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Generalized Relative Essential Spectra

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Abstract. The aim of the present paper is to give some spectral results about generalized Fredholm operators and the so called S-generalized Fredholm operators, where S is a given bounded linear operator acting on a Banach space X. When X possesses some properties, we provide then some sufficient conditions for which a bounded linear operator will be a generalized Fredholm. The obtained results are applied to characterize the so called generalized S-essential spectrum, in particular the generalized Jeribi S-essential spectrum [17]. These results are formulated by means of measure of weak noncompactness.

1. Introduction

Throughout the paper, X and Y are two Banach spaces. The set of all bounded linear operators acting from X into Y is denoted by $\mathcal{L}(X,Y)$. We denote by $\mathcal{K}(X,Y)$ (resp. $\mathcal{W}(X,Y)$) the subset of compact (resp. weakly compact) operators of $\mathcal{L}(X,Y)$. For a linear bounded operator $T : X \to Y$, we denote by $\mathcal{R}(T)$, $Y/\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range, the co-kernel and the kernel of T respectively. The dual (resp. the second dual or bidual) is denoted by X^* (resp. X^{**}), T^* is the conjugate of an operator T and T^{**} is the second conjugate. For a non negative real number, the disc centered at 0 with radius r shall be referred by $D(0,r)$, its closure is $\overline{D}(0,r)$ and for $r_1 \leq r_2$, we conventionally write $C[r_1, r_2] = \overline{D}(0, r_2)\backslash D(0, r_1)$. We denote by $\to$ for the strong convergence and by $\rightharpoonup$ for the weak convergence. Now $T \in \mathcal{L}(X,Y)$ is called tauberian if $T^{**-1}(Y) \subset X$, also T is co-tauberian when its conjugate T^* is tauberian. This definition cannot be reversed as proved by T. Álvarez and M. Gonzàlez in [2]. Another definition of tauberian operator consists in T is tauberian if and only if, T^{co} is injective where T^{co} is the residuum operator of T [15]. For more details on tauberian operators we refer readers to the book of M. González and A. Martinez-Abejón [16] and for more examples to [2, 3, 15]. Tauberian operators have been useful in the study of real interpolation theory of (non reflexive) Banach spaces and some questions related to the preservation of isomorphic properties between those spaces [22], they have also some similar actions on weakly compact sets like upper semi Fredhlom operators whose act on compact sets. The classes of tauberian and co-tauberian operators from X into Y are respectively denoted by $\mathcal{T}(X,Y)$ and $\mathcal{T}^d(X,Y)$.

In 1976, K. W. Yang [26] introduced the class of generalized Fredholm operators for linear bounded operators acting on a Banach space as some extension of the class of Fredholm operators. The sets of upper generalized

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semi-Fredholm operators and lower generalized semi-Fredholm operators are respectively defined and denoted by $\Phi_{g+}(X,Y)$ and $\Phi_{g-}(X,Y)$ as:

$$\Phi_{g+}(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \mathcal{N}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\},$$

$$\Phi_{g-}(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } Y/\mathcal{R}(T) \text{ is reflexive and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

The set $\Phi_g(X,Y) := \Phi_{g+}(X,Y) \cap \Phi_{g-}(X,Y)$ is formed by all generalized Fredholm operators and $\Phi_{g\pm}(X,Y) := \Phi_{g+}(X,Y) \cup \Phi_{g-}(X,Y)$. Now, for $S \in \mathcal{L}(X,Y)$, a complex number λ is in $\Phi_{g+,T,S}(X,Y)$, $\Phi_{g-,T,S}(X,Y)$, $\Phi_{g\pm,T,S}(X,Y)$ or $\Phi_{g,T,S}(X,Y)$ if $\lambda S - T$ is in $\Phi_{g+}(X,Y)$, $\Phi_{g-}(X,Y)$, $\Phi_{g\pm}(X,Y)$ or $\Phi_g(X,Y)$, respectively. Clearly, similarity brings out the correspondence between reflexive Banach spaces and finite-dimensional spaces, generalized Fredholm operators and Fredholm operators, tauberian operators with closed range and semi-Fredholm operators. Identity operator is the simplest example of generalized Fredholm operator. We will present some nontrivial examples of generalized Fredholm operators at the end of section 3.
Recently, C. Schmoeger [24, 25] presented different definitions of generalized Fredholm operators, we mention the following ones: T is a generalized Fredholm operator if and only if $T = T_1 \oplus T_2$, where T_1 is a Fredholm operator with vanish jump and T_2 is a finite-dimensional nilpotent operator, many other definitions are also given but will be dropped in this paper.
In a Banach space which has no reflexive infinite dimensional subspace, the upper semi-Fredholm operators (operators with closed range and finite dimensional kernel) are trivial examples of tauberian operators [20]. Furthermore, the class of upper generalized semi-Fredholm operators acting on a Banach space is strictly contained in the class of tauberian operators. Indeed, it was proved in [19] that there exists a tauberian operator (with non closed range) which cannot be an upper generalized semi-Fredholm. Finally, notice that $\Phi(X,Y) \subset \Phi_g(X,Y)$, where $\Phi(X,Y)$ is the set of Fredholm operators from X to Y [17]. When $X = Y$, all the sets $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{W}(X,Y)$, $\Phi(X,Y)$, $\Phi_g(X,Y)$, $\Phi_{g+}(X,Y)$, $\Phi_{g-}(X,Y)$, $\mathcal{T}(X,Y)$, $\mathcal{T}^d(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{W}(X)$, $\Phi(X)$, $\Phi_g(X)$, $\Phi_{g+}(X)$, $\Phi_{g-}(X)$, $\mathcal{T}(X)$, $\mathcal{T}^d(X)$ respectively.

We recall the S-resolvent set and the S-spectrum of a closed linear operator T acting on a Banach space X, when $S \neq 0$ is a bounded linear operator on X, respectively by

$$\rho_S(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \text{ has a bounded inverse}\},$$

and

$$\sigma_S(T) := \mathbb{C} \backslash \rho_S(T).$$

Subsequently, the operator S should be taken as non invertible. Otherwise the S-resolvent coincides with usual resolvent of the operator $S^{-1}T$. Notice that relative spectra (or S-spectra) is introduced in [13] in order to show the characterization of essential spectrum of the pencil operators. Finally, we recall that an operator $T \in \mathcal{L}(X)$ is weakly compact if $T(M)$ is relatively weakly compact set for every bounded subset $M \subset X$. Note that $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$, we refer to [10, 14] for more details.

Our motivation to use the class of generalized Fredholm operators is to describe the generalized S-essential spectra of a bounded linear operator acting on a (non necessary reflexive) Banach space. In particular, the generalized Wolf and the generalized Gustafson S-essential spectrum. These results may extend some results established by Jeribi in [17] for the Fredholm theory's frame.

The paper is organized as follow. In Section 2, we present some basic facts of weakly compact operators and their connections with tauberian operators and some measurement tools. We suggest in Section 3, some Banach spaces X having some properties, denoted for instance by (H_1) and (H_2) or one of them to get some results concerning generalized Fredholm operators. Therefore, we obtain extensions of some results in [1, 6, 11, 21]. In particular, we provide some sufficient conditions for a linear bounded operator to be generalized Fredholm. In Section 4, we introduce and investigate the generalized S-essential spectrum and so we provide a characterization of the generalized S-essential spectral radius via the concept of measure of weak noncompactness.

2. Basic facts

The notion of a measure of weak noncompactness was introduced by De Blasi [8] and subsequently used in topology, functional analysis and theory of differential and integral equations, (see [4, 5, 12] for instance). Let X be a Banach space, and $\mathcal{M}_X$ (resp. $\mathcal{K}^w(X)$) stands to the set of bounded sets of X (resp. the set of all weakly compact subsets of X). Let $B_r = B(0,r)$ be the open ball centered at 0 and with radius r. Finally we denote by $conv(A)$ the convex hull of the set A, $A \subset X$.
The De Blasi measure of noncompactness of a non empty bounded subset $A \subset X$, denoted by $\omega : \mathcal{M}_X \longrightarrow [0,+\infty[$ is defined as follow:

$$\omega(A) = \inf\{r > 0, \text{ there exists } N \in \mathcal{K}^w(X) \text{ such that } A \subset N + \overline{B}_r\}.$$

This definition can be also reformulated as axiomatic statements [5, 18].

Definition 2.1. A function $\mu : \mathcal{M}_X \longrightarrow [0,+\infty[$ is said to be a measure of weak noncompactness in X if it satisfies the following conditions.
(*i*) $\mu(A) = 0$ if, and only if, A is relatively weakly compact set,
(*ii*) if $A \subset B$, then $\mu(A) \leq \mu(B)$,
(*iii*) $\mu(\overline{conv(A)}) = \mu(A)$,
(*iv*) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$,
(*v*) $\mu(A + B) \leq \mu(A) + \mu(B)$,
(*vi*) $\mu(\lambda A) = |\lambda|\mu(A)$, for $\lambda \in \mathbb{C}$.
From [5], the measure of weak noncompactness, guarantees the Cantor intersection condition, and the following inequality for any measure μ

$$\mu(A) \leq \mu(B_r)\omega(A).$$

We also can define the measure of weak noncompactness of a bounded linear operator T, denoted by $\overline{\omega}(T)$, as follow:

$$\overline{\omega}(T) = \inf\{k \text{ such that } \omega(T(A)) \leq k\omega(A), \text{ for all } A \in \mathcal{M}_X\}.$$

The following proposition collects some similar properties of $\overline{\omega}(\cdot)$ showed in Definition 2.1.

