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Killing Magnetic Curves in Non-Flat Lorentzian-Heisenberg Spaces

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Abstract. We obtain some explicit formulas for Killing magnetic curves in non-flat Lorentzian-Heisenberg spaces.

1. Introduction

Let (M, g) be a three-dimensional (semi-)Riemannian manifold. The magnetic curves γ on M are generalizations of geodesics which satisfy the following differential equation:

$$\nabla_{\gamma'}\gamma' = \varphi(\gamma'), \tag{1}$$

where ∇ is the Levi-Civita connection of g and φ is a skew-symmetric $(1,1)-$tensor field. The tensor field φ is known as the Lorentz force and equation (1) is said to be the Lorentz equation. Magnetic curves were investigated by several authors in Riemannian and semi-Riemannian manifolds (see [10], [11], [12], [13]).

Moreover, when the magnetic fields (which will be explained later) correspond to a Killing vector, the curves γ which fulfill equation (1) are said to be Killing magnetic curves. Studying Killing magnetic curves is an actual topic of research in pure mathematics and theoretical physics. In [14], Romaniuc and Munteanu considered Killing magnetic curves in three-dimensional Euclidean space. In [15], the same authors studied these curves in three-dimensional Minkowski space. In [5], Erjavec gave some characterizations about Killing magnetic curves in $SL(2,\mathbb{R})$. In [6] and [7], Killing magnetic curves were investigated in Sol space and almost cosymplectic Sol space, respectively. In [9], Munteanu and Nistor classified Killing magnetic curves in $S^2 \times \mathbb{R}$. In [3], Calvaruso, Munteanu and Perrone obtained a complete classification for the Killing magnetic curves in three-dimensional almost paracontact manifolds. In [2], Bejan and Romaniuc proved that equipped with a Killing vector field V, any arc length parameterized spacelike or timelike curve, normal to V, is a magnetic trajectory associated to V in a Walker manifold. And finally, in [4], Derkaoui and Hathout occured explicit formulas for Killing magnetic curves in Heisenberg group.

In this paper, we determine the Killing magnetic curves in the three-dimensional Lorentzian-Heisenberg space. It is known that Lorentzian-Heisenberg space can be equipped with three non-isometric metrics. We will consider two of them which are non-flat.

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2. Preliminaries

Let (M, g) be a three-dimensional semi-Riemannian manifold. The magnetic curves on (M, g) are trajectories of charged particles, moving on M under the action of a magnetic field F. A magnetic field on M is a closed 2-form F on M, to which one can associate a skew-symmetric $(1, 1)-$tensor field φ on M, uniquely determined by

$$F(X, Y) = g(\varphi(X), Y),$$

for all $X, Y \in \chi(M)$. Here, the tensor field φ is called the Lorentz force.

The magnetic trajectories of F are regular curves γ in M which satisfy the Lorentz equation

$$\nabla_{\mathbf{t}}\mathbf{t} = \varphi(\mathbf{t}), \tag{2}$$

where $\mathbf{t} = \gamma'$ is the speed vector of γ.

Furthermore, to have a positively oriented orthonormal frame field $\{e_1, e_2, e_3\}$ and represent the vectors X and Y as $X = x_1e_1+x_2e_2+x_3e_3$ and $Y = y_1e_1+y_2e_2+y_3e_3$, the vector product of two vector fields $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ is given by

$$X \wedge Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2).$$

The mixed product of the vector fields $X, Y, Z \in \chi(M)$ is then defined by

$$g(X \wedge Y, Z) = dv_g(X, Y, Z),$$

where dv_g denotes a volume on M.

A vector field V is called a Killing vector field if it satisfies the Killing equation

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0,$$

for all $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection of the metric g.

Let $F_V = i_V dvg$ be the Killing magnetic force corresponding to the Killing magnetic vector field V on M, where i denotes the inner product. The Lorentz force of F_V is described as

$$\varphi(X) = V \wedge X,$$

for all $X \in \chi(M)$. Therefore, equation (2) can be rewritten as

$$\nabla_{\mathbf{t}}\mathbf{t} = V \wedge \mathbf{t}, \tag{3}$$

and solutions of above equation are called Killing magnetic curves corresponding to the Killing vector fields V.

