Filomat 36:8 (2022), 2605–2616
https://doi.org/10.2298/FIL2208605L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Unification of Geraghty Type and Ćirić Type Fixed Point Theorems

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Abstract. In the framework of metric spaces, we introduce the concept of Geraghty-Ćirić type contractions and show the existence and uniqueness of the fixed point of such mappings. This result improves and unifies those obtained by Geraghty (Proc. Amer. Math. Soc. **40**, 604-608 (1973)) and Ćirić (Proc. Amer. Math. Soc. **45**, 267-273, (1974)). Several technical lemmas are employed to ensure that a Picard sequence is a Cauchy sequence. In addition, two illustrative examples are provided to indicate the validity of the obtained results.

1. Introduction

In 1922, Banach [1] proved a fixed point theorem for metric spaces known as the Banach contraction principle, which is one of the central component parts of fixed point theory. Since then, several researchers devoted to extending this theorem to different directions by changing the conditions of the mappings, see e.g., [2–4]. In particular, one of the notable generalizations of this celebrate principle is Geraghty type fixed point theorem, presented by Geraghty [5]. In [5], Geraghty introduced the definition of a new nonlinear contraction and established some fixed point results for such mappings. Thereafter, Amini-Harandi and Rmami [6] characterized the result of Geraghty in the context of a partially ordered complete metric space. Futhermore, Dukić et al. [7] extended fixed point theorems concerning Geraghty type contractions to the frame of partial metric spaces, ordered partial metric spaces and metric type spaces. In recent years, a number of authors studied this kind of nonlinear contraction and its generalized forms in various metric spaces (see e.g. [8–16] and references therein).

In what follows, we recall the fixed point theorem proved by Geraghty [5]. The following concept is a class of nonlinear functions prepared for the theorem.

Let $\mathcal{B}$ be the family of all functions $\beta : [0, +\infty) \to [0, 1)$ which satisfy the condition

$$\beta(t_n) \to 1 \Longrightarrow t_n \to 0.$$

Theorem 1.1 ([5]). *Let $f : X \to X$ be a contraction of a complete metric space satisfying*

$$d(fx, fy) \le \beta(d(x, y)) \cdot d(x, y) \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
Keywords. Geraghty type fixed point theroem; Ćirić type fixed point theroem; unified form
Received: 10 July 2021; Revised: 30 November 2021; Accepted: 22 December 2021
Communicated by Erdal Karapınar
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Research supported by the National Natural Science Foundation of China (12061050, 11561049) and the Natural Science Foundation of Inner Mongolia (2020MS01004)
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where $\beta \in \mathcal{B}$. Then for any choice of initial point x_0, the iteration $x_n = f(x_{n-1})$, $n > 0$, converges to the unique fixed point x^* of f in X.

Throughout this paper, we denote by ω, $\mathbb{N}^+$ the sets of all nonnegative integers and all positive integers, respectively.

In addition, there are many other types of fixed point theorems that extended the Banach contraction principle, such as Kannan type [17], Reich type [18] and Chatterjea type [19]. Particularly, in 1974 Ćirić [20] established the famous fixed point theorem in the setting of metric spaces, which was called Ćirić type fixed point theorem. It is worth recalling that this theorem is an actual generalization of the theorems mentioned above. In some ways, Ćirić type fixed point theorem can be deemed as a unified form of fundamental fixed point theorem for linear contractive mapping.

Theorem 1.2 ([20]). *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \le \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\} \tag{2}$$

for all $x, y \in X$, then T has a unique fixed point $x^* \in X$.

In the past decades, several published papers dealing with various types of Ćirić type contractions can be found in the literature (refer to [21–37]). One of the most interesting results on generalization was presented by Kumam [26] in 2013. Kumam et al. in [26] proved the new fixed point theorem which is a general case of the Ćirić fixed point theorem. Very recently, another remarkable generalization of Ćirić type fixed point theorem was given by Karapınar [32] in 2017. In the literature of this topic, Karapınar [32] investigated the Ćirić type nonunique fixed point results in the context of Branciari metric spaces.

In recent years, some researchers tried to establish a theorem to unify Geraghty type and Ćirić type fixed point theorem. In 2019, Faraji et al. [38] gave a new fixed point theorem concerning Geraghty type contractive mappings in b-metric spaces. Obviously, this theorem can not extend the result of Ćirić. Indeed, the obtained result by Faraji et al. [38] can not deduce the theorem of Geraghty, since $\beta \in \mathcal{B}$ is a function without monotonicity. Now, we give the version of the Faraji's result in metric spaces as follows.

