$ of differentiable vector fields on $\overline{M}$. In this case, we say that $\overline{J}$ is a metallic structure on $\overline{M}$ and the pair $\left(\overline{M},\overline{J}\right)$ is named a metallic manifold [23]. In particular, if $p = q = 1$, then we obtain that $\overline{J}$ is a golden structure and $\left(\overline{M},\overline{J}\right)$ is a golden manifold [14, 21, 22]. If a metallic manifold (resp., a golden manifold) $\left(\overline{M},\overline{J}\right)$ admits a Riemannian metric $\overline{g}$ such that

$$\overline{g}(\overline{J}X,Y) = \overline{g}(X,\overline{J}Y), \tag{2}$$

or equivalently

$$\overline{g}\left(\overline{J}X,\overline{J}Y\right) = p\overline{g}\left(\overline{J}X,Y\right) + q\overline{g}\left(X,Y\right) \ (\text{resp.}, \overline{g}\left(\overline{J}X,\overline{J}Y\right) = \overline{g}\left(\overline{J}X,Y\right) + \overline{g}\left(X,Y\right)) \tag{3}$$

for any vector fields $X,Y \in \Gamma\left(T\overline{M}\right)$, then it is called a metallic Riemannian manifold (resp., a golden Riemannian manifold) and denoted by the triple $\left(\overline{M},\overline{g},\overline{J}\right)$. The n-th power of the metallic structure $\overline{J}$ is given by

$$\overline{J}^n = G_n\overline{J} + qG_{n-1}I, \tag{4}$$

where (G_n) is the generalized secondary Fibonacci sequence defined by $G_{n+1} = pG_n + qG_{n-1}$ for $1 \leq p, q \in \mathbb{N}$ with $G_0 = 0$ and $G_1 = 1$. The eigenvalues of the metallic structure $\overline{J}$ are the numbers $\sigma_{p,q}$ and $p - \sigma_{p,q}$ being the roots of the algebraic equation $x^2 = px + q$. The inverse $\overline{J}^{-1}$ of the metallic structure $\overline{J}$ is given by

$$\overline{J}^{-1} = \frac{1}{q}\overline{J} - \frac{p}{q}I \tag{5}$$

and satisfies the equation

$$\left(\overline{J}^{-1}\right)^2 = -\frac{p}{q}\overline{J}^{-1} + \frac{1}{q}I;$$

however, it is not a metallic structure [23].

Let $\left(\overline{M}, \overline{g}, \overline{J}\right)$ be a metallic Riemannian manifold (resp., a golden Riemannian manifold). We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$. Then the covariant derivative of the metallic structure (resp., the golden structure) $\overline{J}$ is defined by

$$\left(\overline{\nabla}_X \overline{J}\right) Y = \overline{\nabla}_X \overline{J} Y - \overline{J}\, \overline{\nabla}_X Y \tag{6}$$

for any vector fields $X, Y \in \Gamma\left(T\overline{M}\right)$. We say that the metallic structure (resp., the golden structure) $\overline{J}$ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ if its covariant derivative $\overline{\nabla}\,\overline{J}$ vanishes identically. In this case, the triple $\left(\overline{M}, \overline{g}, \overline{J}\right)$ is called a locally decomposable metallic Riemannian manifold (resp., a locally decomposable golden Riemannian manifold) [18].

If $\left(\overline{M}, \overline{g}, \overline{J}\right)$ is a locally decomposable metallic Riemannian manifold, then we have the following relations [12]:

$$\overline{R}(X, Y)\overline{J}Z = \overline{J}\,\overline{R}(X, Y)Z, \tag{7}$$

$$\overline{R}\left(\overline{J}X, Y\right) = \overline{R}\left(X, \overline{J}Y\right), \tag{8}$$

$$\overline{R}\left(\overline{J}X, \overline{J}Y\right) = p\overline{R}\left(\overline{J}X, Y\right) + q\overline{R}(X, Y) \tag{9}$$

and

$$\overline{R}\left(\overline{J}^{n+1}X, Y\right) = G_{n+1}\overline{R}\left(\overline{J}X, Y\right) + qG_n\overline{R}(X, Y) \tag{10}$$

for any vector fields $X, Y \in \Gamma\left(T\overline{M}\right)$, where $\overline{R}$ is the Riemannian curvature tensor of $\overline{M}$. Also, using the basic properties of the Riemann-Christoffel curvature tensor $\overline{K}$ of $\overline{M}$, it follows from (2), (7) and (8) that the following relations hold:

$$\overline{K}\left(\overline{J}X, Y, Z, W\right) = \overline{K}\left(X, \overline{J}Y, Z, W\right) \tag{11}$$

and

$$\overline{K}\left(X, Y, \overline{J}Z, W\right) = \overline{K}\left(X, Y, Z, \overline{J}W\right) \tag{12}$$

for any vector fields $X, Y, Z, W \in \Gamma\left(T\overline{M}\right)$.