For shortness, we will call these curves as $V-$magnetic curves in this paper.

3. Metrics on Lorentzian-Heisenberg space

Each left-invariant Lorentzian metric on the 3-dimensional Heisenberg group H_3 is isometric to one of the following metrics:

$$\begin{aligned} g_1 &= -\frac{1}{\lambda^2}dx^2 + dy^2 + (xdy + dz)^2, \\ g_2 &= \frac{1}{\lambda^2}dx^2 + dy^2 - (xdy + dz)^2,\ \lambda > 0, \\ g_3 &= dx^2 + (xdy + dz)^2 - ((1 - x)dy - dz)^2. \end{aligned}$$

Furthermore, the Lorentzian metrics g_1, g_2, g_3 are non-isometric and the Lorentzian metric g_3 is flat [1]. We will deal with the metrics g_1 and g_2 (i.e. non-flat cases).

Remark 3.1. *According to the coordinate change $u = \lambda^{-1}x,\ v = y,\ w = z + 2xy$, we rewrite the metrics as*

$$\begin{aligned} g_1 &= -du^2 + dv^2 + \lambda^2(vdu - udv)^2, \\ g_2 &= du^2 + dv^2 - \lambda^2(vdu - udv)^2,\ \lambda > 0. \end{aligned} \tag{4}$$

4. The metric g_1

An orthonormal basis on (H_3, g_1) is given by

$$e_1 = \frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ e_3 = \lambda\frac{\partial}{\partial x}, \tag{5}$$

where the vector e_3 is timelike. The non-zero components of the Levi-Civita connection ∇ of the metric g_1 are given by

$$\begin{aligned} \nabla_{e_1} e_2 &= \nabla_{e_2} e_1 = \frac{\lambda}{2} e_3, \\ \nabla_{e_1} e_3 &= \nabla_{e_3} e_1 = \frac{\lambda}{2} e_2, \\ \nabla_{e_2} e_3 &= -\nabla_{e_3} e_2 = \frac{\lambda}{2} e_1. \end{aligned} \tag{6}$$

The Lie algebra of Killing vector fields of (H_3, g_1) admits as basis

$$\begin{aligned} V_1 &= \frac{\partial}{\partial z}, V_2 = \frac{\partial}{\partial y},\ V_3 = \lambda\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial z}, \\ V_4 &= \lambda^2 y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - \frac{1}{2}(x^2 + \lambda^2 y^2)\frac{\partial}{\partial z}. \end{aligned} \tag{7}$$

Using (5), we rewrite equations (7) as follows:

$$\begin{aligned} V_1 &= e_1,\ V_2 = xe_1 + e_2, V_3 = -\lambda y e_1 + e_3, \\ V_4 &= \frac{1}{2}(x^2 - \lambda^2 y^2)e_1 + xe_2 + \lambda y e_3. \end{aligned}$$

If $\gamma : I \to (H_3, g_1),\ \gamma(t) = (x(t), y(t), z(t))$ is a regular curve, then its speed vector is described as

$$\mathbf{t} = \gamma'(t) = (x'(t), y'(t), z'(t))$$

and

$$\mathbf{t} = \gamma'(t) = (z' + xy')e_1 + y'e_2 + \frac{x'}{\lambda}e_3. \tag{8}$$

From equations (6), we have

$$\nabla_{\mathbf{t}}\mathbf{t} = (z' + xy')'e_1 + (y'' + x'(z' + xy'))e_2 + (\frac{x''}{\lambda} + \lambda y'(z' + xy'))e_3. \tag{9}$$

In the following subsections, we obtain some formulas for V_i −magnetic curves ($i = 1, ..., 4$) in (H_3, g_1). To solve the differential equations, we need help of *Wolfram Mathematica.*