Theorem 1.3. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying:*

$$d(Tx, Ty) \le \beta(M((x, y))M(x, y), \ x, y \in X,$$

where:

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\},$$

and $\beta \in \mathcal{B}$. Then T has a unique fixed point.

Due to the existing results mentioned above and application potential, it is significant to unify these two types of fixed point theorems. In this paper, the notion of Geraghty-Ćirić type contractions and a fixed point theorem for this type of mappings in the setting of metric spaces have been initiated. The result unifies both Geraghty type fixed point theorem and Ćirić type fixed point theorem. The proof of this result depends on a technique of how to choose the proper subsequence to prove that a sequence $\{x_n\}$ is a Cauchy sequence. Moreover, a new fixed point theorem is also given as a corollary of our main result. Finally, we give two examples to illustrate our results: one shows that the corollary is truly weaker than the main result, and another indicates that our main result is an actual generalization of Geraghty type and Ćirić type fixed point theorems.

2. Main results

In this section, we shall show a new fixed point theorem, which is called Geraghty-Ćirić type fixed point theorem. Before stating and proving our main results, we start to prove the following useful lemmas which play an important role in the proofs of our results.

Lemma 2.1. *Let (X,d) be a metric space and $\{x_n\}$ be a bounded sequence in X. If $\delta_n := \sup\{d(x_i,x_j) : i, j \geq n\} \to t$ as $n \to \infty$, then there exist two subsequences $\{x_{i_k}\}$ and $\{x_{j_k}\}$ of $\{x_n\}$ such that $i_k, j_k \geq k$ and*

$$d(x_{i_k}, x_{j_k}) \longrightarrow t \quad \text{as } k \to \infty.$$

Proof. Since $\delta_0 = \sup\{d(x_i, x_j) : i, j \in \omega\}$, there exist $i_0, j_0 \in \omega$ such that $i_0 \leq j_0$ and

$$\delta_0 - 1 \leq d(x_{i_0}, x_{j_0}) \leq \delta_0.$$

Let $n_1 = j_0 + 1$. By the definition of δ_{n_1}, there exist integers $i_1, j_1 \geq n_1$ such that $i_1 \leq j_1$ and

$$\delta_{n_1} - \frac{1}{2} \leq d(x_{i_1}, x_{j_1}) \leq \delta_{n_1}.$$

Continuing this process, there exist positive integer sequences $\{i_k\}$, $\{j_k\}$ and $\{n_k\}$ such that $n_k = j_{k-1} + 1$, $k \leq n_k \leq i_k \leq j_k$ and

$$\delta_{n_k} - \frac{1}{k+1} \leq d(x_{i_k}, x_{j_k}) \leq \delta_{n_k}.$$

Since $\delta_n \to t$ $(n \to \infty)$, we can see that $d(x_{i_k}, x_{j_k}) \to t$ as $k \to \infty$. □

Lemma 2.2. *Let $\{a_n^{(i)}\}$ be real number sequences for $i = 1, 2, 3, 4, 5$. Denote $M_n = \max\{a_n^{(i)} : i = 1, 2, 3, 4, 5\}$. Then there exist $i_0 \in \{1, 2, 3, 4, 5\}$ and a subsequence $\{M_{n_k}\}$ of $\{M_n\}$ such that*

$$M_{n_k} = a_{n_k}^{(i_0)}. \tag{3}$$

Proof. The conclusion follows immediately from the fact that there exist infinite terms in sequence $\{M_n\}$ such that each of them is equal to $a_n^{(i_0)}$ for some $i_0 \in \{1, 2, 3, 4, 5\}$. □

Now we introduce the notion of Geraghty-Ćirić type contraction, which is a unification of Geraghty type contraction and Ćirić type contraction.

Definition 2.3. *Let (X,d) be a metric space and $T : X \to X$ be a mapping. The mapping T is called a* Geraghty-Ćirić type contraction *if there exists a function $\beta \in \mathcal{B}$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq M(x, y), \tag{4}$$

where $M(x,y) = \max\{\beta(d(x,y))d(x,y), \beta(d(x,Tx))d(x,Tx), \beta(d(y,Ty))d(y,Ty), \beta(d(x,Ty))d(x,Ty), \beta(d(Tx,y))d(Tx,y)\}$.

The following lemmas are crucial in this paper.

