Filomat 36:8 (2022), 2637–2644
https://doi.org/10.2298/FIL2208637L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nonlinear Mixed Jordan triple ∗-Derivations on Factor von Neumann Algebras

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Abstract. Let $\mathcal{A}$ be a factor von Neumann algebra with dim$\mathcal{A} \geq 2$. In this paper, it is proved that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed Jordan triple ∗-derivation if and only if Φ is an additive ∗-derivation.

1. Introduction

Let $\mathcal{A}$ be a ∗-algebra over the complex field $\mathbb{C}$. For $A, B \in \mathcal{A}$, we call the product $A \bullet B = AB + BA^*$ the Jordan ∗-product and $[A, B]_* = AB - BA^*$ the skew Lie product. These two new products are very important and meaningful in some research topics, which have attracted many scholars to study (see [1–3, 5, 7–12, 16, 21–24]). Let Φ be a map (without the additivity assumption) on $\mathcal{A}$. Recall that Φ is said to be a derivation if $\Phi(AB) = \Phi(A)B + A\Phi(B)$ for all $A, B \in \mathcal{A}$. More generally, we say that Φ is a nonlinear Jordan ∗-derivation or skew Lie derivation if $\Phi(A \bullet B) = \Phi(A) \bullet B + A \bullet \Phi(B)$ or $\Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$ for all $A, B \in \mathcal{A}$. Many authors have paid more attentions on the problem about Jordan ∗-derivations, skew Lie derivations and triple derivations, such as Jordan triple ∗-derivations and skew Lie triple derivations (see [6, 14, 15, 18–20, 25, 28, 29, 31, 32]).

Recently, many authors have studied the isomorphisms and derivations corresponding to the new products of the mixture of (skew) Lie product and Jordan ∗-product (see [17, 26, 27, 30, 33, 34]). Z. Yang and J. Zhang [26, 27] studied the nonlinear maps preserving the mixed skew Lie triple product $[[A, B]_*, C]$ and $[[A, B], C]_*$ on factor von Neumann algebras, where $[A, B] = AB - BA$ is the usual Lie product of A and B. Y. Zhou, Z. Yang and J. Zhang [34] studied the structure of the nonlinear mixed Lie triple derivations on prime ∗-algebras. They proved any map Φ from a unital ∗-algebra $\mathcal{A}$ containing a non-trivial projection to itself satisfying

$$\Phi([[A, B]_*, C]) = [[\Phi(A), B]_*, C] + [[A, \Phi(B)]_*, C] + [[A, B]_*, \Phi(C)]$$

for all $A, B, C \in \mathcal{A}$, is an additive ∗-derivation. C. Li, Y. Zhao and F. Zhao [17] studied the nonlinear maps preserving the mixed product $[A \bullet B, C]_*$ on von Neumann algebras. F. Zhang [30] studied the nonlinear maps preserving the mixed product $[A, B]_* \bullet C$ on factor von Neumann algebras. Motivated by the above mentioned works, in this paper, we will consider the derivations corresponding to the new product of the

2020 *Mathematics Subject Classification*. Primary 16W25; Secondary 46L10
Keywords. mixed Jordan triple ∗-derivations; ∗-derivations; factor von Neumann algebras
Received: 14 July 2021; Revised: 12 February 2022; Accepted: 14 February 2022
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mixture of the skew Lie product and the Jordan $*$-product. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear mixed Jordan triple $*$-derivation if

$$\Phi([A, B]_* \bullet C) = [\Phi(A), B]_* \bullet C + [A, \Phi(B)]_* \bullet C + [A, B]_* \bullet \Phi(C)$$

for all $A, B, C \in \mathcal{A}$. We prove that Φ is a nonlinear mixed Jordan triple $*$-derivation on factor von Neumann algebras if and only if Φ is an additive $*$-derivation.