4.1. V_1−magnetic curves

Using $(7)_1$ and (8), we occur

$$V_1 \wedge \mathbf{t} = -\frac{x'}{\lambda}e_2 - y'e_3. \tag{10}$$

The equation $\nabla_{\mathbf{t}}\mathbf{t} = V_1 \wedge \mathbf{t}$ gives us the following system:

$$S_1 : \begin{cases} y'' + x'((z' + xy') + \frac{1}{\lambda}) = 0, \\ x'' + y'(\lambda^2(z' + xy') + \lambda) = 0, \\ (z' + xy')' = 0. \end{cases} \tag{11}$$

By integrating $(S_1)_3$ and putting it in $(S_1)_{1,2}$, we obtain

$$S_1 : \begin{cases} y'' + x'(c + \frac{1}{\lambda}) = 0, \\ x'' + y'\lambda(\lambda c + 1) = 0, \\ z' + xy' = c \text{ (constant)}. \end{cases}$$

Solution of the system $(S_1)_{1,2}$ is

$$\begin{cases} x(t) = -\frac{\lambda}{\lambda c+1}[k_1 \cosh((\lambda c + 1)t) + k_2 \sinh((\lambda c + 1)t)] + k_3, \\ y(t) = \frac{1}{\lambda c+1}[k_1 \sinh((\lambda c + 1)t) + k_2 \cosh((\lambda c + 1)t)] + k_4, \end{cases} \tag{12}$$

where $k_i,\ i = 1, ..., 4$ are constants. Setting equations (12) in $(S_1)_3$ and by integration, we get

$$\begin{aligned} z(t) \quad = \quad & (c + \frac{\lambda}{2(\lambda c + 1)}(k_1^2 - k_2^2))t + \frac{\lambda}{(\lambda c + 1)^2}[\frac{(k_1^2 + k_2^2)}{4} \sinh(2(\lambda c + 1)t) \\ & + \frac{k_1k_2}{2} \cosh(2(\lambda c + 1)t)] + \frac{1}{\lambda c + 1}[k_1k_3 \sinh((\lambda c + 1)t) \\ & + k_2k_3 \cosh((\lambda c + 1)t)] + k_5, \end{aligned}$$

where k_5 is a constant.

If $c = -\frac{1}{\lambda}$, the system S_1 reduces to

$$S_1 : \begin{cases} y'' = 0, \\ x'' = 0, \\ z' + xy' = -\frac{1}{\lambda}. \end{cases}$$

Its general solution

$$S_1 : \begin{cases} x(t) = k_1t + k_2, \\ y(t) = k_3t + k_4, \\ z(t) = -\frac{k_1k_3}{2}t^2 - (\frac{1}{\lambda} + k_2k_3)t + k_5, \end{cases}$$

where $k_i,\ i = 1, ..., 5$ are constants. Therefore, we state the following theorem.

Theorem 4.1. *All V_1-magnetic curves of (H_3, g_1) satisfy the following equations:*
(i) If $c = -\frac{1}{\lambda}$, then

$$\gamma(t) = \begin{pmatrix} x(t) = k_1t + k_2, \\ y(t) = k_3t + k_4, \\ z(t) = -\frac{k_1k_3}{2}t^2 - (\frac{1}{\lambda} + k_2k_3)t + k_5 \end{pmatrix}.$$

(ii) If $c \neq -\frac{1}{\lambda}$, then

$$\gamma(t) = \begin{pmatrix} x(t) = -\frac{\lambda}{\lambda c+1}[k_1 \cosh((\lambda c + 1)t) + k_2 \sinh((\lambda c + 1)t)] + k_3, \\ y(t) = \frac{1}{\lambda c+1}[k_1 \sinh((\lambda c + 1)t) + k_2 \cosh((\lambda c + 1)t)] + k_4, \\ z(t) = (c + \frac{\lambda}{2(\lambda c+1)}(k_1^2 - k_2^2))t + \frac{\lambda}{(\lambda c+1)^2}[\frac{(k_1^2+k_2^2)}{4} \sinh(2(\lambda c + 1)t) \\ + \frac{k_1k_2}{2} \cosh(2(\lambda c + 1)t)] + \frac{1}{(\lambda c+1)}[k_1k_3 \sinh((\lambda c + 1)t) \\ + k_2k_3 \cosh((\lambda c + 1)t)] + k_5 \end{pmatrix},$$

where $k_i,\ i = 1, ..., 5$ are constants.