Lemma 2.4. *Let (X,d) be a metric space, $T : X \to X$ be a mapping and $x_0 \in X$. Let $\{x_n\}_{n\in\omega}$ be a sequence such that $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}^+$. Denote*

$$D_n = \max\{d(x_i, x_j) : 0 \leq i, j \leq n \text{ and } i, j \in \omega\}$$

for $n \in \omega$. If T is a Geraghty-Ćirić type contraction, then $\{D_n\}$ is bounded.

Proof. Let $n \in \mathbb{N}^+$ be fixed. For any $i, j \in \mathbb{N}^+$ with $1 \le i, j \le n$, by (4) we have

$$\begin{aligned} d(x_i, x_j) &= d(Tx_{i-1}, Tx_{j-1}) \le M(x_{i-1}, x_{j-1}) \\ &= \max\{\beta(d(x_{i-1}, x_{j-1}))d(x_{i-1}, x_{j-1}), \beta(d(x_{i-1}, x_i))d(x_{i-1}, x_i), \beta(d(x_{j-1}, x_j)) \\ &\quad d(x_{j-1}, x_j), \beta(d(x_{i-1}, x_j))d(x_{i-1}, x_j), \beta(d(x_i, x_{j-1}))d(x_i, x_{j-1})\} \\ &< \max\{d(x_{i-1}, x_{j-1}), d(x_{i-1}, x_i), d(x_{j-1}, x_j), d(x_{i-1}, x_j), d(x_i, x_{j-1})\} \\ &\le D_n. \end{aligned}$$

This means that $\max\{d(x_i, x_j) : 1 \le i, j \le n \text{ and } i, j \in \omega\} < D_n$. Hence, we deduce that there exists $l_n \in \mathbb{N}^+$ with $1 \le l_n \le n$ such that

$$D_n = d(x_0, x_{l_n}).$$

Assume that, on the contrary, the sequence $\{D_n\}$ is unbounded. Note that $0 \le D_n \le D_{n+1}$. Then we have $D_n \to +\infty$ as $n \to \infty$. Applying (4), we derive

$$\begin{aligned} D_n &= d(x_0, x_{l_n}) \\ &\le d(x_0, x_1) + d(x_1, x_{l_n}) \\ &\le d(x_0, x_1) + M(x_0, x_{l_n-1}), \end{aligned} \tag{5}$$

where

$$\begin{aligned} M(x_0, x_{l_n-1}) = \max\{&\beta(d(x_0, x_{l_n-1}))d(x_0, x_{l_n-1}), \beta(d(x_0, x_1))d(x_0, x_1), \beta(d(x_{l_n-1}, x_{l_n}))d(x_{l_n-1}, x_{l_n}), \\ &\beta(d(x_0, x_{l_n}))d(x_0, x_{l_n}), \beta(d(x_1, x_{l_n-1}))d(x_1, x_{l_n-1})\}. \end{aligned}$$

By Lemma 2.2, we consider the following five cases.
Case 1. If there exists a subsequence $\{M(x_0, x_{l_{n_k}-1})\}$ of $M(x_0, x_{l_n-1})$ such that

$$\begin{aligned} M(x_0, x_{l_{n_k}-1}) &= \max\{\beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}), \beta(d(x_0, x_1))d(x_0, x_1), \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))d(x_{l_{n_k}-1}, x_{l_{n_k}}), \\ &\qquad \beta(d(x_0, x_{l_{n_k}}))d(x_0, x_{l_{n_k}}), \beta(d(x_{l_{n_k}-1}, x_1))d(x_{l_{n_k}-1}, x_1)\} \\ &= \beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}). \end{aligned} \tag{6}$$

By means of (5) and (6), we get

$$D_{n_k} \le d(x_0, x_1) + \beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}) \tag{7}$$
$$\le d(x_0, x_1) + \beta(d(x_0, x_{l_{n_k}-1}))D_{n_k}. \tag{8}$$

It follows from (8) that

$$1 - \frac{d(x_0, x_1)}{D_{n_k}} \le \beta(d(x_0, x_{l_{n_k}-1})) < 1.$$

Since $D_{n_k} \to +\infty$ as $k \to +\infty$, we can see that $1 - \frac{d(x_0,x_1)}{D_{n_k}} \to 1$ and $\beta(d(x_0, x_{l_{n_k}-1})) \to 1$. From $\beta \in \mathcal{B}$, we have $d(x_0, x_{l_{n_k}-1}) \to 0$. By (7), we deduce that

$$D_{n_k} \le d(x_0, x_1) + \beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}) < d(x_0, x_1) + d(x_0, x_{l_{n_k}-1}).$$