Recall that $\mathcal{A}$ is a von Neumann algebra if it is a weakly closed and self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. A von Neumann algebra $\mathcal{A}$ is a factor von Neumann algebra if its center only contains the scalar operators. We know that the factor von Neumann algebra $\mathcal{A}$ is prime, that is, $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

2. The main result and its proof

To complete the proof of the main theorem, we need some lemmas.

Lemma 2.1. *[15] Let $\mathcal{A}$ be a factor von Neumann algebra and $A \in \mathcal{A}$. If $AB = BA^*$ for all $B \in \mathcal{A}$, then $A \in \mathbb{R}I$, where $\mathbb{R}$ is the real field.*

Lemma 2.2. *[13] Let $\mathcal{A}$ be a factor von Neumann algebra and $A \in \mathcal{A}$. If $AB + BA^* = 0$ for all $B \in \mathcal{A}$, then $A \in i\mathbb{R}I$, where i is the imaginary number unit.*

Lemma 2.3. *([4, Problem 230]) Let $\mathcal{A}$ be a Banach algebra with the identity I. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ are such that $[A, B] = \lambda I$, where $[A, B] = AB - BA$, then $\lambda = 0$.*

Our main result in this paper reads as follows.

Theorem 2.4. *Let $\mathcal{A}$ be a factor von Neumann algebra with $\dim\mathcal{A} \geq 2$. Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies $\Phi([A, B]_* \bullet C) = [\Phi(A), B]_* \bullet C + [A, \Phi(B)]_* \bullet C + [A, B]_* \bullet \Phi(C)$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.*

Proof. Let P be a nontrivial projection in $\mathcal{A}$. Let $P_1 = P$ and $P_2 = I - P$. Denote $\mathcal{A}_{jk} = P_j\mathcal{A}P_k$, $j, k = 1, 2$. Then $\mathcal{A} = \sum_{j,k=1}^{2} \mathcal{A}_{jk}$. Clearly, we only need to prove the necessity. We will prove the theorem by several claims.

Claim 1. $\Phi(0) = 0$.

Indeed, we have

$$\Phi(0) = \Phi([0, 0]_* \bullet 0) = [\Phi(0), 0]_* \bullet 0 + [0, \Phi(0)]_* \bullet 0 + [0, 0]_* \bullet \Phi(0) = 0.$$

Claim 2. Φ is additive.

We will prove Claim 2 by several steps.

Step 2.1. For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

We only need show that

$$T = \Phi(A_{11} + B_{12} + C_{21} + D_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21}) - \Phi(D_{22}) = 0.$$

It follows from Claim 1 that

$$\begin{aligned}
&[\Phi(P_1), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_2 + [P_1, \Phi(A_{11} + B_{12} + C_{21} + D_{22})]_* \bullet P_2 \\
&+ [P_1, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_2) \\
&= \Phi([P_1, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_2) \\
&= \Phi([P_1, B_{12}]_* \bullet P_2) \\
&= \Phi([P_1, A_{11}]_* \bullet P_2) + \Phi([P_1, B_{12}]_* \bullet P_2) + \Phi([P_1, C_{21}]_* \bullet P_2) + \Phi([P_1, D_{22}]_* \bullet P_2) \\
&= [\Phi(P_1), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_2 + [P_1, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})]_* \bullet P_2 \\
&+ [P_1, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_2).
\end{aligned}$$

From this, we get $[P_1, T]_* \bullet P_2 = 0$. So $T_{12} = 0$. Similarly, we can prove $T_{21} = 0$.