4.2. V_2-magnetic curves

According to $(7)_2$ and (8), we have

$$V_2 \wedge \mathbf{t} = \frac{x'}{\lambda} e_1 - \frac{xx'}{\lambda} e_2 - (xy' - (z' + xy'))e_3. \tag{13}$$

From the equation $\nabla_{\mathbf{t}} \mathbf{t} = V_2 \wedge \mathbf{t}$, we get

$$S_2 : \begin{cases} y'' + x'(z' + xy') = -\frac{xx'}{\lambda}, \\ (\lambda y' - 1)(z' + xy') = -xy' - \frac{x''}{\lambda}, \\ (z' + xy')' = \frac{x'}{\lambda}. \end{cases} \tag{14}$$

By integrating $(S_2)_3$, we deduce

$$z' + xy' = \frac{x}{\lambda} + c,$$

where c is a constant. Putting the last equation in $(S_2)_{1,2}$ we get

$$\bar{S}_2 : \begin{cases} y'' = -x'(\frac{2x}{\lambda} + c), \\ (\lambda y' - 1)(\frac{x}{\lambda} + c) = -xy' - \frac{x''}{\lambda}, \end{cases}$$

and

$$\bar{S}_2 : \begin{cases} y' = -\frac{x^2}{\lambda} - xc, \\ x'' - 2x^3 - x - \lambda c(3x^2 + c\lambda x + 1) = 0. \end{cases} \tag{15}$$

Without loss of generality, we can suppose that $c = 0$. In this case, the equation $x'' - 2x^3 - x = 0$ involves Jacobi elliptic functions as solutions. So, we can express the following proposition.

Proposition 4.2. *The Killing magnetic curves in (H_3, g_1) corresponding to the Killing vector field $V_2 = xe_1 + e_2$ are solutions of the system of differential equations (14).*

4.3. V_3-magnetic curves

From $(7)_3$ and (8), we have

$$V_3 \wedge \mathbf{t} = -y' e_1 + (yx' + (z' + xy'))e_2 + \lambda yy' e_3. \tag{16}$$

Using the equation $\nabla_{\mathbf{t}} \mathbf{t} = V_3 \wedge \mathbf{t}$, we obtain

$$S_3 : \begin{cases} y'' + x'(z' + xy') = yx' + (z' + xy'), \\ \frac{x''}{\lambda} + \lambda y'(z' + xy') = \lambda yy', \\ (z' + xy')' = -y'. \end{cases} \tag{17}$$

By integrating $(S_3)_3$, we occur

$$z' + xy' = -y + c,$$

where c is a constant. Putting the last equation in $(S_3)_{1,2}$, we get

$$\bar{S}_3 : \begin{cases} x' = \lambda^2 y^2 - \lambda^2 yc, \\ y'' - 2x'y + y + c(x' - 1) = 0. \end{cases} \tag{18}$$

Without loss of generality, we can assume that $c = 0$. In this case, when we try to solve the system $\bar{S}_3$, *i.e.*, the equation $y'' - 2\lambda^2 y^3 + y = 0$, we encounter Jacobi elliptic functions. Therefore, we write the following proposition.

Proposition 4.3. *The Killing magnetic curves in (H_3, g_1) corresponding to the Killing vector field $V_3 = -\lambda ye_1 + e_3$ are solutions of the system of differential equations (17).*

4.4. V_4-magnetic curves

From $(7)_3$ and (8), we write

$$V_4 \wedge \mathbf{t} = (\frac{xx'}{\lambda} - \lambda yy')e_1 - (\frac{1}{2}(x^2-\lambda^2y^2)\frac{x'}{\lambda} - \lambda y(z'+xy'))e_2 \\ -(\frac{1}{2}(x^2-\lambda^2y^2)y' - x(z'+xy'))e_3. \tag{19}$$