This leads to $\{D_{n_k}\}$ is bounded, a contradiction.
Case 2. If there exists a subsequence $\{M(x_0, x_{l_{n_k}-1})\}$ of $M(x_0, x_{l_n-1})$ such that

$$\begin{aligned} M(x_0, x_{l_{n_k}-1}) &= \max\{\beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}), \beta(d(x_0, x_1))d(x_0, x_1), \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))d(x_{l_{n_k}-1}, x_{l_{n_k}}), \\ &\qquad \beta(d(x_0, x_{l_{n_k}}))d(x_0, x_{l_{n_k}}), \beta(d(x_{l_{n_k}-1}, x_1))d(x_{l_{n_k}-1}, x_1)\} \\ &= \beta(d(x_0, x_1))d(x_0, x_1). \end{aligned} \tag{9}$$

By virtue of (5) and (9), we get

$$D_{n_k} \leq d(x_0, x_1) + \beta(d(x_0, x_1))d(x_0, x_1) < 2d(x_0, x_1),$$

which contradicts that $D_{n_k} \to \infty$ as $k \to \infty$.

Case 3. If there exists a subsequence $\{M(x_0, x_{l_{n_k}-1})\}$ of $M(x_0, x_{l_n-1})$ such that

$$\begin{aligned} M(x_0, x_{l_{n_k}-1}) &= \max\{\beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}), \beta(d(x_0, x_1))d(x_0, x_1), \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))d(x_{l_{n_k}-1}, x_{l_{n_k}}), \\ &\quad \beta(d(x_0, x_{l_{n_k}}))d(x_0, x_{l_{n_k}}), \beta(d(x_{l_{n_k}-1}, x_1))d(x_{l_{n_k}-1}, x_1)\} \\ &= \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))d(x_{l_{n_k}-1}, x_{l_{n_k}}). \end{aligned} \tag{10}$$

Combining (5) and (10), we derive

$$\begin{aligned} D_{n_k} &\leq d(x_0, x_1) + \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))d(x_{l_{n_k}-1}, x_{l_{n_k}}) \\ &\leq d(x_0, x_1) + \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))D_{n_k}, \end{aligned}$$

leading to that

$$1 - \frac{d(x_0, x_1)}{D_{n_k}} \leq \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}})) < 1.$$

Following a similar argument as in Case 1, we can deduce that $d(x_{l_{n_k}-1}, x_{l_{n_k}}) \to 0$ and $\{D_{n_k}\}$ is bounded, which contradicts that $D_n \to +\infty$.

Case 4. If there exists a subsequence $\{M(x_0, x_{l_{n_k}-1})\}$ of $M(x_0, x_{l_n-1})$ such that

$$\begin{aligned} M(x_0, x_{l_{n_k}-1}) &= \max\{\beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}), \beta(d(x_0, x_1))d(x_0, x_1), \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))d(x_{l_{n_k}-1}, x_{l_{n_k}}), \\ &\quad \beta(d(x_0, x_{l_{n_k}}))d(x_0, x_{l_{n_k}}), \beta(d(x_{l_{n_k}-1}, x_1))d(x_{l_{n_k}-1}, x_1)\} \\ &= \beta(d(x_0, x_{l_{n_k}}))d(x_0, x_{l_{n_k}}). \end{aligned} \tag{11}$$

In light of (5) and (11), we have

$$\begin{aligned} D_{n_k} &\leq d(x_0, x_1) + \beta(d(x_0, x_{l_{n_k}}))d(x_0, x_{l_{n_k}}) \\ &\leq d(x_0, x_1) + \beta(d(x_0, x_{l_{n_k}}))D_{n_k}. \end{aligned}$$

The above inequality implies that

$$1 - \frac{d(x_0, x_1)}{D_{n_k}} \leq \beta(d(x_0, x_{l_{n_k}})) < 1.$$

In a similar way as in Case 1, we can prove that $d(x_0, x_{l_{n_k}}) \to 0$ and $\{D_{n_k}\}$ is bounded, which contradicts that $D_n \to +\infty$.