Let M be any isometrically immersed submanifold of a metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$ and we denote by the same symbol $\overline{g}$ the Riemannian metric induced on M. Then the Gauss and Weingarten formulas of M in $\overline{M}$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{13}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{14}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, where ∇ is the induced connection, h is the second fundamental form, A_V is the shape operator with respect to V and $\nabla^{\perp}$ is the normal connection. Also, there exists a relation between the second fundamental form h and the shape operator A such that

$$\overline{g}\,(h\,(X, Y)\,, V) = \overline{g}\,(A_V X, Y) \tag{15}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$. Furthermore, the Codazzi equation is as follows:

$$\left(\overline{R}\,(X, Y)\,Z\right)^{\perp} = (\nabla_X h)\,(Y, Z) - (\nabla_Y h)\,(X, Z) \tag{16}$$

for any vector fields $X, Y, Z \in \Gamma\,(TM)$. The submanifold M is said to be totally geodesic in $\overline{M}$ if its second fundamental form vanishes identically, i.e., $h = 0$. If $H = 0$, we say that M is a minimal submanifold, where H is the mean curvature vector of M. The submanifold M is called totally umbilical if $h\,(X, Y) = \overline{g}\,(X, Y)\,H$ for any vector fields $X, Y \in \Gamma(TM)$. If M is a totally umbilical submanifold in $\overline{M}$, then the Gauss and Weingarten formulas take the following forms:

$$\overline{\nabla}_X Y = \nabla_X Y + \overline{g}\,(X, Y)\,H \tag{17}$$

and

$$\overline{\nabla}_X V = -\overline{g}\,(H, V)\,X + \nabla^{\perp}_X V \tag{18}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, respectively. Also, we note that the Codazzi equation is expressed by

$$\overline{R}\,(X, Y, Z, V) = \overline{g}\,(Y, Z)\,\overline{g}\left(\nabla^{\perp}_X H, V\right) - \overline{g}\,(X, Z)\,\overline{g}\left(\nabla^{\perp}_Y H, V\right) \tag{19}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$ [6]. The submanifold M such that $\dim M \geq 2$ is said to be an extrinsic sphere if it is totally umbilical and has the non-zero parallel mean curvature vector H, i.e., $\nabla^{\perp}_X H = 0$ for any vector field $X \in \Gamma(TM)$ [27].

We define four operators T, N, t and n as follows:

$$TX = \left(\bar{J}X\right)^{\top}, \tag{20}$$

$$NX = \left(\bar{J}X\right)^{\perp}, \tag{21}$$

$$tU = \left(\bar{J}U\right)^{\top} \tag{22}$$

and

$$nU = \left(\bar{J}U\right)^{\perp} \tag{23}$$

for any vector fields $X \in \Gamma\,(TM)$ and $U \in \Gamma\,(TM^{\perp})$, where $\left(\bar{J}X\right)^{\top}, \left(\bar{J}U\right)^{\top} \in \Gamma\,(TM)$ and $\left(\bar{J}X\right)^{\perp}, \left(\bar{J}U\right)^{\perp} \in \Gamma\,(TM^{\perp})$. Hence, for any vector field X tangent to M, the vector field $\bar{J}X$ is given by the form

$$\bar{J}X = TX + NX. \tag{24}$$

Similarly, for any vector field U normal to M, we have

$$\bar{J}U = tU + nU. \tag{25}$$

Besides, it obviously seems that the operators $T : \Gamma(TM) \to \Gamma(TM)$ and $n : \Gamma(TM^{\perp}) \to \Gamma(TM^{\perp})$ are an endomorphism, and the operators $N : \Gamma(TM) \to \Gamma(TM^{\perp})$ and $t : \Gamma(TM^{\perp}) \to \Gamma(TM)$ are a bundle-valued 1-form. In addition, the operators T and n are $\overline{g}$-symmetric [10], that is,