For every $X_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}
&[\Phi(X_{12}), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_2 + [X_{12}, \Phi(A_{11} + B_{12} + C_{21} + D_{22})]_* \bullet P_2 \\
&+ [X_{12}, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_2) \\
&= \Phi([X_{12}, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_2) \\
&= \Phi([X_{12}, D_{22}]_* \bullet P_2) \\
&= \Phi([X_{12}, A_{11}]_* \bullet P_2) + \Phi([X_{12}, B_{12}]_* \bullet P_2) + \Phi([X_{12}, C_{21}]_* \bullet P_2) + \Phi([X_{12}, D_{22}]_* \bullet P_2) \\
&= [\Phi(X_{12}), A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet P_2 + [X_{12}, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})]_* \bullet P_2 \\
&+ [X_{12}, A_{11} + B_{12} + C_{21} + D_{22}]_* \bullet \Phi(P_2).
\end{aligned}$$

Then $[X_{12}, T]_* \bullet P_2 = 0$, that is $X_{12}TP_2 + P_2T^*X_{12}^* = 0$. So $X_{12}TP_2 = 0$ for every $X_{12} \in \mathcal{A}_{12}$. By the primeness of $\mathcal{A}$, we have $T_{22} = 0$. Similarly, we can prove $T_{11} = 0$, proving the step.

Step 2.2. For every $A_{jk}, B_{jk} \in \mathcal{A}_{jk}, 1 \le j \ne k \le 2$, we have

$$\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk}).$$

Since

$$[-\frac{i}{2}I, i(P_j + A_{jk})]_* \bullet (P_k + B_{jk}) = (A_{jk} + B_{jk}) + A_{jk}^* + B_{jk}A_{jk}^*,$$

we get from Step 2.1 that

$$\begin{aligned}
&\Phi(A_{jk} + B_{jk}) + \Phi(A_{jk}^*) + \Phi(B_{jk}A_{jk}^*) \\
&= \Phi([-\frac{i}{2}I, i(P_j + A_{jk})]_* \bullet (P_k + B_{jk})) \\
&= [\Phi(-\frac{i}{2}I), i(P_j + A_{jk})]_* \bullet (P_k + B_{jk}) + [-\frac{i}{2}I, \Phi(i(P_j + A_{jk}))]_* \bullet (P_k + B_{jk}) \\
&+ [-\frac{i}{2}I, i(P_j + A_{jk})]_* \bullet \Phi(P_k + B_{jk}) \\
&= [\Phi(-\frac{i}{2}I), i(P_j + A_{jk})]_* \bullet (P_k + B_{jk}) + [-\frac{i}{2}I, \Phi(iP_j) + \Phi(iA_{jk})]_* \bullet (P_k + B_{jk}) \\
&+ [-\frac{i}{2}I, i(P_j + A_{jk})]_* \bullet (\Phi(P_k) + \Phi(B_{jk})) \\
&= \Phi([-\frac{i}{2}I, iP_j]_* \bullet P_k) + \Phi([-\frac{i}{2}I, iP_j]_* \bullet B_{jk}) + \Phi([-\frac{i}{2}I, iA_{jk}]_* \bullet P_k) + \Phi([-\frac{i}{2}I, iA_{jk}]_* \bullet B_{jk}) \\
&= \Phi(B_{jk}) + \Phi(A_{jk} + A_{jk}^*) + \Phi(B_{jk}A_{jk}^*) \\
&= \Phi(B_{jk}) + \Phi(A_{jk}) + \Phi(A_{jk}^*) + \Phi(B_{jk}A_{jk}^*).
\end{aligned}$$

Hence $\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk})$.

Step 2.3. For every $A_{jj}, B_{jj} \in \mathcal{A}_{jj}, 1 \le j \le 2$, we have

$$\Phi(A_{jj} + B_{jj}) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

Let $T = \Phi(A_{jj} + B_{jj}) - \Phi(A_{jj}) - \Phi(B_{jj})$. For $1 \le j \ne k \le 2$, it follows that

$$\begin{aligned}
&[\Phi(P_j), A_{jj} + B_{jj}]_* \bullet P_k + [P_j, \Phi(A_{jj} + B_{jj})]_* \bullet P_k + [P_j, A_{jj} + B_{jj}]_* \bullet \Phi(P_k) \\
&= \Phi([P_j, A_{jj} + B_{jj}]_* \bullet P_k) \\
&= \Phi([P_j, A_{jj}]_* \bullet P_k) + \Phi([P_j, B_{jj}]_* \bullet P_k) \\
&= [\Phi(P_j), A_{jj} + B_{jj}]_* \bullet P_k + [P_j, \Phi(A_{jj}) + \Phi(B_{jj})]_* \bullet P_k + [P_j, A_{jj} + B_{jj}]_* \bullet \Phi(P_k).
\end{aligned}$$

From this, we get $[P_j, T]_* \bullet P_k = 0$. So $T_{jk} = 0$. Similarly, we can prove $T_{kj} = 0$.