From the equation $\nabla_{\mathbf{t}}\mathbf{t} = V_4 \wedge \mathbf{t}$, we get

$$S_4 : \begin{cases} y'' + x'(z'+xy') = -\frac{1}{2}(x^2-\lambda^2y^2)\frac{x'}{\lambda} + \lambda y(z'+xy'), \\ \frac{x''}{\lambda} + \lambda y'(z'+xy') = -\frac{1}{2}(x^2-\lambda^2y^2)y' + x(z'+xy'), \\ (z'+xy')' = (\frac{xx'}{\lambda} - \lambda yy'). \end{cases} \tag{20}$$

By integrating $(S_4)_3$, we obtain

$$z' + xy' = \frac{x^2}{2\lambda} - \frac{\lambda y^2}{2} + c, \tag{21}$$

where c is a constant. Putting the last equation in $(S_4)_{1,2}$, we get

$$\bar{S}_4 : \begin{cases} y'' + \frac{x'}{\lambda}(x^2-\lambda^2y^2) = \frac{1}{2}y(x^2-\lambda^2y^2) + c(\lambda y - x'), \\ x'' + \lambda y'(x^2-\lambda^2y^2) = \frac{1}{2}x(x^2-\lambda^2y^2) + c(\lambda x - \lambda^2 y'). \end{cases} \tag{22}$$

It seems very difficult to solve the system $\bar{S}_4$ in general case. For a particular case $x = \lambda y$, we deduce

$$\bar{S}_4 : \begin{cases} x'' + c\lambda x' - c\lambda x = 0, \\ y'' + c\lambda y' - c\lambda y = 0. \end{cases} \tag{23}$$

By solving the second equation of the above system, we get

$$y(t) = k_1 e^{-\frac{t}{2}(c\lambda + \sqrt{c\lambda(4+c\lambda)})} + k_2 e^{\frac{t}{2}(-c\lambda + \sqrt{c\lambda(4+c\lambda)})},$$

where k_1 and k_2 are constants. From (21), we obtain

$$z(t) = ct - \frac{\lambda y^2}{2} \\ = ct - \frac{\lambda}{2}(k_1 e^{-\frac{t}{2}(c\lambda + \sqrt{c\lambda(4+c\lambda)})} + k_2 e^{\frac{t}{2}(-c\lambda + \sqrt{c\lambda(4+c\lambda)})})^2.$$

Therefore, a solution of the system $\bar{S}_4$ is given by

$$\bar{S}_4 : \begin{cases} x(t) = \lambda\left(k_1 e^{-\frac{t}{2}(c\lambda + \sqrt{c\lambda(4+c\lambda)})} + k_2 e^{\frac{t}{2}(-c\lambda + \sqrt{c\lambda(4+c\lambda)})}\right), \\ y(t) = k_1 e^{-\frac{t}{2}(c\lambda + \sqrt{c\lambda(4+c\lambda)})} + k_2 e^{\frac{t}{2}(-c\lambda + \sqrt{c\lambda(4+c\lambda)})}, \\ z(t) = ct - \frac{\lambda}{2}(k_1 e^{-\frac{t}{2}(c\lambda + \sqrt{c\lambda(4+c\lambda)})} + k_2 e^{\frac{t}{2}(-c\lambda + \sqrt{c\lambda(4+c\lambda)})})^2. \end{cases} \tag{24}$$

Hence, we write the following proposition.

Proposition 4.4. *The Killing magnetic curves in (H_3, g_1) corresponding to the Killing vector field $V_4 = \frac{1}{2}(x^2 - \lambda^2y^2)e_1 + xe_2 + \lambda y e_3$ are solutions of the system of differential equations (20). Moreover, the space curves given by parametric equations (24) are V_4-magnetic curves in (H_3, g_1).*

In the last section, we follow the steps explained in the strategy mentioned in this section for the metric g_2.