$$\overline{g}(TX, Y) = \overline{g}(X, TY) \tag{26}$$

and

$$\overline{g}(nU, V) = \overline{g}(U, nV) \tag{27}$$

for any vector fields $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(TM^{\perp})$. Using the definition of the metallic structure $\overline{J}$ in (1), we get from (24) and (25) that the following relations are valid [20]:

$$pT + qI = T^2 + tN, \tag{28}$$

$$pN = NT + nN, \tag{29}$$

$$pt = Tt + tn \tag{30}$$

and

$$pn + qI = n^2 + Nt. \tag{31}$$

3. Totally Umbilical Semi-Invariant Submanifolds

In this section, we find three classification theorems and one absence theorem for totally umbilical semi-invariant submanifolds of a locally decomposable metallic Riemannian manifold.

We start by mentioning the concept of a semi-invariant submanifold in metallic Riemannian manifolds.

Semi-invariant submanifolds of a metallic Riemannian manifold were defined for the first time by C. E. Hreţcanu and A. M. Blaga as a particular case of semi-slant submanifolds in [19].

Any semi-invariant submanifold in metallic Riemannian manifolds is a semi-slant submanifold with the slant angle $\theta = \frac{\pi}{2}$ [19] or a hemi-slant submanifold with the slant angle $\theta = 0$ [20]. Thus, we have the following explicit definition:

Definition 3.1. *[19, 20] Let M be any isometrically immersed submanifold of a metallic Riemannian manifold $(\overline{M}, \overline{g}, \overline{J})$. Then M is called a semi-invariant submanifold if there exist two orthogonal complementary distributions D and $D^{\perp}$ on M satisfying the following conditions:*

(**a**) $\overline{J}(D_P) = D_P \subseteq T_P M$,
(**b**) $\overline{J}\left(D_P^{\perp}\right) \subseteq T_P M^{\perp}$

for each point $P \in M$, where D and $D^{\perp}$ are said to be $\overline{J}$-invariant distribution and $\overline{J}$-anti-invariant distribution, respectively. If neither $D = \{0\}$ nor $D^{\perp} = \{0\}$, then M is named a proper semi-invariant submanifold.

Let M be any semi-invariant submanifold of a metallic Riemannian manifold $(\overline{M}, \overline{g}, \overline{J})$. Then the following expressions are correct:

$$TD^{\perp} = \{0\}, \tag{32}$$

$$ND^{\perp} = \overline{J}D^{\perp}, \tag{33}$$

$$ND = \{0\} \tag{34}$$

and

$$TD = D. \tag{35}$$

Furthermore, it is clear that $D = \ker N$ and $D^{\perp} = \ker T$.

Now, we give some facts on semi-invariant submanifolds in metallic Riemannian manifolds to prove our results.

Proposition 3.2. *Any isometrically immersed submanifold M of a metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$ is semi-invariant if and only if*

$$NT = 0. \tag{36}$$

Proof. We assume that the submanifold M is semi-invariant. Then the tangent bundle TM admits the decomposition $TM = D \oplus D^{\perp}$, where D is $\overline{J}$-invariant distribution and $D^{\perp}$ is $\overline{J}$-anti invariant distribution. We denote by r and s the projection operators of the tangent bundle TM onto the distributions D and $D^{\perp}$, respectively. In this case, the following relations hold:

$$r + s = I,\ r^2 = r,\ s^2 = s \text{ and } rs = sr = 0.$$

Thus, every vector field X in $\Gamma(TM)$ can be written in the form

$$X = rX + sX. \tag{37}$$

From (37), the vector field $\overline{J}X$ is given by

$$\overline{J}X = \overline{J}rX + \overline{J}sX$$

for any vector field $X \in \Gamma(TM)$. Then by means of (24), we obtain

$$TX + NX = TrX + NrX + TsX + NsX \tag{38}$$

for any vector field $X \in \Gamma(TM)$. On the other hand, we conclude from (32) and (34) that

$$Ts = 0 \text{ and } Nr = 0.$$

Hence, comparing the tangential and normal parts of both sides of (38), we get

$$T = Tr \text{ and } N = Ns.$$

As a result, we derive from (29) and (34) that

$$NT = 0.$$

Conversely, let us assume that M is an arbitrary submanifold of the metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$ and $NT = 0$. Applying the endomorphism T from the right hand side to (28), we obtain

$$T^3 = pT^2 + qT. \tag{39}$$

Now, we define two operators on M as follows:

$$r = \frac{1}{q}\left(T^2 - pT\right) \text{ and } s = \frac{1}{q}\left(-T^2 + pT + qI\right). \tag{40}$$