Proposition 2.2. Let X be a Banach space, T and $S \in \mathcal{L}(X)$ and let $B \in \mathcal{M}_X$. Then, we have the following properties:

(*i*) $\overline{\omega}(T) = 0$ if, and only if, T is weakly compact.

(*ii*) $\omega(T(B)) \leq \overline{\omega}(T)\omega(B)$.

(*iii*) $\overline{\omega}(TS) \leq \overline{\omega}(T)\overline{\omega}(S)$.

(*iv*) $\overline{\omega}(T + S) \leq \overline{\omega}(T) + \overline{\omega}(S)$.

(*v*) $\overline{\omega}(\lambda T) = |\lambda|\overline{\omega}(T)$, for $\lambda \in \mathbb{C}$.

Definition 2.3. Let X be a Banach space. We say that X has the property (H_1) (resp. (H_2)) if every closed reflexive subspace admits a closed complementary subspace (resp. if every closed subspace with reflexive quotient space admits a closed complementary subspace).
We say that X has the property (H), if it satisfies both properties (H_1) and (H_2).

A basic example of space having the property (H_1) is given in [23]. Indeed, for $1 < p < \infty$, $p \neq 2$, $L_p(0,1)$ has the property (H_1). Moreover, $L_1(\mu)$ and $C(S)$ the space of all bounded (real or complex valued) continuous functions on an infinite compact Hausdorff space S do not have it [10, 16].
Now, let us recall a characterization of a generalized Fredholm operator using a definition due to K. W. Yang [26].

Theorem 2.4. Let X and Y be two Banach spaces satisfying the properties (H_1) and (H_2) respectively, and let $T \in \mathcal{L}(X,Y)$. Then the following assertions are equivalent.

(*i*) T is a generalized Fredholm operator .
(*ii*) There exist weakly compact operators $W_1 \in \mathcal{W}(X)$, $W_2 \in \mathcal{W}(Y)$ and an operator $T_0 \in \mathcal{L}(Y,X)$ such that $T_0T = I + W_1$ and $TT_0 = I + W_2$, and $\mathcal{R}(T)$ is closed in Y.

Definition 2.5. [1] Let X and Y be two Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is said to be strictly singular if there is no infinite dimensional subspace M of X such that $T : M \to T(M)$, the restriction of T to M, is an isomorphism.

Notice that singular operators need not to be compact, as shown in [1] where the natural embedding $J : l_p \longrightarrow l_r$ (for $1 \leq p < r < \infty$) is a non-compact singular operator.

The following is a simplest characterization of singular operators [1].

Theorem 2.6. For $T \in \mathcal{L}(X,Y)$, the following statements are equivalent.
(*i*) T is strictly singular.
(*ii*) For every infinite dimensional closed subspace X_1 of X, there exists an infinite dimensional closed subspace X_2 of X_1 such that $T : X_2 \to Y$ is a compact operator.

3. Main results

Let X be a Banach space and let $T \in \mathcal{L}(X)$. Recall that $\mathcal{M}_X$ is the set of bounded sets of X. We introduce the following non-negative quantities:

$$\begin{aligned}\overline{\alpha}(T) &:= \sup\left\{\frac{\omega(T(A))}{\omega(A)} \text{ such that } A \in \mathcal{M}_X \text{ and } \omega(A) > 0\right\} = \overline{\omega}(T),\\ \overline{\beta}(T) &:= \inf\left\{\frac{\omega(T(A))}{\omega(A)} \text{ such that } A \in \mathcal{M}_X \text{ and } \omega(A) > 0\right\}.\end{aligned}$$

We set $\overline{\alpha}_0(T)$ (resp. $\overline{\beta}_0(T)$) as the limit of the sequence $\overline{\alpha}(T^n)^{\frac{1}{n}}$ (resp. $\overline{\beta}(T^n)^{\frac{1}{n}}$). These limits exist (see [[21], Lemma 1.21]). The following result gives some properties of these functions.

Proposition 3.1. Let X be a Banach space. Let $T, S \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$, then the following properties hold:
(*i*) $\overline{\alpha}(\lambda T) = |\lambda|\overline{\alpha}(T)$ and $\overline{\beta}(\lambda T) = |\lambda|\overline{\beta}(T)$.
(*ii*) $|\overline{\alpha}(T) - \overline{\alpha}(S)| \leq \overline{\alpha}(T+S) \leq \overline{\alpha}(T) + \overline{\alpha}(S)$.
(*iii*) $\overline{\beta}(T) - \overline{\alpha}(S) \leq \overline{\beta}(T+S) \leq \overline{\beta}(T) + \overline{\alpha}(S)$.
(*iv*) $\overline{\alpha}(TS) \leq \overline{\alpha}(T)\overline{\alpha}(S)$ and $\overline{\beta}(TS) \geq \overline{\beta}(T)\overline{\beta}(S)$.
(*v*) $\overline{\alpha}(T) = 0 \Leftrightarrow T$ is weakly compact.

Proof. Proofs of (*i*), (*ii*) and (*iii*) are standard when applying definitions of $\overline{\alpha}$ and $\overline{\beta}$ together with Definition 2.1 and Proposition 2.2.
(*iv*) From the properties of ω (see Definition 2.1 and Proposition 2.2), we have

$$\omega((TS)(A)) \leq \overline{\omega}(T)\overline{\omega}(S)\omega(A),$$

then,

$$\sup\left\{\frac{\omega((TS)(A))}{\omega(A)},\ \omega(A) > 0\right\} \leq \overline{\omega}(T)\overline{\omega}(S),$$

thus,

$$\overline{\alpha}(TS) \leq \overline{\alpha}(T)\overline{\alpha}(S).$$

It follows that

$$\begin{aligned}\frac{\omega((TS)(A))}{\omega(A)} &\leq \frac{\omega(T(S(A)))}{\omega(A)}\\ &\leq \overline{\alpha}(T)\frac{\omega(S(A))}{\omega(A)},\end{aligned}$$

then

$$\inf\left\{\frac{\omega((TS)(A))}{\omega(A)},\ \omega(A) > 0\right\} \geq \overline{\alpha}(T)\inf\left\{\frac{\omega(S(A))}{\omega(A)},\ \omega(A) > 0\right\},$$

thus,

$$\overline{\beta}(TS) \geq \overline{\alpha}(T)\overline{\beta}(S). \quad (1)$$

Consequently,

$$\overline{\beta}(TS) \geq \overline{\beta}(T)\overline{\beta}(S).$$

(v) Let A be a bounded set. $\omega(A) = 0$ implies that A is relatively weakly compact, hence $T(A)$ is relatively weakly compact. Now for $\omega(A) > 0$, if $\overline{\alpha}(T) = 0$, then $\frac{\omega(T(A))}{\omega(A)} = 0$, implies that $\omega(T(A)) = 0$ which also implies that $T(A)$ is relatively weakly compact. Conversely, let A be a bounded set such that $\omega(A) > 0$. If $T(A)$ is relatively weakly compact, then $\frac{\omega(T(A))}{\omega(A)} = 0$, and so

$$\sup\left\{\frac{\omega(T(A))}{\omega(A)},\ A \in \mathcal{M}_X \text{ and } \omega(A) > 0\right\} = \overline{\alpha}(T) = 0.$$

Which concludes the proof. □

Remark 3.2. In relation (1), we can deduce that $\overline{\omega}(TS) \geq \overline{\beta}(TS) \geq \overline{\omega}(T)\overline{\beta}(S)$.

Now, let us introduce the following definition needed in the sequel.

Definition 3.3. Let X and Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. We say that T is weakly proper if for every $K \in \mathcal{K}^w(Y)$, $T^{-1}(K) \in \mathcal{K}^w(X)$.

The following lemma gives a characterization of weakly proper operator see also (Definition 28, [7]).

Lemma 3.4. Let X be a Banach space, $T \in \mathcal{L}(X)$. If $\overline{\beta}(T) > 0$, then T is weakly proper on bounded sets.

Proof. Let K be weakly compact set of X and B a bounded subset of X. We prove that $A = B \cap T^{-1}(K)$ is weakly compact. Suppose the opposite, we have then $\omega(A) \neq 0$. Observe that

$$T(A) = T(B \cap T^{-1}(K)) \subset T(B) \cap T(T^{-1}(K)) \subset T(B) \cap K,$$

then

$$\begin{aligned} \omega(T(A)) &\leq \omega(T(B) \cap K) \\ &\leq \omega(K)(= 0). \end{aligned}$$

It follows that $\omega(T(A)) = 0$. Now, since $\overline{\beta}(T)\omega(A) \leq \omega(T(A))$ and $\omega(A) \neq 0$, then $\overline{\beta}(T) = 0$, which is a contradiction whence the result. □

Theorem 3.5. Let X be a Banach space and $T \in \mathcal{L}(X)$. If for any bounded subset B of X and for any compact K of X, the set $\{x \in B : Tx \in K\}$ is weakly compact, then $T \in \Phi_{g+}(X)$.