Case 5. If there exists a subsequence $\{M(x_0, x_{l_{n_k}-1})\}$ of $M(x_0, x_{l_n-1})$ such that

$$\begin{aligned} M(x_0, x_{l_{n_k}-1}) &= \max\{\beta(d(x_0, x_{l_{n_k}-1}))d(x_0, x_{l_{n_k}-1}), \beta(d(x_0, x_1))d(x_0, x_1), \beta(d(x_{l_{n_k}-1}, x_{l_{n_k}}))d(x_{l_{n_k}-1}, x_{l_{n_k}}), \\ &\quad \beta(d(x_0, x_{l_{n_k}}))d(x_0, x_{l_{n_k}}), \beta(d(x_{l_{n_k}-1}, x_1))d(x_{l_{n_k}-1}, x_1)\} \\ &= \beta(d(x_1, x_{l_{n_k}-1}))d(x_1, x_{l_{n_k}-1}). \end{aligned} \tag{12}$$

Following a similar argument as in Case 3 or Case 4, we can show that our assumption deduces a contradiction.

Motivated by the above said work, we conclude that $\{D_n\}$ is a bounded sequence. □

Lemma 2.5. *Let (X, d) be a metric space and $T : X \to X$ be a Geraghty-Ćirić type contraction with some $\beta \in \mathcal{B}$. Then for each $x \in X$, $\{T^n x\}_{n\in\omega}$ is a Cauchy sequence in X (here, $T^0 = I$ is the identity map).*

Proof. For any $x_0 \in X$, let $\{x_n\}_{n\in\omega}$ be a sequence defined by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}^+$. We claim that $\{x_n\}_{n\in\omega}$ is a Cauchy sequence in X. Denote

$$D_n = \max\{d(x_i, x_j) : 0 \le i, j \le n \text{ and } i, j \in \omega\}$$

for $n \in \omega$. Then by Lemma 2.4, there exists $L > 0$ such that $D_n \le L$ for all $n \in \mathbb{N}^+$. Note that $\{D_n\}$ is a nondecreasing sequence. Then, we have $\lim\limits_{n\to\infty} D_n \le L$. Denote

$$\delta_n = \sup\{d(x_i, x_j) : i, j \ge n \text{ and } i, j \in \omega\}.$$

It is clear that $0 \le \delta_n \le \delta_{n-1} \le \cdots \le \delta_0 = \lim\limits_{n\to\infty} D_n \le L$ for all $n \in \omega$, and so $\lim_{n\to\infty} \delta_n = t$ exists. Let $\lim_{n\to\infty} \delta_n = t \ge 0$. Assume that $t > 0$. Then, by Lemma 2.1, there exist two subsequences $\{x_{i_k}\}$ and $\{x_{j_k}\}$ of $\{x_n\}$ such that $i_k, j_k \ge k$ and

$$d(x_{i_k}, x_{j_k}) \longrightarrow t \quad \text{as } k \to \infty.$$

Taking account of (4), we have

$$d(x_{i_k}, x_{j_k}) \le M(x_{i_k-1}, x_{j_k-1}), \tag{13}$$

where

$$\begin{aligned} M(x_{i_k-1}, x_{j_k-1}) = \max\{&\beta(d(x_{i_k-1}, x_{j_k-1}))d(x_{i_k-1}, x_{j_k-1}), \beta(d(x_{i_k-1}, x_{i_k}))d(x_{i_k-1}, x_{i_k}), \beta(d(x_{j_k-1}, x_{j_k}))d(x_{j_k-1}, x_{j_k}),\\ &\beta(d(x_{i_k-1}, x_{j_k}))d(x_{i_k-1}, x_{j_k}), \beta(d(x_{j_k-1}, x_{i_k}))d(x_{j_k-1}, x_{i_k})\}. \end{aligned}$$

By means of Lemma 2.2, there exists a subsequence of $\{M(x_{i_k-1}, x_{j_k-1})\}_k$ matching one of the five terms above. Now we prove the case of the first term, and other cases can be similarly proved. Suppose that $M(x_{i_k-1}, x_{j_k-1}) = \beta(d(x_{i_k-1}, x_{j_k-1}))d(x_{i_k-1}, x_{j_k-1})$ for all $k \in \omega$. Then (13) turns into

$$d(x_{i_k}, x_{j_k}) \le \beta(d(x_{i_k-1}, x_{j_k-1}))d(x_{i_k-1}, x_{j_k-1}) \tag{14}$$

$$\le \beta(d(x_{i_k-1}, x_{j_k-1}))\delta_{k-1}. \tag{15}$$

It follows from (15) that

$$\frac{d(x_{i_k}, x_{j_k})}{\delta_{k-1}} \le \beta(d(x_{i_k-1}, x_{j_k-1})) < 1.$$