Then it is easy to show that the operators r and s satisfy the following relations:

$$r + s = I,\ r^2 = r,\ s^2 = s \text{ and } rs = sr = 0. \tag{41}$$

In other words, r and s are orthogonal complementary projection operators. Thus, we have two orthogonal complementary distributions D and $D^{\perp}$ corresponding to the projection operators r and s, respectively. Taking account of the assumption that $NT = 0$, we deduce from (40) and (41) that

$$Tr = T,\ Ts = 0,\ sTr = sT = 0 \text{ and } Nr = 0,$$

which show that the distribution D is $\overline{J}$-invariant and the distribution $D^{\perp}$ is $\overline{J}$-anti-invariant. Consequently, there exist two orthogonal complementary distributions $\overline{J}$-invariant D and $\overline{J}$-anti-invariant $D^{\perp}$ on the submanifold M. That is, M is a semi-invariant submanifold. □

Proposition 3.3. *Let M be any totally umbilical proper semi-invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$. Then we have the following expressions:*

(**a**) *The invariant distribution D is integrable,*
(**b**) *The anti-invariant distribution $D^{\perp}$ is integrable.*

Proof. Firstly, we note that

$$h\left(X, Y\right) = \overline{g}\left(X, Y\right) H \tag{42}$$

for any vector fields $X, Y \in \Gamma\left(TM\right)$ because of the totally umbilicality of the submanifold M. Taking into account that the semi-invariant submanifold M is a special case of any semi-slant submanifold from the point of the slant angle θ, it seems from [19, Remark 34] that the integrability of the invariant distribution D is equivalent to the condition

$$h\left(\overline{J}X, Y\right) = h\left(X, \overline{J}Y\right) \tag{43}$$

for any vector fields $X, Y \in \Gamma\left(D\right)$. Hence, by virtue of (43), we get from (2) and (42) that (a) holds. At the same time, because the semi-invariant submanifold M is also a hemi-slant submanifold with the slant angle $\theta = 0$, it follows from [20, Theorem 4.9] that the anti-invariant distribution $D^{\perp}$ is integrable if and only if

$$A_{\overline{J}D^{\perp}} D^{\perp} = \{0\},$$

which implies that

$$\overline{g}\left(h\left(X, Y\right), \overline{J}Z\right) = 0 \tag{44}$$

for any vector fields $X \in \Gamma\left(D\right)$ and $Y, Z \in \Gamma\left(D^{\perp}\right)$. Thus, using again (2) and (42), we infer from (44) that (b) is correct. Therefore, the proof has been shown. □

We write $\mathfrak{D}^{\perp} = \overline{J}D^{\perp}$ and denote by $\mathfrak{D}$ its orthogonal complement in the normal bundle $TM^{\perp}$, so we have $TM^{\perp} = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$.

Let us consider a tensor field $\overline{\mathfrak{J}}$ of type $(1, 1)$ on the metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$ defined by

$$\overline{\mathfrak{J}} = -q\overline{J}^{-1}. \tag{45}$$

In this case, it can be easily shown that $\overline{\mathfrak{J}}$ is a metallic structure. Also, the Riemannian metric $\overline{g}$ is $\overline{\mathfrak{J}}$-compatible, i.e.,

$$\overline{g}\left(\overline{\mathfrak{J}}X, Y\right) = \overline{g}\left(X, \overline{\mathfrak{J}}Y\right) \tag{46}$$

for any vector fields $X, Y \in \Gamma\left(T\overline{M}\right)$.