For every $X_{jk} \in \mathcal{A}_{jk}$, $j \neq k$, on the one hand, we have

$$\begin{aligned}
&[\Phi(X_{jk}), A_{jj} + B_{jj}]_* \bullet P_k + [X_{jk}, \Phi(A_{jj} + B_{jj})]_* \bullet P_k + [X_{jk}, A_{jj} + B_{jj}]_* \bullet \Phi(P_k) \\
&= \Phi([X_{jk}, A_{jj} + B_{jj}]_* \bullet P_k) \\
&= \Phi([X_{jk}, A_{jj}]_* \bullet P_k) + \Phi([X_{jk}, B_{jj}]_* \bullet P_k) \\
&= [\Phi(X_{jk}), A_{jj} + B_{jj}]_* \bullet P_k + [X_{jk}, \Phi(A_{jj}) + \Phi(B_{jj})]_* \bullet P_k + [X_{jk}, A_{jj} + B_{jj}]_* \bullet \Phi(P_k),
\end{aligned}$$

which implies that $[X_{jk}, T]_* \bullet P_k = 0$. So $X_{jk}T_{kk} = 0$ for all $X_{jk} \in \mathcal{A}_{jk}$. By the primeness of $\mathcal{A}$, we have $T_{kk} = 0$.

On the other hand, it follows from Steps 2.1 and 2.2 that

$$\begin{aligned}
&[\Phi(A_{jj} + B_{jj}), X_{jk}]_* \bullet P_k + [A_{jj} + B_{jj}, \Phi(X_{jk})]_* \bullet P_k + [A_{jj} + B_{jj}, X_{jk}]_* \bullet \Phi(P_k) \\
&= \Phi([A_{jj} + B_{jj}, X_{jk}]_* \bullet P_k) \\
&= \Phi(A_{jj}X_{jk}) + \Phi(B_{jj}X_{jk}) + \Phi(X_{jk}^*A_{jj}^*) + \Phi(X_{jk}^*B_{jj}^*) \\
&= \Phi([A_{jj}, X_{jk}]_* \bullet P_k) + \Phi([B_{jj}, X_{jk}]_* \bullet P_k) \\
&= [\Phi(A_{jj}) + \Phi(B_{jj}), X_{jk}]_* \bullet P_k + [A_{jj} + B_{jj}, \Phi(X_{jk})]_* \bullet P_k + [A_{jj} + B_{jj}, X_{jk}]_* \bullet \Phi(P_k).
\end{aligned}$$

Hence $[T_{jj}, X_{jk}]_* \bullet P_k = 0$, and then $T_{jj}X_{jk} = 0$ for all $X_{jk} \in \mathcal{A}_{jk}$. By the primeness of $\mathcal{A}$, we have $T_{jj} = 0$. Then $\Phi(A_{jj} + B_{jj}) = \Phi(A_{jj}) + \Phi(B_{jj})$.

Now, it follows from Steps 2.1, 2.2 and 2.3 that Φ is additive, proving the Claim 2.

Claim 3.

(1) $\Phi(iI)^* = \Phi(iI)$;

(2) $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I, \Phi(\mathbb{R}I) \subseteq \mathbb{R}I$;

(3) $\Phi(A) = \Phi(A)^*$ for all $A = A^* \in \mathcal{A}$.