5. The metric g_2

We have an orthonormal basis on (H_3, g_2)

$$e_1 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ e_2 = \lambda\frac{\partial}{\partial x}, e_3 = \frac{\partial}{\partial z}, \tag{25}$$

where the vector e_3 is timelike. The non-zero components of the Levi-Civita connection ∇ of the metric g_2 are given by

$$\begin{aligned} \nabla_{e_1}e_2 &= -\nabla_{e_2}e_1 = \frac{\lambda}{2}e_3, \\ \nabla_{e_1}e_3 &= \nabla_{e_3}e_1 = \frac{\lambda}{2}e_2, \\ \nabla_{e_2}e_3 &= \nabla_{e_3}e_2 = -\frac{\lambda}{2}e_1. \end{aligned} \tag{26}$$

The Lie algebra of Killing vector fields of (H_3, g_2) admits as basis

$$\begin{aligned} V_1 &= \frac{\partial}{\partial z}, V_2 = \frac{\partial}{\partial y}, V_3 = \lambda\frac{\partial}{\partial x} - \lambda y\frac{\partial}{\partial z}, \\ V_4 &= -\lambda^2 y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{1}{2}(-x^2 + \lambda^2 y^2)\frac{\partial}{\partial z}. \end{aligned} \tag{27}$$

Using (25), we rewrite equations (27) as follows:

$$\begin{aligned} V_1 &= e_3,\ V_2 = e_1 + xe_3, V_3 = e_2 - \lambda ye_3, \\ V_4 &= xe_1 - \lambda ye_2 + \frac{1}{2}(x^2 + \lambda^2 y^2)e_3. \end{aligned}$$

If $\gamma : I \to (H_3, g_2),\ \gamma(t) = (x(t), y(t), z(t))$ is a regular curve, then its speed vector is described as

$$\mathbf{t} = \gamma'(t) = (x'(t), y'(t), z'(t))$$

and

$$\mathbf{t} = \gamma'(t) = y'e_1 + \frac{x'}{\lambda}e_2 + (z' + xy')e_3. \tag{28}$$

From equations (26), we have

$$\nabla_{\mathbf{t}}\mathbf{t} = (y'' - x'(z' + xy'))e_1 + (\frac{x''}{\lambda} + \lambda y'(z' + xy'))e_2 + (z' + xy')'e_3. \tag{29}$$

5.1. V_1–*magnetic curves*

We have

$$V_1 \wedge \mathbf{t} = -\frac{x'}{\lambda}e_1 + y'e_2. \tag{30}$$

From the equation $\nabla_{\mathbf{t}}\mathbf{t} = V_1 \wedge \mathbf{t}$, we get

$$S_1 : \begin{cases} y'' - x'((z' + xy') - \frac{1}{\lambda}) = 0, \\ x'' + y'(\lambda^2(z' + xy') - \lambda) = 0, \\ (z' + xy')' = 0. \end{cases} \tag{31}$$

By integrating $(S_1)_3$ and putting it in $(S_1)_{1,2}$, we obtain

$$S_1 : \begin{cases} y'' - x'(c - \frac{1}{\lambda}) = 0, \\ x'' + y'\lambda(\lambda c - 1) = 0, \\ z' + xy' = c \text{ (constant)}. \end{cases}$$

Solution of the system $(S_1)_{1,2}$ is

$$S_1 : \begin{cases} x(t) = \frac{\lambda}{\lambda c - 1}[k_1 \cos((\lambda c - 1)t) + k_2 \sin((\lambda c - 1)t)] + k_3, \\ y(t) = \frac{1}{\lambda c - 1}[k_1 \sin((\lambda c - 1)t) - k_2 \cos((\lambda c - 1)t)] + k_4, \end{cases} \tag{32}$$

where $k_i,\ i = 1, ..., 4$ are constants. Setting equations (31) in $(S_1)_3$ and by integration, we get

$$\begin{aligned} z(t) \quad = \quad & (c - \frac{\lambda}{2(\lambda c - 1)}(k_1^2 - k_2^2))t - \frac{\lambda}{(\lambda c - 1)^2}[\frac{(k_1^2 - k_2^2)}{4} \sin(2(\lambda c - 1)t) \\ & - \frac{k_1 k_2}{2} \cos(2(\lambda c - 1)t)] + \frac{1}{\lambda c - 1}(k_1 k_3 \sin((\lambda c - 1)t) \\ & - k_2 k_3 \cos((\lambda c - 1)t)) + k_5, \end{aligned}$$

where k_5 is a constant. If $c = \frac{1}{\lambda}$, the system S_1 reduces to

$$S_1 : \begin{cases} y'' = 0, \\ x'' = 0, \\ z' + xy' = \frac{1}{\lambda}. \end{cases}$$

Its general solution

$$S_1 : \begin{cases} x(t) = k_1 t + k_2, \\ y(t) = k_3 t + k_4, \\ z(t) = -\frac{k_1 k_3}{2} t^2 + (\frac{1}{\lambda} - k_2 k_3)t + k_5, \end{cases}$$

where $k_i,\ i = 1, ..., 5$ are constants. So, we have proved the theorem below.