Proposition 3.4. *Let M be any semi-invariant submanifold of a metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$. Then we have the following expressions:*

(**a**) *$\mathfrak{D}$ is a $\overline{\mathfrak{J}}$-invariant distribution,*
(**b**) *$\mathfrak{D}^{\perp}$ is a $\overline{\mathfrak{J}}$-anti-invariant distribution.*

Proof. Let U be in $\Gamma\left(\mathfrak{D}\right)$. Then we obtain from (5), (24), (45) and (46) that

$$\overline{g}\left(\overline{\mathfrak{J}}U, X\right) = -\overline{g}\left(U, NX\right) \tag{47}$$

for any vector field $X \in \Gamma(TM)$. By means of (33) and (34), it follows from (47) that

$$\overline{g}\left(\overline{\mathfrak{J}}U, X\right) = 0,$$

from which we have

$$\overline{\mathfrak{J}}U \in \Gamma\left(TM^{\perp}\right). \tag{48}$$

Now, we suppose that V belongs to $\Gamma(\mathfrak{D}^{\perp})$. Then it means from (33) that there is a vector field $Y \in \Gamma(D^{\perp})$ such that $V = \overline{J}Y$. Hence, we get

$$\overline{\mathfrak{J}}V = -qY \in \Gamma\left(D^{\perp}\right). \tag{49}$$

We infer from (46) and (49) that

$$\overline{g}\left(\overline{\mathfrak{J}}U, V\right) = 0,$$

which implies

$$\overline{\mathfrak{J}}U \in \Gamma(\mathfrak{D}). \tag{50}$$

Thus, it results from (48) and (50) that $\mathfrak{D}$ is a $\overline{\mathfrak{J}}$-invariant distribution, i.e., (a) holds. We remark that $\mathfrak{D}^{\perp} = \overline{J}D^{\perp} \subseteq TM^{\perp}$. If $U \in \Gamma(\mathfrak{D}^{\perp})$, then there exists a vector field Z in $\Gamma(D^{\perp})$ such that $U = \overline{J}Z$. Hence, we obtain

$$\overline{\mathfrak{J}}U = -qZ \in \Gamma\left(D^{\perp}\right) \subseteq \Gamma(TM),$$

which shows that $\mathfrak{D}^{\perp}$ is a $\overline{\mathfrak{J}}$-anti-invariant distribution. That is, (b) is correct. Therefore, the proof has been completed. □

Theorem 3.5. *Let M be any totally umbilical proper semi-invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$. Then the mean curvature vector H belongs to the $\overline{\mathfrak{J}}$-invariant distribution $\mathfrak{D}$, i.e.,*

$$H \in \Gamma(\mathfrak{D}).$$

Proof. Since M is a totally umbilical semi-invariant submanifold, we recall from Proposition 3.3 that the anti-invariant distribution $D^{\perp}$ is integrable. Thus, we obtain

$$\overline{g}\left(H, \overline{J}X\right)Y = A_{\overline{J}X}Y = 0 \tag{51}$$

for any vector fields $X, Y \in \Gamma(D^{\perp})$, from which we have

$$\overline{g}\left(H, \overline{J}X\right) = 0. \tag{52}$$

As it is seen, (52) proves our assertion. Consequently, the proof has been shown. □

Proposition 3.6. *Let M be any totally umbilical proper semi-invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$. Then the covariant derivative of the endomorphism T is identically zero.*

Proof. We remark from [19, Proposition 9] that the covariant derivative of the endomorphism T is given by

$$\left(\overline{\nabla}_X T\right)Y = A_{NY}X + th(X, Y) \tag{53}$$

for any vector fields $X, Y \in \Gamma(TM)$. Taking into account that M is a totally umbilical semi-invariant submanifold, (53) is expressed as follows:

$$\left(\overline{\nabla}_X T\right) Y = \overline{g}(H, NY) X + \overline{g}(X, Y) tH \tag{54}$$

for any vector fields $X, Y \in \Gamma(TM)$. If $H = 0$, the proof is trivial. Now, we suppose that $H \neq 0$. We denote by r and s the projection operators corresponding to the distributions D and $D^{\perp}$, respectively. Hence, (54) takes the form

$$\left(\overline{\nabla}_X T\right) Y = \overline{g}(H, NrY) X + \overline{g}(H, NsY) X + \overline{g}(X, Y) tH \tag{55}$$

for any vector fields $X, Y \in \Gamma(TM)$. On the other hand, using the fact that the distribution $\mathfrak{D}$ is $\overline{\mathfrak{J}}$-invariant in Proposition 3.4, we deduce from (25) that

$$t\mathfrak{D} = \{0\}. \tag{56}$$

In addition, we recall from Theorem 3.5 that

$$H \in \Gamma(\mathfrak{D}). \tag{57}$$

Therefore, using (33), (34), (56) and (57), it results from (55) that $\overline{\nabla} T = 0$. □