Proof. Let $B := \overline{B}(0, 1)$ denote the closed unit ball. Obviously, $\mathcal{N}(T) \cap B = T^{-1}\{0\} \cap B$ is weakly compact, then $\mathcal{N}(T)$ is reflexive. Suppose that $\mathcal{R}(T)$ is not closed, then there exists a sequence $(x_n)_n$ in X such that $(Tx_n)_n$ converge to a point $y \in X \backslash \mathcal{R}(T)$. Note we may suppose for all $n \in \mathbb{N}$ that $x_n \notin \mathcal{N}(T)$. For each $n \in \mathbb{N}$, put $d_n = dist\,(x_n, \mathcal{N}(T))$. Clearly, $d_n > 0$. We claim that $d_n \to \infty$ as $n \to \infty$. To see this, suppose otherwise. Then there exist $A > 0$ and a subsequence $(d_{\varphi(n)})_n$ of $(d_n)_n$ such that $d_{\varphi(n)} < A$ for all $n \in \mathbb{N}$. Since $d_{\varphi(n)} = \inf_{y \in \mathcal{N}(T)} \|x_{\varphi(n)} - y\|$, there exists a sequence $(y_{\varphi(n)})_n \subset \mathcal{N}(T)$ such that for all $n \in \mathbb{N}$, $\|x_{\varphi(n)} - y_{\varphi(n)}\| < 2A$. Hence, $(x_{\varphi(n)} - y_{\varphi(n)})_n \subset B(0, 2A)$. Now, observe that

$$\{T(x_{\varphi(n)} - y_{\varphi(n)}), n \in \mathbb{N}\} \subset \{Tx_{\varphi(n)}, n \in \mathbb{N}\} \cup \{y\} := K.$$

It follows that

$$(x_{\varphi(n)} - y_{\varphi(n)})_n \subset T^{-1}(K) \cap \overline{B}(0, 2A).$$

Since K is compact, then $T^{-1}(K) \cap \overline{B}(0, 2A)$ is weakly compact, then there exist a subsequence $(x_{\varphi\circ\psi(n)} - y_{\varphi\circ\psi(n)})_{n\in\mathbb{N}}$ of $(x_{\varphi(n)} - y_{\varphi(n)})_n$ and $x \in X$ such that $x_{\varphi\circ\psi(n)} - y_{\varphi\circ\psi(n)} \rightharpoonup x$. Thus,

$$T(x_{\varphi\circ\psi(n)} - y_{\varphi\circ\psi(n)}) \rightharpoonup y.$$

It follows that $y = Tx$, which is a contradiction. Hence, $d_n \to \infty$.
Now let us define the sequence $(z_n)_n$ by:

$$z_n = \frac{(x_{\varphi(n)} - y_{\varphi(n)})}{\|x_{\varphi(n)} - y_{\varphi(n)}\|}, \ n \in \mathbb{N}.$$

Clearly,

$$\begin{aligned} \|Tz_n\| &= \frac{1}{\|x_{\varphi(n)} - y_{\varphi(n)}\|}\|Tx_{\varphi(n)}\| \\ &\leq \frac{1}{d_{\varphi(n)}}\|Tx_{\varphi(n)}\|. \end{aligned}$$

Thus,

$$\|Tz_n\| \leq \frac{1}{d_{\varphi(n)}}\|T\|(\|x_{\varphi(n)} - y_{\varphi(n)}\| + \|y_{\varphi(n)}\|). \tag{2}$$

Since $(y_{\varphi(n)})_n \subset B(x_{\varphi(n)}, 2A) \cap \mathcal{N}(T)$ is weakly compact, then there exist a subsequence $(y_{\varphi\circ\gamma(n)})_{n\in\mathbb{N}}$ and $M > 0$ such that $\|y_{\varphi\circ\gamma(n)}\| \leq M$ for all $n \in \mathbb{N}$. By Inequality (2), we obtain

$$\|Tz_{\gamma(n)}\| \leq \frac{1}{d_{\varphi\circ\gamma(n)}}\|T\|(2A + M).$$

Thus, $Tz_{\gamma(n)} \to 0$. Since $\|z_{\gamma(n)}\| = 1$ and $Tz_{\gamma(n)} \to 0$, then $(z_{\gamma(n)})_{n\in\mathbb{N}} \subset \overline{B}(0,1) \cap T^{-1}(K')$ where K' is the compact set defined by $K' := \{Tz_{\gamma(n)}, \ n \in \mathbb{N}\} \cup \{0\}$. Taking into account that $\overline{B}(0,1) \cap T^{-1}(K')$ is weakly compact, then there exist a subsequence $(z_{\gamma\circ\xi(n)})_n$ of $(z_{\gamma(n)})_n$ and $z \in X$ such that $z_{\gamma\circ\xi(n)} \rightharpoonup z$, then $Tz_{\gamma\circ\xi(n)} \rightharpoonup Tz$. It follows that $z \in \mathcal{N}(T)$.
But for all $n \in \mathbb{N}$,

$$\begin{aligned} \|z_{\gamma\circ\xi(n)} - z\| &= \left\|\frac{x_{\varphi\circ\gamma\circ\xi(n)} - y_{\varphi\circ\gamma\circ\xi(n)}}{\|x_{\varphi\circ\gamma\circ\xi(n)} - y_{\varphi\circ\gamma\circ\xi(n)}\|} - z\right\| \\ &= \frac{1}{\|x_{\varphi\circ\gamma\circ\xi(n)} - y_{\varphi\circ\gamma\circ\xi(n)}\|}\|x_{\varphi\circ\gamma\circ\xi(n)} - y_{\varphi\circ\gamma\circ\xi(n)} - z\|x_{\varphi\circ\gamma\circ\xi(n)} - y_{\varphi\circ\gamma\circ\xi(n)}\|\| \\ &\geq \frac{1}{2A}(d_{\varphi\circ\gamma\circ\xi(n)} - \|z\|d_{\varphi\circ\gamma\circ\xi(n)}) \\ &\geq \frac{1}{2A}(1 - \|z\|)d_{\varphi\circ\gamma\circ\xi(n)}. \end{aligned}$$

Since $d_{\varphi\circ\gamma\circ\xi(n)} \to \infty$, it follows that $\|z_{\varphi\circ\gamma(n)} - z\| \to \infty$, which is a contradiction. □

Theorem 3.6. Let X be a non-reflexive Banach space and $T \in \mathcal{L}(X)$. The following statements hold.

(*i*) If $\overline{\beta}(T) > 0$, then $T \in \Phi_{g+}(X)$.
(*ii*) Assume that X satisfies the property (H_1) and $T \in \Phi_{g+}(X)$, then $\overline{\beta}(T) > 0$.

Proof. (*i*) By Lemma 3.4 and Theorem 3.5 we conclude the result.
(*ii*) Let X be a non-reflexive Banach space having the property (H_1) and $T \in \mathcal{L}(X)$. Since T is an upper generalized semi-Fredholm operator then it is left invertible modulo a weakly compact operator and so there exist $T_0 \in \mathcal{L}(X)$ and $W \in \mathcal{W}(X)$ such that $T_0T = I + W$.
If $\overline{\beta}(T) = 0$, then for all $\varepsilon > 0$, there exists a bounded subset D_ε with $\omega(D_\varepsilon) > 0$ and $\frac{\omega(T(D_\varepsilon))}{\omega(D_\varepsilon)} \leq \varepsilon$. Take $\varepsilon < \frac{1}{\overline{\omega}(T_0)}$. Then,

$$\begin{aligned}
\omega(T(D_\varepsilon)) &\leq \varepsilon\omega(D_\varepsilon) \\
&\leq \varepsilon\omega((T_0T - W)D_\varepsilon) \\
&\leq \varepsilon\omega(T_0TD_\varepsilon) + \varepsilon\omega(WD_\varepsilon) \\
&\leq \varepsilon\overline{\omega}(T_0)\omega(T(D_\varepsilon)) + \varepsilon\overline{\omega}(W)\omega(D_\varepsilon) \\
&\leq \varepsilon\overline{\omega}(T_0)\omega(T(D_\varepsilon)).
\end{aligned}$$

It follows that

$$(1 - \varepsilon\overline{\omega}(T_0))\omega(T(D_\varepsilon)) \leq 0.$$

Since $\varepsilon\overline{\omega}(T_0) < 1$, then $\omega(T(D_\varepsilon)) = 0$ i.e., $T(D_\varepsilon)$ is relatively weakly compact set. Since T is left invertible modulo weakly compact operator, then $T_0TD_\varepsilon = ((I + W)D_\varepsilon)$ is relatively weakly compact. Hence D_ε is relatively weakly compact i.e., $\omega(D_\varepsilon) = 0$, which is a contradiction. □

Now, we present the following remark.

Remark 3.7. It easy to check that, $T \in \Phi_{g+}(X)$ if and only if, $T^* \in \Phi_{g-}(X^*)$, where X is a Banach space.

Theorem 3.8. Let X be a non-reflexive Banach space and $T \in \mathcal{L}(X)$. The following statements hold.

(*i*) If $\overline{\beta}(T^*) > 0$, then $T \in \Phi_{g-}(X)$.
(*ii*) If $\overline{\beta}(T) > 0$ and $\overline{\beta}(T^*) > 0$, then $T \in \Phi_g(X)$.