It follows from Claim 2 that

$$\begin{aligned}
-4\Phi(iI) &= \Phi([iI, iI]_* \bullet (iI)) \\
&= [\Phi(iI), iI]_* \bullet (iI) + [iI, \Phi(iI)]_* \bullet (iI) + [iI, iI]_* \bullet \Phi(iI) \\
&= 4\Phi(iI)^* - 8\Phi(iI).
\end{aligned}$$

So $\Phi(iI)^* = \Phi(iI)$.

Let $\lambda \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
0 &= \Phi([\lambda I, A]_* \bullet I) = [\Phi(\lambda I), A]_* \bullet I \\
&= \Phi(\lambda I)(A - A^*) - (A - A^*)\Phi(\lambda I)^*
\end{aligned}$$

holds true for any $A \in \mathcal{A}$. So $\Phi(\lambda I)B = B\Phi(\lambda I)^*$ holds true for all $B = -B^* \in \mathcal{A}$. Since for every $B \in \mathcal{A}$, $B = B_1 + iB_2$ with $B_1 = \frac{B-B^*}{2}$ and $B_2 = \frac{B+B^*}{2i}$, it follows that $\Phi(\lambda I)B = B\Phi(\lambda I)^*$ holds true for all $B \in \mathcal{A}$. It follows from Lemma 2.1 that $\Phi(\lambda I) \in \mathbb{R}I$. Since $\lambda \in \mathbb{R}$ is arbitrary, we obtain $\Phi(\mathbb{R}I) \subseteq \mathbb{R}I$.

For any $A = A^* \in \mathcal{A}$, we have

$$0 = \Phi([A, iI]_* \bullet I) = [\Phi(A), iI]_* \bullet I = 2i(\Phi(A) - \Phi(A)^*),$$

which implies that $\Phi(A) = \Phi(A)^*$.

Let $\lambda \in \mathbb{C}$ be arbitrary. For all $A = A^* \in \mathcal{A}$ and $B \in \mathcal{A}$, it follows from $\Phi(A) = \Phi(A)^*$ that

$$0 = \Phi([A, \lambda I]_* \bullet B) = [A, \Phi(\lambda I)]_* \bullet B.$$

It follows from Lemma 2.2 that $[A, \Phi(\lambda I)]_* = [A, \Phi(\lambda I)] = i\lambda I$ for some $\lambda \in \mathbb{R}$. By Lemma 2.3, we have $[A, \Phi(\lambda I)] = 0$, that is $A\Phi(\lambda I) = \Phi(\lambda I)A$ for all $A = A^* \in \mathcal{A}$. Since for every $B \in \mathcal{A}$, $B = B_1 + iB_2$ with $B_1 = \frac{B+B^*}{2}$

and $B_2 = \frac{B-B^*}{2i}$, it follows that $B\Phi(\lambda I) = \Phi(\lambda I)B$ holds true for all $B \in \mathcal{A}$. So $\Phi(\lambda I) \in \mathbb{C}I$. Now we obtain $\Phi(\mathbb{C}I) \subseteq \mathbb{C}I$.

Claim 4. For $1 \leq j \neq k \leq 2$, we have $P_j\Phi(P_j)P_k = -P_j\Phi(P_k)P_k$ and $P_j\Phi(P_k)P_j = 0$.

On the one hand, it follows from Claim 3 that

$$\begin{aligned} 0 &= \Phi([iI, P_j]_* \bullet P_k) \\ &= [iI, \Phi(P_j)]_* \bullet P_k + [iI, P_j]_* \bullet \Phi(P_k) \\ &= 2i(\Phi(P_j)P_k - P_k\Phi(P_j)^* + P_j\Phi(P_k) - \Phi(P_k)P_j). \end{aligned}$$

Multiplying both sides of the above equation by P_j and P_k from the left and right respectively, we obtain that $P_j\Phi(P_j)P_k = -P_j\Phi(P_k)P_k$.