Theorem 3.7. *Let M be any totally umbilical semi-invariant submanifold of a locally decomposable metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$. If $\dim \overline{M} = \dim M + \dim D^{\perp}$, then the submanifold M is totally geodesic.*

Proof. If $\dim D^{\perp} = 0$, then the proof is clear. Now, we assume that $\dim D^{\perp} \neq 0$. Theorem 3.5 shows us that

$$H \in \Gamma(\mathfrak{D}). \tag{58}$$

On the other hand, since $\dim \overline{M} = \dim M + \dim D^{\perp}$ in the hypothesis, it clearly seems that

$$\dim D^{\perp} = \dim TM^{\perp}, \tag{59}$$

which implies from the injectiveness of the metallic structure $\overline{J}$ that

$$\overline{J} D^{\perp} = TM^{\perp}. \tag{60}$$

Taking into account the fact that $\mathfrak{D} \subseteq TM^{\perp}$, it follows from (60) that $\mathfrak{D} = \{0\}$. Hence, it is clear from (58) that

$$H = 0,$$

which proves that the submanifold M is totally geodesic. □

Theorem 3.8. *Let M be any totally umbilical proper semi-invariant submanifold which has the non-zero mean curvature vector H in a locally decomposable metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$. If $\dim D \geq 2$, then the submanifold M is an extrinsic sphere.*

Proof. Since $\dim D \geq 2$ in the hypothesis, there exist two non-zero vector fields $X, Y \in \Gamma(D)$ such that

$$\overline{g}(X, Y) = 0. \tag{61}$$

Hence, taking $Z = TY$ in (7), we have

$$\overline{R}(X, Y) \overline{J} TY = \overline{J}\, \overline{R}(X, Y) TY. \tag{62}$$

Using (24), it results from the characterization for semi-invariant submanifolds given in Proposition 3.2 that (62) is written in the form

$$\overline{R}(X,Y)T^2Y = \overline{J}\,\overline{R}(X,Y)TY. \tag{63}$$

On the other hand, Proposition 3.6 shows that the following relation holds:

$$\overline{R}(X,Y)TZ = T\overline{R}(X,Y)Z \tag{64}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$. Thus, taking account of (28), (32), (34) and (64), we deduce from (63) that

$$pT\overline{R}(X,Y)Y + q\overline{R}(X,Y)Y = \overline{J}T\overline{R}(X,Y)Y \tag{65}$$

for any vector fields $X, Y \in \Gamma(D)$. In view of the fact that the distribution D is invariant, applying (35) to (65), we obtain

$$\overline{g}\left(\overline{R}(X,Y)Y, V\right) = 0 \tag{66}$$

for any vector fields $X, Y \in \Gamma(D)$ and $V \in \Gamma(TM^{\perp})$. Furthermore, by means of (19) and (61), (66) is reduced to

$$\overline{g}(Y,Y)\,\overline{g}\left(\nabla^{\perp}_X H, V\right) = 0,$$

which implies that $\nabla^{\perp}_X H = 0$ for any vector field $X \in \Gamma(D)$. In a similar way, it can be shown that $\nabla^{\perp}_X H = 0$ for any vector field $X \in \Gamma(D^{\perp})$. Consequently, $\nabla^{\perp}_X H = 0$ for any vector field $X \in \Gamma(TM)$, in other words, M is an extrinsic sphere. □

Theorem 3.9. *There exist no totally umbilical proper semi-invariant submanifolds in a posivitely or negatively curved locally decomposable metallic Riemannian manifold* $\left(\overline{M}, \overline{g}, \overline{J}\right)$.

Proof. We assume that M is a totally umbilical proper semi-invariant submanifold of a posivitely or negatively curved locally decomposable metallic Riemannian manifold $\left(\overline{M}, \overline{g}, \overline{J}\right)$. If $H = 0$, the proof is clear. Now, let us consider the case $0 \neq H \in \Gamma(\mathfrak{D})$. We choose two vector fields X and V such that $X \in \Gamma(D)$ and $V \in \Gamma\left(\overline{J}D^{\perp}\right)$. In this case, we have

$$\overline{g}(V,H) = 0. \tag{67}$$

Differentiating (67) with respect to the normal connection $\nabla^{\perp}$, we get

$$\overline{g}\left(\nabla^{\perp}_X V, H\right) = -\overline{g}\left(V, \nabla^{\perp}_X H\right). \tag{68}$$