Proof. (*i*) $\overline{\beta}(T^*) > 0$, then by Theorem 3.6 (*i*), we conclude that $T^* \in \Phi_{g+}(X^*)$. Hence, $T \in \Phi_{g-}(X)$.
(*ii*) Combining assumption (*i*) and Theorem 3.6 (*i*), we obtain the result. □

As proved in ([21], Theorems 5,6,9), we can extend these results to generalized Fredholm operators.

Theorem 3.9. Let X be a non-reflexive Banach space having the property (H_1) and let $T, S \in \mathcal{L}(X)$. The following statements hold.

(*i*) If $ST \in \Phi_{g+}(X)$, then $T \in \Phi_{g+}(X)$.
(*ii*) If $ST \in \Phi_{g-}(X)$ and X^* satisfies the property (H_1), then $S \in \Phi_{g-}(X)$.
(*iii*) If $ST \in \Phi_g(X)$ and X^* satisfies the property (H_1), then $S \in \Phi_{g-}(X)$ and $T \in \Phi_{g+}(X)$.

Proof. (*i*) Let $ST \in \Phi_{g+}(X)$. Since X satisfying the property (H_1), then there exist $A \in \mathcal{L}(X)$ and $W_1 \in \mathcal{W}(X)$ such that $AST = I - W_1$ which implies that $\overline{\beta}(T) > 0$ and so by Theorem 3.6 (*i*) we infer that $T \in \Phi_{g+}(X)$.
(*ii*) The proof of (*ii*) can be checked in a dual way as (*i*).
(*iii*) The proof of (*iii*) follows from (*i*) and (*ii*). □

As a consequence result of Theorems 3.6 and 3.9 (*i*), we have the following remark.

Remark 3.10. Let X be a non-reflexive Banach space having the property (H_1) and let $T \in \mathcal{L}(X)$. If T is left and right invertible modulo the weakly compact operators, then T is a generalized Fredholm operator on X.

We can get now, some perturbation results under properties (H_1), (H_2) and both of them.

Theorem 3.11. Let X, Y and Z be three non-reflexive Banach spaces and let $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$. The following statements hold.
(i) Assume that X, Y and Z satisfy the properties (H_1), (H) and (H_2) respectively. If $T \in \Phi_g(X, Y)$ and $S \in \Phi_g(Y, Z)$, then $ST \in \Phi_g(X, Z)$.
(ii) Assume that X and Y satisfy the properties (H_1) and (H_2) respectively. If $T \in \Phi_g(X, Y)$ and $W \in \mathcal{W}(X, Y)$, then $(T + W) \in \Phi_g(X, Y)$.
For $X = Y = Z$, we have the following assertions:
(iii) Assume that X has the property (H_1). If $T \in \Phi_{g+}(X)$ and $W \in \mathcal{W}(X)$, then $(T + W) \in \Phi_{g+}(X)$.
(iv) Assume that X^* satisfies the property (H_1). If $T \in \Phi_{g-}(X)$ and $W \in \mathcal{W}(X)$, then $(T + W) \in \Phi_{g-}(X)$.
(v) Assume that X has the property (H_1). If $T \in \Phi_{g+}(X)$ and $S \in \Phi_{g+}(X)$, then $ST \in \Phi_{g+}(X)$.
(vi) Assume that X^* has the property (H_1). If $T \in \Phi_{g-}(X)$ and $S \in \Phi_{g-}(X)$, then $ST \in \Phi_{g-}(X)$.

Proof. (i) Since X, Y and Z satisfy the properties (H_1), (H) and (H_2) respectively and $T \in \Phi_g(X, Y)$ and $S \in \Phi_g(Y, Z)$, then by Theorem 2.4 there exist $T_0 \in \mathcal{L}(Y, X)$, $S_0 \in \mathcal{L}(Z, Y)$ and $W_1 \in \mathcal{W}(X)$, $W_2, W_3 \in \mathcal{W}(Y)$ and $W_4 \in \mathcal{W}(Z)$ such that

$$T_0T = I_X + W_1 \quad \text{on } X, \qquad TT_0 = I_Y + W_2 \quad \text{on } Y.$$
$$S_0S = I_Y + W_3 \quad \text{on } Y, \qquad SS_0 = I_Z + W_4 \quad \text{on } Z.$$

Then,

$$T_0S_0ST = T_0(I_Y + W_3)T = I_X + W_1 + T_0W_3T = I_X + W_5, \ W_5 \in \mathcal{W}(X),$$

and

$$STT_0S_0 = S(I_Y + W_2)S_0 = I_Z + W_4 + SW_2S_0 = I_Z + W_6, \ W_6 \in \mathcal{W}(Z).$$

Now, by using Remark 3.10, we deduce that $ST \in \Phi_g(X, Z)$.
(ii) The proof of (ii) is similar to (i).
(iii) Since X has the property (H_1), $T \in \Phi_{g+}(X)$ and $W \in \mathcal{W}(X)$, then by using assertions (iii) and (v) from Proposition 3.1, we get $\overline{\beta}(T+W) = \overline{\beta}(T) > 0$. It follows from assertion (i) of Theorem 3.6 that $(T+W) \in \Phi_{g+}(X)$.
(iv) Since X^* has the property (H_1) and $T^* \in \Phi_{g+}(X^*)$, then there exist $T_0 \in \mathcal{L}(X^*)$ and $W_1 \in \mathcal{W}(X^*)$ such that $T_0T^* = I + W_1$. Hence,

$$T_0(T + W)^* = I + W_1 + T_0W^*,$$

where $W^* \in \mathcal{W}(X^*)$. Implies that $(T + W)T_0^* \in \Phi_{g-}(X)$. It follows from Theorem 3.9 (ii) that $(T + W) \in \Phi_{g-}(X)$.
(v) Since X having the property (H_1) and $T, S \in \Phi_{g+}(X)$, then by assertion (ii) of Theorem 3.6, we conclude that $\overline{\beta}(T) > 0$ and $\overline{\beta}(S) > 0$. Taking into account that $\overline{\beta}(ST) \geq \overline{\beta}(T)\overline{\beta}(S) > 0$, we deduce that $ST \in \Phi_{g+}(X)$.
(vi) Assume that X^* having the property (H_1). Since $T^*, S^* \in \Phi_{g+}(X^*)$, then by the previous assertion (v), we deduce that $T^*S^* \in \Phi_{g+}(X^*)$ and consequently, $ST \in \Phi_{g-}(X)$. □

Theorem 3.12. Let X and Y be two non-reflexive Banach spaces having the properties (H_1) and (H_2) respectively. Assume that $S \in \Phi_g(X, Y)$. Then there is an $\eta > 0$ such that for any $T \in \mathcal{L}(X, Y)$ satisfying $\|T\| < \eta$, one has

$$(S + T) \in \Phi_g(X, Y).$$

Proof. Since $S \in \Phi_g(X, Y)$, then by Theorem 2.4 there exist $S_0 \in \mathcal{L}(Y, X)$, $W_1 \in \mathcal{W}(X)$ and $W_2 \in \mathcal{W}(Y)$ such that $S_0S = I + W_1$ and $SS_0 = I + W_2$ and so,

$$S_0(S + T) = I + W_1 + S_0T \quad \text{on } X,$$

$$(S + T)S_0 = I + W_2 + TS_0 \quad \text{on } Y.$$

We have

$$\|S_0T\| \leq \|S_0\|\|T\| \quad \text{and} \quad \|TS_0\| \leq \|T\|\|S_0\|.$$

Take $\eta = \|S_0\|^{-1}$, hence $\|S_0T\| < 1$ and $\|TS_0\| < 1$. Thus the operators $I + S_0T$ and $I + TS_0$ have bounded inverses and so,

$$(I + S_0T)^{-1}S_0(S + T) = I + (I + S_0T)^{-1}W_1 \quad \text{on } X,$$

$$(S+T)S_0(I+TS_0)^{-1} = I + W_2(I+TS_0)^{-1} \quad \text{on } Y.$$

Thus, by using Remark 3.10 we infer that $(S+T) \in \Phi_g(X,Y)$. □

Proposition 3.13. Let X and Y be two non-reflexive Banach spaces having the properties (H_1) and (H_2) respectively and let $T, S \in \mathcal{L}(X,Y)$ where S is a nonzero operator. Then, $\Phi_{g,T,S}$ is open.

Proof. Let $\lambda_0 \in \Phi_{g,T,S}$, then by Theorem 3.12 there exists $\eta > 0$ such that for all $\alpha \in \mathbb{C}$ with $|\alpha| < \frac{\eta}{\|S\|}$, the operator $(\lambda_0 S - \alpha S - T) \in \Phi_g(X,Y)$. Consider $|\lambda - \lambda_0| < \frac{\eta}{\|S\|}$, then $(\lambda S - T) \in \Phi_g(X,Y)$. So, $\Phi_{g,T,S}$ is open. □

The following result is an extension to weakly proper operator of the Lemma 4.2 in [11].

Lemma 3.14. Let X be a Banach space and let $T \in \mathcal{L}(X)$. Suppose that for some $n \in \mathbb{N}\backslash\{0\}$, $\overline{\omega}(T^n) < 1$. Then for any closed bounded $B \subset X$ and for any weakly compact $K \subset X$, the set $\{x \in B : (I-T)x \in K\}$ is weakly compact, i.e., $I - T$ is weakly proper on closed bounded sets.