On the other hand, we have

$$\begin{aligned} 0 &= \Phi([iP_j, iI]_* \bullet P_k) \\ &= [\Phi(iP_j), iI]_* \bullet P_k + [iP_j, \Phi(iI)]_* \bullet P_k + [iP_j, iI]_* \bullet \Phi(P_k) \\ &= i(\Phi(iP_j)P_k - \Phi(iP_j)^*P_k - P_k\Phi(iP_j)^* + P_k\Phi(iP_j)) - 2P_j\Phi(P_k) - 2\Phi(P_k)P_j. \end{aligned}$$

Multiplying both sides of the above equation by P_j, we obtain that $P_j\Phi(P_k)P_j = 0$.

Now, let $T = P_1\Phi(P_1)P_2 - P_2\Phi(P_1)P_1$. By Claim 3 (3), we have $T^* = -T$. Defining a map $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(A) = \Phi(A) - (AT - TA)$ for all $A \in \mathcal{A}$. It is easy to verify that δ has the following properties.

Claim 5.

(1) For all $A, B, C \in \mathcal{A}$, $\delta([A, B]_* \bullet C) = [\delta(A), B]_* \bullet C + [A, \delta(B)]_* \bullet C + [A, B]_* \bullet \delta(C)$;

(2) $\delta(P_j) = P_j\Phi(P_j)P_j \in \mathcal{A}_{jj}$, $j = 1, 2$;

(3) $\delta(iI)^* = \delta(iI)$;

(4) $\delta(A) = \delta(A)^*$ for all $A = A^* \in \mathcal{A}$;

(5) δ is additive;

(6) δ is a $*$-derivation if and only if Φ is a $*$-derivation.

Claim 6. $\delta(P_j) = 0$ and $\delta(\mathcal{A}_{jk}) \subseteq \mathcal{A}_{jk}$, $j, k = 1, 2$.

Let $A_{jk} \in \mathcal{A}_{jk}$, $1 \leq j \neq k \leq 2$. On the one hand, it follows from Claim 5 that

$$\begin{aligned} \delta(iA_{jk}) &= \delta([\frac{i}{2}I, P_j]_* \bullet A_{jk}) \\ &= [\frac{i}{2}I, \delta(P_j)]_* \bullet A_{jk} + [\frac{i}{2}I, P_j]_* \bullet \delta(A_{jk}) \\ &= i(\delta(P_j)A_{jk} - A_{jk}\delta(P_j)^* + P_j\delta(A_{jk}) - \delta(A_{jk})P_j) \\ &= i(\delta(P_j)A_{jk} + P_j\delta(A_{jk}) - \delta(A_{jk})P_j). \end{aligned}$$

Hence $P_j\delta(iA_{jk})P_j = P_k\delta(iA_{jk})P_k = 0$, and then $\delta(iA_{jk}) = P_j\delta(iA_{jk})P_k + P_k\delta(iA_{jk})P_j$. On the other hand, for all $B \in \mathcal{A}$, we have

$$0 = \delta([iA_{jk}, P_j]_* \bullet B) = [\delta(iA_{jk}), P_j]_* \bullet B.$$

It follows from Lemma 2.2 that $[\delta(iA_{jk}), P_j]_* = P_k\delta(iA_{jk})P_j - P_j\delta(iA_{jk})^*P_k \in i\mathbb{R}I$, and then $P_k\delta(iA_{jk})P_j = 0$. Now we obtain $\delta(iA_{jk}) = P_j\delta(iA_{jk})P_k$. Since A_{jk} is arbitrary, we have $\delta(\mathcal{A}_{jk}) \subseteq \mathcal{A}_{jk}$, $j \neq k$.