On the other hand, it is not difficult to show that since the metallic structure $\overline{J}$ is parallel with respect to the Levi-Civita connection $\overline{\nabla}$, the metallic structure $\overline{\mathfrak{J}}$ defined by $\overline{\mathfrak{J}} = -q\overline{J}^{-1}$ in (45) is so. Hence, we have

$$\overline{\nabla}_X \overline{\mathfrak{J}} V = \overline{\mathfrak{J}}\,\overline{\nabla}_X V$$

for the vector fields $X \in \Gamma(D)$ and $V \in \Gamma\left(\overline{J}D^{\perp}\right)$. Using the Gauss and Weingarten formulas for totally umbilical submanifolds given in (17) and (18), respectively, we obtain

$$\nabla_X \overline{\mathfrak{J}} V + \overline{g}\left(X, \overline{\mathfrak{J}} V\right) H = -\overline{g}(V,H)\,\overline{\mathfrak{J}} X + \overline{\mathfrak{J}} \nabla^{\perp}_X V. \tag{69}$$

Taking into account of the chosen of the vector fields X and V, with the help of (67) and (69), we obtain

$$\nabla_X \overline{\mathfrak{J}} V = \overline{\mathfrak{J}} \nabla^{\perp}_X V,$$

which implies from the $\overline{\mathfrak{J}}$-anti-invariance of the distribution $\mathfrak{D}^{\perp}$ given in Proposition 3.4 that

$$\nabla^{\perp}_{X}V \in \Gamma\left(\mathfrak{D}^{\perp}\right) \tag{70}$$

for the vector field $X \in \Gamma(D)$. Therefore, it follows from (68) and (70) that

$$\nabla^{\perp}_{X}H \in \Gamma\left(\mathfrak{D}\right) \tag{71}$$

for the vector field $X \in \Gamma(D)$. Also, since the vector field V lies in $\Gamma\left(\overline{J}D^{\perp}\right)$, there is a vector field $Y \in \Gamma\left(D^{\perp}\right)$ such that

$$V = \overline{J}Y. \tag{72}$$

At the same time, by a similar argument used in the proof of Theorem 3.8, it can be demonstrated that

$$\overline{g}\left(\nabla^{\perp}_{Y}H, \overline{J}Y\right) = 0 \tag{73}$$

for the vector fields $X \in \Gamma(D)$ and $Y \in \Gamma\left(D^{\perp}\right)$. On the other hand, we deduce from (12) and (45) that

$$\overline{K}\left(X, Y, X, Y\right) = -\frac{1}{q}\overline{K}\left(X, Y, \overline{\mathfrak{J}}X, \overline{J}Y\right) \tag{74}$$

for the vector fields $X \in \Gamma(D)$ and $Y \in \Gamma\left(D^{\perp}\right)$. Taking account of (74), by the help of (19), (71) and (72), the sectional curvature $\overline{K}$ of the plane section spanned by the vector fields $X \in \Gamma(D)$ and $Y \in \Gamma\left(D^{\perp}\right)$ of the ambient manifold $\overline{M}$ is given by

$$\overline{K}\left(X \wedge Y\right) = -\frac{1}{q}\left\{\overline{g}\left(Y, \overline{\mathfrak{J}}X\right)\overline{g}\left(\nabla^{\perp}_{X}H, \overline{J}Y\right) - \overline{g}\left(X, \overline{\mathfrak{J}}X\right)\overline{g}\left(\nabla^{\perp}_{Y}H, \overline{J}Y\right)\right\}. \tag{75}$$

Hence, using (71) and (73) in (75), we obtain

$$\overline{K}\left(X \wedge Y\right) = 0$$

for the vector fields $X \in \Gamma(D)$ and $Y \in \Gamma\left(D^{\perp}\right)$, which contradicts our assumption. Consequently, the proof has been completed. □

Corollary 3.10. *There exist no totally geodesic proper semi-invariant submanifolds in a posivitely or negatively curved locally decomposable metallic Riemannian manifold* $\left(\overline{M}, \overline{g}, \overline{J}\right)$.

Proof. Taking into account that any totally geodesic submanifold is totally umbilical, the proof is obvious from Theorem 3.9. □

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