Proof. Set $M := \{x \in B : (I-T)x \in K\}$, M is closed and bounded and K is weakly compact. It remains to prove that $\omega(M) = 0$. For this purpose let $x \in M$, then there exists $y \in K$ such that $x = Tx + y$. Thus, $(I-T)(Tx+y) = Tx + y - T^2x - Ty = y$. Hence,

$$x = T^2x + Ty + y.$$

When calculate $(I-T)(T^2x + Ty + y) = y$, we get

$$x = T^3x + T^2y + Ty + y.$$

More generally, for any $n \in \mathbb{N}$, we obtain:

$$x = T^n x + \sum_{j=0}^{n-1} T^j y. \tag{3}$$

Let $K_1 := \sum_{j=0}^{n-1} T^j(K)$. Since T is continuous then K_1 is weakly compact, and from Eq. (3), we deduce that

$$M \subset T^n(M) + K_1,$$

then,

$$\omega(M) \leq \omega(T^n(M)) \leq \overline{\omega}(T^n)\omega(M).$$

Thus, $(1 - \overline{\omega}(T^n))\omega(M) \leq 0$ and consequently, $\omega(M) = 0$. □

We can now present a new generalization of Theorem 4.4 in [11]. Before that, let us present the following useful lemma. The proof can be found in [16].

Lemma 3.15. Let X, Y be two Banach spaces and $Z \subset X$. For an operator $T \in \mathcal{L}(X,Y)$, the following statements are equivalent.

(*i*) T is co-tauberian.

(*ii*) Every operator $S \in \mathcal{L}(Y,Z)$ is weakly compact whenever ST is weakly compact.

Theorem 3.16. Let X be a Banach space and let $T \in \mathcal{L}(X)$ such that $\overline{\omega}(T^n) < 1$ for some $n \in \mathbb{N}\backslash\{0\}$. Then, $(I-T) \in \Phi_g(X)$.

Proof. By Theorem 3.5 and Lemma 3.14, we show that

$$(I - T) \in \Phi_{g+}(X). \tag{4}$$

Let $S \in \mathcal{L}(X)$ such that $S(I - T)$ is weakly compact. We have to prove that S is weakly compact. $S(I - T^n)$ is weakly compact thanks to $S(I - T^n) = S(I - T)(I + T + \cdots + T^{n-1})$, where $(I + T + \cdots + T^{n-1})$ is bounded. From the expression $S = S - ST^n + ST^n = S(I - T^n) + ST^n$, we deduce that

$$\overline{\omega}(S) \leq \overline{\omega}(ST^n) \leq \overline{\omega}(S)\overline{\omega}(T^n)$$

and thus, $(1 - \overline{\omega}(T^n))\overline{\omega}(S) \leq 0$, which implies that $\overline{\omega}(S) = 0$ and so S is weakly compact. Then, from Lemma 3.15, we infer that $(I - T)$ is co-tauberian and according to Eq. (4) we conclude that $\mathcal{R}(I - T)$ is closed. Hence,

$$(I - T) \in \Phi_{g-}(X). \tag{5}$$

In view of Eqs. (4) and (5), we obtain the desired result. □

Theorem 3.17. Let X be a non-reflexive Banach space and let $T, S \in \mathcal{L}(X)$. If $\overline{\alpha}(T) < \overline{\beta}(S)$, then $(T+S) \in \Phi_{g+}(X)$.

Proof. By statement (*iii*) of Proposition 3.1, we prove that $\overline{\beta}(T + S) > 0$, the result is then obtained from statement (*i*) of Theorem 3.6. □

Definition 3.18. Let X be a Banach space and let $T, S \in \mathcal{L}(X)$, we define the non-negative quantity:

$$\overline{\Psi}(T) = \sup\{\overline{\beta}(T + S) \text{ such that } \overline{\beta}(S) = 0\}.$$

We denote by $\overline{\Psi}_0(T)$ the limit of the sequence $\overline{\Psi}(T^n)^{\frac{1}{n}}$.

Theorem 3.19. Let X be a non-reflexive Banach space having the property (H_1) and let T, S be two commuting bounded linear operators on X.

(*i*) If $\overline{\psi}(T) < \overline{\beta}(S)$, then $(T + S) \in \Phi_{g+}(X)$.
(*ii*) If $\overline{\psi}(T^n) < \overline{\beta}(S^n)$, then $(T + S) \in \Phi_{g+}(X)$.

Proof. (*i*) Suppose that $(T + S) \notin \Phi_{g+}(X)$, so by assertion (*i*) of Theorem 3.6 we get $\overline{\beta}(T + S) = 0$. Then,

$$\begin{aligned} \overline{\beta}(S) &= \overline{\beta}(T - (T + S)) \\ &\leq \sup\{\overline{\beta}(T - (T + S)) \text{ such that } \overline{\beta}(T + S) = 0\} \\ &= \sup\{\overline{\beta}(T + (-(T + S))) \text{ such that } \overline{\beta}(-(T + S)) = 0\}. \end{aligned}$$

Put $S' := -(T + S)$, then

$$\begin{aligned} \overline{\beta}(S) &\leq \sup\{\overline{\beta}(T + S') \text{ such that } \overline{\beta}(S') = 0\}. \\ &= \overline{\psi}(T). \end{aligned}$$

Hence,

$$\overline{\beta}(S) \leq \overline{\psi}(T).$$

(*ii*) Suppose $\overline{\beta}(T + S) = 0$. By assertion (*ii*) of Theorem 3.6 and since X satisfies the property (H_1), we infer that $(T + S) \notin \Phi_{g+}(X)$. We have

$$(T^n + S^n) = (T^{n-1} - T^{n-2}S + T^{n-3}S^2 + \cdots + S^{n-1})(T + S).$$

Since $(T+S) \notin \Phi_{g+}(X)$, then from Theorem 3.9 (*i*), we deduce that $(T^n+S^n) \notin \Phi_{g+}(X)$ and so $\overline{\beta}(T^n+S^n)=0$. Therefore,

$$\begin{array}{rcl} \overline{\beta}(S^n) & = & \overline{\beta}(T^n-(T^n+S^n)) \\ & \leq & \sup\{\overline{\beta}(T^n+(-(T^n+S^n))) \text{ such that } \overline{\beta}(T^n+S^n)=0\} \\ & = & \sup\{\overline{\beta}(T^n+(-(T^n+S^n))) \text{ such that } \overline{\beta}(-(T^n+S^n))=0\}. \end{array}$$

Take $A := -(T^n+S^n)$, then

$$\begin{array}{rcl} \overline{\beta}(S^n) & \leq & \sup\{\overline{\beta}(T^n+A) \text{ such that } \overline{\beta}(A)=0\} \\ & = & \overline{\psi}(T^n). \end{array}$$

Thus,

$$\overline{\beta}(S^n) \leq \overline{\psi}(T^n). \quad \square$$

The following result consists to a generalization of [[6], Proposition 3.1].

Proposition 3.20. Let X be a non-reflexive Banach space having the property (H) and let $T \in \mathcal{L}(X)$ and $S \in \Phi_g(X)$. If $S_0 \in \mathcal{L}(X)$ is an inverse of S modulo weakly compact operators, then we have the following.
(*i*) If $\overline{\omega}(TS_0) < 1$, then $(T-S) \in \Phi_g(X)$.
(*ii*) If $\overline{\omega}(S_0T) < 1$, then $(T-S) \in \Phi_g(X)$.

Proof. (*i*) According to Theorem 3.16, we show that $(I-TS_0) \in \Phi_g(X)$. Since X has the property (H) and $S \in \Phi_g(X)$, then by Theorem 2.4, there exist $S_0 \in \mathcal{L}(X)$ and $W_1 \in \mathcal{W}(X)$ such that $S_0S = I + W_1$. Thus, $(I-TS_0)S \in \Phi_g(X)$. This implies that $(I-TS_0)S = (S-TS_0S) = (S-T-TW_1) \in \Phi_g(X)$, hence $(T-S) \in \Phi_g(X)$.
(*ii*) The proof of (*ii*) is similar to (*i*). $\square$

We are able now to give a singular perturbation result of generalized Fredholm operators which extends the Theorem 4.63 in [1].

Theorem 3.21. Let X be a non-reflexive Banach space having the property (H_1) and let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(X)$. If T is a generalized Fredholm and S is strictly singular, then $T+S$ is a generalized Fredholm operator.