Let $A_{jk} \in \mathcal{A}_{jk}$, $1 \leq j \neq k \leq 2$. Then

$$\begin{aligned} \delta(A_{jk}) + \delta(A_{jk}^*) &= \delta([A_{jk}, P_k]_* \bullet P_k) \\ &= [\delta(A_{jk}), P_k]_* \bullet P_k + [A_{jk}, \delta(P_k)]_* \bullet P_k + [A_{jk}, P_k]_* \bullet \delta(P_k) \\ &= \delta(A_{jk}) + \delta(A_{jk})^* + 2A_{jk}\delta(P_k) + \delta(P_k)^*A_{jk}^* + \delta(P_k)A_{jk}^*. \end{aligned}$$

Multiplying both sides of the above equation by P_j and P_k from the left and right respectively, we obtain that $A_{jk}\delta(P_k)P_k = 0$ for all $A_{jk} \in \mathcal{A}_{jk}$. Then $\delta(P_k) = P_k\delta(P_k)P_k = 0, k = 1,2$.

Let $A_{jj} \in \mathcal{A}_{jj}$, $j = 1,2$ and $i \neq j$. On the one hand, we have

$$0 = \delta([P_i, A_{jj}]_* \bullet P_j) = [P_i, \delta(A_{jj})]_* \bullet P_j = P_i\delta(A_{jj})P_j + P_j\delta(A_{jj})^*P_i$$

and

$$0 = \delta([P_j, A_{jj}]_* \bullet P_i) = [P_j, \delta(A_{jj})]_* \bullet P_i = P_j\delta(A_{jj})P_i + P_i\delta(A_{jj})^*P_j.$$

So $P_i\delta(A_{jj})P_j = P_j\delta(A_{jj})P_i = 0$. On the other hand, for any $T_{ji} \in \mathcal{A}_{ji}$ and $B \in \mathcal{A}$, we have

$$0 = \delta([T_{ji}, A_{jj}]_* \bullet B) = [T_{ji}, \delta(A_{jj})]_* \bullet B.$$

It follows from Lemma 2.2 that $[T_{ji}, \delta(A_{jj})]_* \in i\mathbb{R}I$, and then $T_{ji}\delta(A_{jj})P_i = 0$ for all $T_{ji} \in \mathcal{A}_{ji}$. By the primeness of $\mathcal{A}$, we have $P_i\delta(A_{jj})P_i = 0$. Now we obtain that $\delta(A_{jj}) = P_j\delta(A_{jj})P_j \in \mathcal{A}_{jj}$. Since A_{jj} is arbitrary, we have $\delta(\mathcal{A}_{jj}) \subseteq \mathcal{A}_{jj}$, $j = 1,2$.

Claim 7. $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$.

Let $A_{ij} \in \mathcal{A}_{ij}$ and $B_{ji} \in \mathcal{A}_{ji}$, $1 \leq i \neq j \leq 2$. It follows from Claim 6 that

$$\begin{aligned}\delta(A_{ij}B_{ji}) &= \delta([P_i, A_{ij}]_* \bullet B_{ji}) = [P_i, \delta(A_{ij})]_* \bullet B_{ji} + [P_i, A_{ij}]_* \bullet \delta(B_{ji}) \\ &= \delta(A_{ij})B_{ji} + A_{ij}\delta(B_{ji}).\end{aligned}$$

So

$$\delta(A_{ij}B_{ji}) = \delta(A_{ij})B_{ji} + A_{ij}\delta(B_{ji}). \tag{1}$$

For any $C_{ji} \in \mathcal{A}_{ji}$, it follows from Eq. (1) that

$$\begin{aligned}&\delta(A_{ii}B_{ij})C_{ji} + A_{ii}B_{ij}\delta(C_{ji}) = \delta(A_{ii}B_{ij}C_{ji}) = \delta([A_{ii}, B_{ij}]_* \bullet C_{ji}) \\ &= [\delta(A_{ii}), B_{ij}]_* \bullet C_{ji} + [A_{ii}, \delta(B_{ij})]_* \bullet C_{ji} + [A_{ii}, B_{ij}]_* \bullet \delta(C_{ji}) \\ &= \delta(A_{ii})B_{ij}C_{ji} + A_{ii}\delta(B_{ij})C_{ji} + A_{ii}B_{ij}\delta(C_{ji}).\end{aligned}$$