Theorem 5.1. *All V_1–magnetic curves of (H_3, g_2) satisfy the following equations:*
(i) If $c = \frac{1}{\lambda}$, then

$$\gamma(t) = \begin{pmatrix} x(t) = k_1 t + k_2, \\ y(t) = k_3 t + k_4, \\ z(t) = -\frac{k_1 k_3}{2} t^2 + (\frac{1}{\lambda} - k_2 k_3)t + k_5 \end{pmatrix}.$$

(ii) If $c \neq \frac{1}{\lambda}$, then

$$\gamma(t) = \begin{pmatrix} x(t) = \frac{\lambda}{\lambda c - 1}[k_1 \cos((\lambda c - 1)t) + k_2 \sin((\lambda c - 1)t)] + k_3, \\ y(t) = \frac{1}{\lambda c - 1}[k_1 \sin((\lambda c - 1)t) - k_2 \cos((\lambda c - 1)t)] + k_4, \\ z(t) = (c - \frac{\lambda}{2(\lambda c - 1)}(k_1^2 - k_2^2))t - \frac{\lambda}{(\lambda c - 1)^2}[\frac{(k_1^2 - k_2^2)}{4} \sin(2(\lambda c - 1)t) \\ - \frac{k_1 k_2}{2} \cos(2(\lambda c - 1)t)] + \frac{1}{\lambda c - 1}(k_1 k_3 \sin((\lambda c - 1)t) \\ - k_2 k_3 \cos((\lambda c - 1)t)) + k_5 \end{pmatrix},$$

where $k_i,\ i = 1, ..., 5$ are constants.

Remark 5.2. *These curves was considered by Lee in [8] according to corresponding metric g_2 in (4).*

5.2. V_2−magnetic curves

Direct computations give

$$V_2 \wedge \mathbf{t} = -\frac{xx'}{\lambda}e_1 + (xy' - (z' + xy'))e_2 - \frac{x'}{\lambda}e_3. \tag{33}$$

The equation $\nabla_{\mathbf{t}}\mathbf{t} = V_2 \wedge \mathbf{t}$ concludes

$$S_2 : \begin{cases} y'' - x'(z' + xy') = -\frac{xx'}{\lambda}, \\ (\lambda y' + 1)(z' + xy') = xy' - \frac{x''}{\lambda}, \\ (z' + xy')' = -\frac{x'}{\lambda}. \end{cases} \tag{34}$$

By integrating $(S_2)_3$, we obtain

$$z' + xy' = -\frac{x}{\lambda} + c,$$

where c is a constant. Putting the last equation in $(S_2)_{1,2}$, we get

$$\bar{S}_2 : \begin{cases} y' = -\frac{x^2}{\lambda} + xc, \\ x'' + 2x^3 - x - \lambda c(3x^2 - c\lambda x - 1) = 0. \end{cases} \tag{35}$$

Without loss of generality, we can assume that $c = 0$. This system $\bar{S}_2$, *i.e.*, the equation $x'' + 2x^3 - x = 0$ involves Jacobi elliptic functions. So, we write the following proposition.