Proof. S is strictly singular, then from Theorem 2.6, there exist closed subspaces X_1 of X and X_2 of X_1 such that $S_1 : X_2 \to X$ is compact. Consider S as a composite mapping as follow:

$$X \xrightarrow{P_1} X_1 \xrightarrow{P_2} X_2 \xrightarrow{S_1} X,$$

in order to obtain $S = S_1 \circ P_2 \circ P_1$. Then,

$$\overline{\alpha}(S) = \overline{\alpha}(S_1P_2P_1) \leq \overline{\alpha}(S_1)\overline{\alpha}(P_2P_1). \tag{6}$$

Since $S_1 \in \mathcal{K}(X) \subset \mathcal{W}(X)$, we get $\overline{\alpha}(S_1) = 0$. This result, combined with the use of Eq. (6) allow us to conclude that $\overline{\alpha}(S) = 0$. Using the fact that $T \in \Phi_{g+}(X)$, then by Theorem 3.6 (*ii*), we deduce that $\overline{\beta}(T) > 0$. To prove that $(T+S) \in \Phi_{g+}(X)$, it suffices to show that $\overline{\beta}(T+S) > 0$. For this purpose, we have from Proposition 3.1 (*iii*) that

$$\overline{\beta}(T+S) \geq \overline{\beta}(T) - \overline{\alpha}(S).$$

Since $\overline{\beta}(T) > 0$ and $\overline{\alpha}(S) = 0$, then we infer that $\overline{\beta}(T+S) > 0$. Therefore, by assertion (*i*) from Theorem 3.6 we obtain that

$$(T+S) \in \Phi_{g+}(X). \tag{7}$$

Now, by using Lemma 3.15, it remains to prove that $T+S$ is co-tauberian. For this purpose let $A \in \mathcal{L}(X)$ such that $A(T+S)$ is weakly compact. It is enough to show that A is weakly compact. We have,

$$\overline{\omega}(A(T+S)) \geq \overline{\omega}(A)\overline{\beta}(T+S).$$

Since $\overline{\beta}(T+S) > 0$ and $\overline{\omega}(A(T+S)) = 0$, then $\overline{\omega}(A) = 0$. Hence $(T+S) \in \mathcal{T}^d(X)$. From Eq. (7), it follows that $\mathcal{R}(T+S)$ is closed. Consequently,

$$(T+S) \in \Phi_{g-}(X). \tag{8}$$

According to Eqs. (7) and (8), we deduce the result. □

Now, we shall present an example of tauberian and co-tauberian operator from [15] which is not a generalized Fredholm.

Example 3.22. Let X be a non-reflexive Banach space and let T an operator in X defined by

$$\begin{array}{rcl} T : l_2(X) & \longrightarrow & l_2(X) \\ (x_n) & \mapsto & (x_n/n). \end{array}$$

is not a generalized Fredholm. Indeed,
firstly, we have that for any normed linear space X, the dual of $l_2(X)$ is $l_2(X^*)$. Hence, T^* maps $l_2(X^*)$ into itself and T^{**} maps $l_2(X^{**})$ into itself. Indeed, the isomorphism P from $l_2(X^*)$ to $l_2(X)^*$ is defined by:

$$\begin{array}{rcl} P : l_2(X^*) & \longrightarrow & l_2(X)^* \\ (f_n)_n & \mapsto & \displaystyle\sum_{n=0}^{\infty}(f_n(x_n)), \end{array}$$

for all $(x_n)_n \in l_2(X)$. It is not difficult to prove that P is linear and bijective.
Secondly, clearly T is bounded. Furthermore, we have

$$(T^{**})^{-1}(l_2(X)) = \{x_n^{**} \in l_2(X^{**}) \text{ such that } (x_n^{**}/n) \in l_2(X)\}.$$

Since $(x_n^{**}/n) \in l_2(X)$, then $x_n^{**} \in X$, for each n. Moreover,

$$x_n^{**} \in l_2(X^{**}) \text{ and } x_n^{**} \in X, \text{ for each } n.$$

Thus, $x_n^{**} \in l_2(X)$ and so T is tauberian. Now, using the fact that T is co-tauberian whenever T^* is tauberian and applying the same way as above, then we conclude that T is co-tauberian. However, the range of T is not closed. Hence, we infer that T is not a generalized Fredholm operator.

4. Generalized S-essential spectra

In this section we investigate the generalized S-essential spectrum of a bounded linear operator acting on a Banach space X and present some generalization of results obtained in [17]. We begin with the a classical definition:

Definition 4.1. Let X be a Banach space and let T and S be two bounded linear operators on X. We respectively define the generalized Wolf S-essential spectrum and the generalized Gustafson S-essential spectrum of T by:

$$\begin{array}{rcl} \sigma_{Se_4,g}(T) & := & \left\{\lambda \in \mathbb{C} \text{ such that } (\lambda S - T) \notin \Phi_g(X)\right\}, \text{ and} \\ \sigma_{Se_1,g}(T) & := & \left\{\lambda \in \mathbb{C} \text{ such that } (\lambda S - T) \notin \Phi_{g+}(X)\right\}. \end{array}$$

The generalized S-resolvent $\rho_{S,g}(T)$ is $\mathbb{C} \backslash \sigma_{Se_4,g}(T)$. The generalized S-essential spectral radius of T is defined by:

$$r_{Se,g}(T) := \sup\{|\lambda| \text{ such that } \lambda \in \sigma_{Se_4,g}(T)\}.$$

Note that for $S = I$, the two previous generalized essential spectra of T will be respectively denoted by $\sigma_{e_4,g}(T)$ and $\sigma_{e_1,g}(T)$ and $\rho_{S,g}(T)$ (resp. $r_{Se,g}(T)$) will simply be denoted by $\rho_g(T)$ (resp. $r_{e,g}(T)$).

Now, let us start with the following classical result concerning the invariance of spectra.

Proposition 4.2. Let X be a non-reflexive Banach space and let T and S be two bounded linear operators on X.

(i) If S is an invertible operator, then $\sigma_{Se_4,g}(T) = \sigma_{e_4,g}(S^{-1}T)$.
(ii) Assume that X having the property (H) and $S \in \Phi_g(X)$. Then,

$$\sigma_{e_4,g}(T) = \sigma_{Se_4,g}(TS) = \sigma_{Se_4,g}(ST).$$

Proof. (i) $\lambda \notin \sigma_{e_4,g}(S^{-1}T)$ implies that $(\lambda I - S^{-1}T) \in \Phi_g(X)$. Since S is invertible, it follows that $S \in \Phi(X) \subset \Phi_g(X)$, then $S(\lambda I - S^{-1}T) = (\lambda S - T) \in \Phi_g(X)$. We infer that $\lambda \notin \sigma_{Se_4,g}(T)$. Conversely, if $\lambda \notin \sigma_{Se_4,g}(T)$, then $(\lambda S - T) \in \Phi_g(X)$. Since S is invertible, we obtain $(\lambda I - S^{-1}T) \in \Phi_g(X)$, thus $\lambda \notin \sigma_{e_4,g}(S^{-1}T)$.

(ii) Let us prove the first equality $\sigma_{e_4,g}(T) = \sigma_{Se_4,g}(ST)$. If $\lambda \notin \sigma_{e_4,g}(T)$, then $(\lambda I - T) \in \Phi_g(X)$. Since $S \in \Phi_g(X)$, then by Theorem 3.11 (i) we deduce that $S(\lambda I - T) \in \Phi_g(X)$. Hence, $\lambda \notin \sigma_{Se_4,g}(ST)$. Conversely, if $\lambda \notin \sigma_{Se_4,g}(ST)$, then $(\lambda S - ST) \in \Phi_g(X)$. Since $S \in \Phi_g(X)$, then by Theorem 2.4 there exist $S_0 \in \mathcal{L}(X)$ and $W_1 \in \mathcal{W}(X)$ such that $S_0S = I + W_1$. Then, $S_0(\lambda S - ST) = (\lambda I - T + \lambda W_1 - W_1T) \in \Phi_g(X)$, and so by statement ($ii$) of Theorem 3.11 we conclude that $(\lambda I - T) \in \Phi_g(X)$. Thus, $\lambda \notin \sigma_{e_4,g}(T)$. Similarly we prove that $\sigma_{e_4,g}(T) = \sigma_{Se_4,g}(TS)$. □

Now, we can give the second main result concerning the generalized S-essential spectrum of T.

Theorem 4.3. Let X be a non-reflexive Banach space having the property (H_1) and T and S be two bounded linear operators on X. Then,

$$\sigma_{Se_1,g}(T) = \bigcap_{W \in \mathcal{W}(X)} \sigma_{Se_4,g}(T + W).$$

Proof. Let $\lambda \notin \sigma_{Se_1,g}(T)$, then $(\lambda S - T) \in \Phi_{g+}(X)$. Since X satisfies the property (H_1) and $(\lambda S - T) \in \Phi_{g+}(X)$, then by Theorem 3.11 ($iii$) we deduce that

$$(\lambda S - T - W_1) \in \Phi_{g+}(X), \tag{9}$$

where, $W_1 \in \mathcal{W}(X)$. It remains to show that $(\lambda S - T - W_1)$ is co-tauberian. Let $A \in \mathcal{L}(X)$ such that $A(\lambda S - T - W_1)$ is weakly compact. By Lemma 3.15, it is enough to show that A is weakly compact.
We have

$$\overline{\omega}(A(\lambda S - T - W_1)) \geq \overline{\beta}(\lambda S - T - W_1)\overline{\omega}(A). \tag{10}$$

Since $(\lambda S - T - W_1) \in \Phi_{g+}(X)$, then by Theorem 3.6 ($ii$), we have

$$\overline{\beta}(\lambda S - T - W_1) > 0.$$

Taking into account that $\overline{\omega}(A(\lambda S - T - W_1)) = 0$, then the use of Eq. (10) leads to $\overline{\omega}(A) = 0$. Hence, A is weakly compact, and then $(\lambda S - T - W_1) \in \mathcal{T}^d(X)$. From Eq. (9), we obtain that $\mathcal{R}(\lambda S - T - W_1)$ is closed. Thus,

$$(\lambda S - T - W_1) \in \Phi_{g-}(X). \tag{11}$$

From Eqs. (9) and (11), we conclude that $(\lambda S - T - W_1) \in \Phi_g(X)$, which yields to $\lambda \notin \bigcap_{W \in \mathcal{W}(X)} \sigma_{Se_4,g}(T + W)$. Conversely, let $\lambda \notin \bigcap_{W \in \mathcal{W}(X)} \sigma_{Se_4,g}(T + W)$, then there exists $W \in \mathcal{W}(X)$ such that $\lambda \in \rho_{S,g}(T + W)$. Thus, $(\lambda S - T - W) \in \Phi_g(X)$ and so $(\lambda S - T - W) \in \Phi_{g+}(X)$. Since X has the property (H_1) and $(\lambda S - T - W) \in \Phi_{g+}(X)$, then by Theorem 3.11 ($iii$) we deduce that $(\lambda S - T) \in \Phi_{g+}(X)$ and consequently, $\lambda \notin \sigma_{Se_1,g}(T)$. □

Now, given a bounded linear operator S acting on a Banach space X, we introduce the following definitions.