So $(\delta(A_{ii}B_{ij}) - \delta(A_{ii})B_{ij} - A_{ii}\delta(B_{ij}))C_{ji} = 0$ for any $C_{ji} \in \mathcal{A}_{ji}$. By the primeness of $\mathcal{A}$, we have

$$\delta(A_{ii}B_{ij}) = \delta(A_{ii})B_{ij} + A_{ii}\delta(B_{ij}). \tag{2}$$

It follows from Eq. (1) that

$$\begin{aligned}&\delta(A_{ij}B_{jj})C_{ji} + A_{ij}B_{jj}\delta(C_{ji}) = \delta(A_{ij}B_{jj}C_{ji}) = \delta([A_{ij}, B_{jj}]_* \bullet C_{ji}) \\ &= [\delta(A_{ij}), B_{jj}]_* \bullet C_{ji} + [A_{ij}, \delta(B_{jj})]_* \bullet C_{ji} + [A_{ij}, B_{jj}]_* \bullet \delta(C_{ji}) \\ &= \delta(A_{ij})B_{jj}C_{ji} + A_{ij}\delta(B_{jj})C_{ji} + A_{ij}B_{jj}\delta(C_{ji}).\end{aligned}$$

In the same manner, we obtain

$$\delta(A_{ij}B_{jj}) = \delta(A_{ij})B_{jj} + A_{ij}\delta(B_{jj}). \tag{3}$$

It follows from Eq. (2) that

$$\begin{aligned}&\delta(A_{jj}B_{jj})C_{ji} + A_{jj}B_{jj}\delta(C_{ji}) = \delta(A_{jj}B_{jj}C_{ji}) = \delta([A_{jj}, B_{jj}]_* \bullet C_{ji}) \\ &= [\delta(A_{jj}), B_{jj}]_* \bullet C_{ji} + [A_{jj}, \delta(B_{jj})]_* \bullet C_{ji} + [A_{jj}, B_{jj}]_* \bullet \delta(C_{ji}) \\ &= \delta(A_{jj})B_{jj}C_{ji} + A_{jj}\delta(B_{jj})C_{ji} + A_{jj}B_{jj}\delta(C_{ji}).\end{aligned}$$

Then

$$\delta(A_{jj}B_{jj}) = \delta(A_{jj})B_{jj} + A_{jj}\delta(B_{jj}). \tag{4}$$

Write $A = \sum_{i,j=1}^{2} A_{ij}, B = \sum_{i,j=1}^{2} B_{ij} \in \mathcal{A}$. Then $AB = A_{11}B_{11} + A_{11}B_{12} + A_{12}B_{21} + A_{12}B_{22} + A_{21}B_{11} + A_{21}B_{12} + A_{22}B_{21} + A_{22}B_{22}$. By Eqs (1)-(4) and the additivity of δ, we obtain that $\delta(AB) = \delta(A)B + A\delta(B)$.

Claim 8. $\delta(A^*) = \delta(A)^*$ for all $A \in \mathcal{A}$.

By Claims 6 and 7, we have

$$0 = -\delta(I) = \delta((iI)(iI)) = 2i\delta(iI).$$

So $\delta(iI) = 0$, and then $\delta(iA) = \delta((iI)A) = i\delta(A)$.

For every $A \in \mathcal{A}$, $A = A_1 + iA_2$, where $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2i}$ are self-adjoint elements. By Claim 5, we have

$$\begin{aligned}\delta(A^*) &= \delta(A_1 - iA_2) = \delta(A_1) - i\delta(A_2)\\ &= \delta(A_1)^* + (i\delta(A_2))^* = (\delta(A_1) + \delta(iA_2))^*\\ &= \delta(A)^*.\end{aligned}$$

Now, by Claims 5, 7 and 8, we obtain that Φ is an additive $*$-derivation. This completes the proof of Theorem 2.4. □

Acknowledgements

The authors are grateful to the anonymous referees and editors for their work.

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