Proposition 5.3. *The Killing magnetic curves in (H_3, g_2) corresponding to the Killing vector field $V_2 = e_1 + xe_3$ are solutions of the system of differential equations (34).*

5.3. V_3−magnetic curves

We have

$$V_3 \wedge \mathbf{t} = (yx' + (z' + xy'))e_1 - \lambda yy'e_2 + y'e_3. \tag{36}$$

From the equation $\nabla_{\mathbf{t}}\mathbf{t} = V_3 \wedge \mathbf{t}$, we get

$$S_3 : \begin{cases} y'' - x'(z' + xy') = yx' + (z' + xy'), \\ \frac{x''}{\lambda} + \lambda y'(z' + xy') = -\lambda yy', \\ (z' + xy')' = y'. \end{cases} \tag{37}$$

By integrating $(S_3)_3$, we deduce

$$z' + xy' = y + c,$$

where c is a constant. Putting the last equation in $(S_3)_{1,2}$, we have

$$\bar{S}_3 : \begin{cases} x' = -\lambda^2 y^2 - \lambda^2 yc, \\ y'' - 2x'y - y - c(x' + 1) = 0. \end{cases} \tag{38}$$

Without loss of generality, we can suppose that $c = 0$. Then, the system $\bar{S}_3$, *i.e.*, the equation $y'' + 2\lambda^2 y^3 - y = 0$ has solutions which include Jacobi elliptic functions. Thus, we give the proposition below.

Proposition 5.4. *The Killing magnetic curves in (H_3, g_2) corresponding to the Killing vector field $V_3 = e_2 - \lambda ye_3$ are solutions of the system of differential equations (37).*

5.4. V_4−magnetic curves

We write

$$V_4 \wedge \mathbf{t} = \Big(-\frac{1}{2}(x^2+\lambda^2 y^2)\frac{x'}{\lambda} - \lambda y(z'+xy')\Big)e_1 + \Big((x^2+\lambda^2 y^2)\frac{y'}{2} - x(z'+xy')\Big)e_2 - \Big(\frac{xx'}{\lambda} + \lambda yy'\Big)e_3. \tag{39}$$

The equation $\nabla_{\mathbf{t}}\mathbf{t} = V_4 \wedge \mathbf{t}$ gives us

$$S_4 : \begin{cases} y'' - x'(z'+xy') = -\frac{1}{2}(x^2+\lambda^2 y^2)\frac{x'}{\lambda} - \lambda y(z'+xy'), \\ \frac{x''}{\lambda} + \lambda y'(z'+xy') = \frac{1}{2}(x^2+\lambda^2 y^2)y' - x(z'+xy'), \\ (z'+xy')' = -\frac{xx'}{\lambda} - \lambda yy'. \end{cases} \tag{40}$$

By integrating $(S_4)_3$, we obtain

$$z' + xy' = -\frac{x^2}{2\lambda} - \frac{\lambda y^2}{2} + c, \tag{41}$$

where c is a constant. Putting the last equation in $(S_4)_{1,2}$, we get

$$\bar{S}_4 : \begin{cases} y'' + \frac{x'}{\lambda}(x^2+\lambda^2 y^2) = \frac{1}{2}y(x^2+\lambda^2 y^2) + c(-\lambda y + x'), \\ x'' - \lambda y'(x^2+\lambda^2 y^2) = \frac{1}{2}x(x^2+\lambda^2 y^2) - c(\lambda x + \lambda^2 y'). \end{cases} \tag{42}$$

It seems a true challenge to solve the system $\bar{S}_4$ in general case. However, we can find a special solution by considering $c = \lambda = 1$. In this case,

$$x(t) = \cos\frac{\sqrt{2}}{2}t, \; y(t) = \sin\frac{\sqrt{2}}{2}t$$

will be a solution for the system $\bar{S}_4$. Using these relations in (41), we get

$$z(t) = \frac{2-\sqrt{2}}{4}t - \frac{1}{4}\sin\sqrt{2}t + k_1$$

where k_1 is a constant. Therefore, we can state the last propositon of the paper.

Proposition 5.5. *The space curves given by the parametric equations*

$$\gamma(t) = \begin{pmatrix} x(t) = \cos\frac{\sqrt{2}}{2}t, \\ y(t) = \sin\frac{\sqrt{2}}{2}t, \\ z(t) = \frac{2-\sqrt{2}}{4}t - \frac{1}{4}\sin\sqrt{2}t + k_1 \end{pmatrix}$$

are V_4−magnetic curves in (H_3, g_2), where k_1 is an arbitrary constant.

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