Definition 4.4. Let $T \in \mathcal{L}(X)$. The generalized Jeribi essential spectrum and the generalized Jeribi S-essential spectrum of T are respectively defined by:

$$\sigma_{j,g}(T) := \bigcap_{W \in \mathcal{W}_*(X)} \sigma_{e_4,g}(T+W) \text{ and } \sigma_{Sj,g}(T) := \bigcap_{W \in \mathcal{W}_*(X)} \sigma_{Se_4,g}(T+W).$$

Here, $\mathcal{W}_*(X)$ stands for each one of the sets $\mathcal{W}(X)$ or $\mathcal{S}(X)$, where $\mathcal{S}(X)$ is the class of strictly singular operators [17].

Definition 4.5. Let X be a Banach space and $T \in \mathcal{L}(X)$. The generalized Jeribi S-essential spectral radius of T is given by:

$$r_{Sj,g}(T) = \sup\{|\lambda| \text{ such that } \lambda \in \sigma_{Sj,g}(T)\}.$$

Note that when $S = I$, $r_{Sj,g}(T)$ is simply denoted by $r_{j,g}(T)$.

As for the essential spectral radius defined in [6], we introduce the generalized essential spectral radius

Definition 4.6. [26] Let X be a Banach space. The generalized Calkin algebra of X (the quotient space $\mathcal{L}(X)/\mathcal{W}(X)$) is denoted by $\Gamma(X)$ and let Π the (canonical) quotient map of $\mathcal{L}(X)$ onto $\Gamma(X)$.

$$\begin{array}{rcl}\Pi : \mathcal{L}(X) & \to & \Gamma(X) \\ T & \to & \Pi(T) = [T] = T + \mathcal{W}(X).\end{array}$$

For instance, We say that an operator $T \in \mathcal{L}(X)$ is generalized essentially invertible if $\Pi(T)$ is invertible in the generalized Calkin algebra. When X is non-reflexive has the property (H_1), we see that the generalized essentially invertible operators are precisely the generalized Fredholm operators. This is simply restatement of Remark 3.10.

Remark 4.7. Let X be a non-reflexive Banach space having the property (H). An operator $T \in \mathcal{L}(X)$ is generalized essentially invertible if and only if, is a generalized Fredholm. Then, the generalized Wolf essential spectrum of T, $\sigma_{e_4,g}(T)$, is the spectrum of $\Pi(T)$ in the algebra $\Gamma(X)$. So, the generalized essential spectral radius $r_{e,g}(T)$ of T is equal to the spectral radius $r(\Pi(T))$ of $\Pi(T)$ that is,

$$r_{e,g}(T) = r(\Pi(T)) = \lim_{n\to\infty} \left[\|T^n\|_W\right]^{\frac{1}{n}}, \text{ where}$$

$$\|T\|_W := \inf\left\{\|T+W\|, W \in \mathcal{W}(X)\right\}.$$

Now, we can give an extended notion of generalized S-essential spectral radius of a bounded operator:
Let X be a non-reflexive Banach space has the property (H). We consider the operators $T \in \mathcal{L}(X)$ and $S \in \Phi_g(X)$. Then, by Theorem 2.4, there exist $S_0 \in \mathcal{L}(X)$ and $W_1, W_2 \in \mathcal{W}(X)$ such that $SS_0 = I + W_1$ and $S_0S = I + W_2$. We start with the following proposition.

Proposition 4.8. Let the space X and the operators T and S defined as above. Then, according to the notations mentioned in the definition above, the generalized S-essential spectral radius of T is given by the formulas:

$$r_{Se,g}(T) = \lim_{n\to\infty} [\overline{\omega}((TS_0)^n)]^{\frac{1}{n}} = \lim_{n\to\infty} [\overline{\omega}((S_0T)^n)]^{\frac{1}{n}}.$$

Proof. Let $\lambda \in \mathbb{C}$. If $|\lambda| > \overline{\omega}(TS_0)$. In view of Proposition 3.20 (*i*), we see that $(T - \lambda S) \in \Phi_g(X)$ i.e., $\lambda \notin \sigma_{Se_4,g}(T)$. Hence, $r_{Se,g}(T) \leq \overline{\omega}(TS_0)$. Now, set $r := \lim_{n\to\infty} [\overline{\omega}((TS_0)^n)]^{\frac{1}{n}}$. The limit of $[\overline{\omega}((TS_0)^n)]^{\frac{1}{n}}$ exists for all $T \in \mathcal{L}(X)$ (see [[11]: Lemma 2.16]). We have, according properties of $\overline{\omega}$: $\overline{\omega}(TS_0) \leq \|TS_0\|_W$, and so

$$\lim_{n\to\infty}[\overline{\omega}((TS_0)^n)]^{\frac{1}{n}} \leq \lim_{n\to\infty}[\|(TS_0)^n\|_W]^{\frac{1}{n}}.$$

Since

$$r(\Pi(TS_0)) = r_{e,g}(TS_0) = r_{Se,g}(TS_0S)$$

and

$$r_{Se,g}(TS_0S) = r_{Se,g}(T + TW_1) = r_{Se,g}(T),\ W_1 \in \mathcal{W}(X),$$

it follows that,

$$\lim_{n\to\infty}[\overline{\omega}((TS_0)^n)]^{\frac{1}{n}} \leq r_{Se,g}(T). \tag{12}$$

In the other side, let $\lambda \in \mathbb{C}$. If $|\lambda| > r$, then there exists $n \in \mathbb{N}^*$ such that $|\lambda|^n > \overline{\omega}((TS_0)^n)$. By assertion (*i*) of Proposition 3.20, we get $(T - \lambda S) \in \Phi_g(X)$ and thus $\lambda \notin \sigma_{Se_4,g}(T)$. This shows that

$$r_{Se,g}(T) \leq \lim_{n\to\infty}[\overline{\omega}((TS_0)^n)]^{\frac{1}{n}}. \tag{13}$$

From Inequalities (12) and (13), we obtain the desired result. The other equality can be proved similarly. □

Following notations of previous results, we give some strong arguments to locate the generalized S-essential spectra.

Theorem 4.9. Let X be a non-reflexive Banach space having the property (H_1) and let $T, S \in \mathcal{L}(X)$ such that S is non vanishing nonzero and non weakly compact operator.

(*i*) If $0 \notin \sigma_{e_1,g}(S)$, then $\sigma_{Se_1,g}(T) \subset \overline{D}\left(0, \frac{\overline{\psi}_0(T)}{\overline{\beta}(S)}\right)$.

(*ii*) If $0 \notin \sigma_{e_1,g}(S) \cup \sigma_{e_1,g}(T)$, then $\sigma_{Se_1,g}(T) \subset C\left[\frac{\overline{\beta}_0(T)}{\overline{\psi}(S)}, \frac{\overline{\psi}_0(T)}{\overline{\beta}(S)}\right]$.

(*iii*) If $0 \in \sigma_{e_4,g}(T)$, then $\sigma_{e_4,g}(T) \subset \overline{D}(0, \overline{\psi}_0(T))$.

Proof. (*i*) Let $\lambda \in \mathbb{C}$, such that $\lambda \notin \overline{D}\left(0, \frac{\overline{\psi}_0(T)}{\overline{\beta}(S)}\right)$, means that $|\lambda|\overline{\beta}(S) > \overline{\psi}_0(T)$, then there exists $n \in \mathbb{N}^*$ such that

$$|\lambda|^n\overline{\beta}(S^n) > \overline{\psi}(T^n).$$

By using Theorem 3.19 (*ii*), we deduce that $(T - \lambda S) \in \Phi_{g+}(X)$, which implies that $\lambda \notin \sigma_{Se_1,g}(T)$.

(*ii*) Let $\lambda \in \mathbb{C}$, such that $\lambda \notin C\left[\frac{\overline{\beta}_0(T)}{\overline{\psi}(S)}, \frac{\overline{\psi}_0(T)}{\overline{\beta}(S)}\right]$, i.e., $|\lambda|\overline{\psi}(S) < \overline{\beta}_0(T)$. Then, there exists $n \in \mathbb{N}^*$ such that

$$|\lambda|^n\overline{\psi}(S^n) < \overline{\beta}(T^n),$$

then using once more Theorem 3.19 (*ii*), we conclude that $(T - \lambda S) \in \Phi_{g+}(X)$ and thus, $\lambda \notin \sigma_{Se_1,g}(T)$.

(*iii*) Let $n \in \mathbb{N}^*$ and suppose that $|\lambda|^n > \overline{\psi}(T^n)$. Since $\overline{\beta}(I) = 1$, then by assertion (*ii*) of Theorem 3.19, we infer that $(\lambda I - T) \in \Phi_{g+}(X)$. This fact and by using Lemma 3.15 we obtain that $(\lambda I - T) \in \mathcal{T}^d(X)$. Hence, $(\lambda I - T) \in \Phi_g(X)$ and consequently $\lambda \notin \sigma_{e_4,g}(T)$. □

Theorem 4.10. Let X be a non-reflexive Banach space satisfying the property (H_1) and let T and S be two bounded linear operators on X such that S is a non weakly compact operator.

(*i*) If $0 \notin \sigma_{e_1,g}(S)$ then $\sigma_{Sj,g}(T) \subset \overline{D}\left(0, \frac{\overline{\psi}_0(T)}{\overline{\beta}(S)}\right)$.

(*ii*) If $0 \notin \sigma_{e_1,g}(S) \cup \sigma_{e_1,g}(T)$ then $\sigma_{Sj,g}(T) \subset C\left[\frac{\overline{\beta}_0(T)}{\overline{\psi}(S)}, \frac{\overline{\psi}_0(T)}{\overline{\beta}(S)}\right]$.

Proof. As $\mathcal{W}_*(X)$ contains $\mathcal{W}(X)$ and by applying Theorem 4.3, we get

$$\bigcap_{W\in\mathcal{W}_*(X)} \sigma_{Se_4,g}(T+W) \subset \bigcap_{W\in\mathcal{W}(X)} \sigma_{Se_4,g}(T+W),$$

then we deduce that

$$\sigma_{Sj,g}(T) \subset \sigma_{Se_1,g}(T). \tag{14}$$

(i) Combining relation (14) and Theorem 4.9 (i), we get that $\sigma_{Sj,g}(T) \subset \overline{D}\Big(0, \frac{\overline{\psi}_0(T)}{\overline{\beta}(S)}\Big)$, if $0 \notin \sigma_{e_1,g}(S)$.
(ii) It suffices to use Eq. (14) together with assertion (ii) of Theorem 4.9 to obtain the desired result. □

Theorem 4.11. Let X be a non-reflexive Banach space having the property (H_1) and let $T, S \in \mathcal{L}(X)$. Then,

$$\sigma_{Sj,g}(T) = \sigma_{Se_1,g}(T).$$

Proof. Clearly, $\sigma_{Sj,g}(T) \subset \sigma_{Se_1,g}(T)$. Now, we will show the following inclusion

$$\sigma_{Se_1,g}(T) \subset \sigma_{Sj,g}(T).$$

Let $\lambda \notin \sigma_{Sj,g}(T)$, then there exists $W \in \mathcal{W}_*(X)$ such that $\lambda \in \rho_{S,g}(T+W)$. This implies that

$$(\lambda S - T - W) \in \Phi_g(X). \tag{15}$$

Taking into account that $\mathcal{W}_*(X)$ is a two-sided ideal of $\mathcal{L}(X)$, we have in the two following cases:
1^{st} case: If $W \in \mathcal{W}(X)$, then we can write

$$\lambda S - T = \lambda S - T - W + W. \tag{16}$$

Since X satisfies the property (H_1) and $(\lambda S - T - W) \in \Phi_{g+}(X)$, then by applying Theorem 3.11 (iii) to Eq. (16), we deduce that $(\lambda S - T) \in \Phi_{g+}(X)$ and then $\lambda \notin \sigma_{Se_1,g}(T)$.
2^{nd} case: If $W \in \mathcal{S}(X)$, then $(\lambda S - T - W)^{-1}W \in \mathcal{S}(X)$, it follows from Theorem 3.21 that $(I + (\lambda S - T - W)^{-1}W) \in \Phi_g(X)$. Implies that

$$(I + (\lambda S - T - W)^{-1}W) \in \Phi_{g+}(X). \tag{17}$$

On the other hand, we have from Eq. (15) that

$$(\lambda S - T - W) \in \Phi_{g+}(X). \tag{18}$$

Furthermore, we have

$$(\lambda S - T) = (\lambda S - T - W)(I + (\lambda S - T - W)^{-1}W). \tag{19}$$

Again, the use of Eqs. (17),(18),(19) and Theorem 3.11 (v), enables us to conclude that $(\lambda S - T) \in \Phi_{g+}(X)$, and so $\lambda \notin \sigma_{Se_1,g}(T)$. □

Remark 4.12. In Theorem 4.11, when $S = I$ we get

$$\sigma_{j,g}(T) = \sigma_{e_1,g}(T).$$

References

[1] Y. Abramovich, C. D. Aliprantis, Invitation to Operator Theory. Grad. Stud. Math. 50, Amer. Math. Soc., Providence, 2002.
[2] T. Álvarez, M. Gonzàlez, Some examples of tauberian operators. Proc. Amer. Math. Soc. 111 (1991), 1023-1027.
[3] J. Araujo, J. M. Mauriva, A note on tauberian operators. Internat. J. Math. Math. Sci. 13(02), 1990, 271-274.
[4] K. Astala, On measures of noncompactness and ideal variations in Banach spaces, Ann. Acad. Sci. Fenn. Ser. A.I. Math. Diss. 29 (1980).
[5] J. Banaś, J. Rivero, On measures of weak noncompactness, Ann. Mat. Pura Appl. 151 (1988) 213-262.
[6] C. Belabbaci, M. Aissani, M. Terbeche, *S*-essential spectra and measure of noncompactness. Math. Slovaca 67 (2017), No. 5, 1203-1212.
[7] P. Constantin, A. Debussche, G. P. Galdi, M. Ružička, G. Seregin, Topics in Mathematical Fluid Mechanics. Lecture Notes in Mathematics 2073. Cetraro, Italy 2010.
[8] F. S. De Blasi, On a property of the unit sphere in a Banach space, Bull. Math. Soc. Sci. Math. R.S. Roumanie, 21 (1977), 259-262.
[9] N. Dunford, J. T. Schwartz, Linear operations on summable functions, Trans. Amer. Math. Soc. 47 (1940) 323-392.
[10] N. Dunford, J. T. Schwartz, Linear operators, Part I. General Theory, Interscience. New York. (1958).
[11] D. Edmunds, W. D. Evans, Spectral Theory and Diferential Operators. Oxford Mathematical Monographs, Clarendon Press, 1987.
[12] G. Emmanuele, Measure of weak noncompactness and fixed point theorems, Boll. Math. Soc. Sci. Math. R.S. Roumanie. 25 (1981) 353-358.
[13] M. Faierman, R. Mennicken, M. Moller, A boundary eigenvalue problem for a system of partial differential operators occuring in magnetohydrodynamics, Math Nachr. 173 141-167 (1995).
[14] S. Goldberg, Unbounded Linear Operators, Theory and Applications, McGraw-Hill Book Co. New York. (1966).
[15] M. Gonzàlez, Tauberian operators. Properties, applications and open problems. In "Concrete operators, spectral theory, operators in harmonic analysis and approximation", pp 231-242. Operator Theory: Advances and application 236. Birkhauser 2014.
[16] M. González, A. Martinez-Abejón, Tauberian operators. Operator Theory: Advances and applications 194. Birkhauser Verlag, Basel. (2010).
[17] A. Jeribi, Spectral Theory and Applications of Linear Operators and Block Operator Matrices, Springer, New York, 2015.
[18] A. Jeribi, B. Krichen, Nonlinear Functional Analysis in Banach Spaces and Banach Algebras: Fixed Point Theory under Weak Topology for Nonlinear Operators and Block Operator Matrices with Applications, Monographs and Research Notes in Mathematics. CRC Press Taylor and Francis. (2015).
[19] W. Johnson, A. Nasseri, G. Schechtman, T. Tkcoz, Injective tauberian operators on L_1 and operators with dense range on l_∞ Canad. Math. Bull. 58 (2015), 276-280.
[20] N. Kalton, A. Wilansky. Tauberian operators on Banach spaces. Proc. Amer. Math. Soc. 57 (1976), 251-255.
[21] V. Müller, Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras. Oper. Theory Adv. Appl. 139, 2nd edition, Birkhäuser, Basel, Boston, Berlin, 2007.
[22] R. Neidinger, Properties of tauberian operators on Banach spaces Ph. D. Thesis. The University of Texas at Austin, Austin TX, 1984.
[23] A. Pelczynski, Projections in certain Banach spaces. Studiu Math. 19 (1960), 209-228.
[24] C. Schmoeger, On a class of generalized Fredholm operators I, Demonstratio Math. 30 (1997), 829-842.
[25] C. Schmoeger, On a class of generalized Fredholm operators II, Demonstratio Math. 31 (1998), 705-722.
[26] K. W. Yang, The generalized Fredholm operators, Trans. Amer. Math. Soc. 216 (1976), 313-326.