Since $\lim_{k\to\infty} d(x_{i_k}, x_{j_k}) = \lim_{k\to\infty} \delta_{k-1} = t > 0$, we see that $\beta(d(x_{i_k-1}, x_{j_k-1})) \to 1$ as $k \to \infty$. Taking the fact that $\beta \in \mathcal{B}$ into account, together with (14), we obtain that

$$\lim_{k\to\infty} d(x_{i_k-1}, x_{j_k-1}) = 0.$$

From (14), we have

$$t = \lim_{k\to\infty} d(x_{i_k}, x_{j_k}) = 0,$$

which contradicts the assumption $t > 0$. Thus, $\lim\limits_{n\to\infty} \delta_n = t = 0$.

Let $m, n \in \omega$ with $m > n$. Then we obtain that

$$d(x_m, x_n) \le \delta_n \longrightarrow 0 \quad \text{as } n \to \infty.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. $\square$

Now, we shall use Lemma 2.4 and Lemma 2.5 to prove the following new fixed point theorem which unifies and generalizes the Geraghty type and Ćirić type fixed point theorems.

Theorem 2.6. *Let (X, d) be a complete metric space and $T : X \to X$ be a Geraghty-Ćirić type contraction with some $\beta \in \mathcal{B}$. Then T has a unique fixed point $z \in X$.*

Proof. Let $x_0 \in X$ be arbitrarily given. Define the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}^+$. By Lemma 2.5, $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists $z \in X$ such that $\{x_n\}$ converges to z. Next, we prove that z is a fixed point of T.

Assume that $Tz \neq z$, i.e. $d(z, Tz) > 0$. From (4), we get

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq M(x_n, z), \tag{16}$$

where

$$M(x_n, z) = \max\{\beta(d(x_n, z))d(x_n, z), \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}), \beta(d(z, Tz))d(z, Tz), \beta(d(x_n, Tz))d(x_n, Tz), \beta(d(x_{n+1}, z))d(x_{n+1}, z)\}.$$

Since these three sequences $\{d(x_n, z)\}$, $\{d(x_n, x_{n+1})\}$ and $\{d(x_{n+1}, z)\}$ all converge to 0 as $n \to \infty$, we deduce that

$$M(x_n, z) = \max\{\beta(d(z, Tz))d(z, Tz), \beta(d(x_n, Tz))d(x_n, Tz)\}$$

as n is large enough.

If there exists a subsequence $\{M(x_{n_k}, z)\}$ of $\{M(x_n, z)\}$ such that $M(x_{n_k}, z) = \beta(d(z, Tz))d(z, Tz)$. Then, (16) yields that

$$d(x_{n_k+1}, Tz) \leq \beta(d(z, Tz))d(z, Tz).$$

Putting $k \to \infty$, we see that $d(z, Tz) \leq \beta(d(z, Tz))d(z, Tz) < d(z, Tz)$. That is a contradiction. Thus, we conclude that

$$M(x_n, z) = \beta(d(x_n, Tz))d(x_n, Tz)$$

as n is large enough.

By means of (16), we obtain that $d(x_{n+1}, Tz) \leq \beta(d(x_n, Tz))d(x_n, Tz)$. Then

$$\frac{d(x_{n+1}, Tz)}{d(x_n, Tz)} \leq \beta(d(x_n, Tz)) < 1.$$

Note that

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = \lim_{n\to\infty} d(x_n, Tz) = d(z, Tz) > 0.$$

So we deduce that $\beta(d(x_n, Tz)) \to 1$ as $n \to \infty$. Since $\beta \in \mathcal{B}$, we can see that $d(x_n, Tz) \to 0$ $(n \to \infty)$. This leads to $d(z, Tz) = 0$, which contradicts the assumption $z \neq Tz$.

Therefore, we obtain that $z = Tz$ and z is a fixed point of T. This completes the proof of the existence of the fixed point of T.

Finally, we prove that z is the unique fixed point of T. Suppose that $\bar{z}$ is another fixed point of T. Then (4) implies that

$$\begin{aligned} d(z, \bar{z}) = d(Tz, T\bar{z}) &\leq M(z, \bar{z}) \\ &\leq \max\{\beta(d(z, \bar{z}))d(z, \bar{z}), \beta(d(z, Tz))d(z, Tz), \beta(d(\bar{z}, T\bar{z}))d(\bar{z}, T\bar{z}), \\ &\quad \beta(d(z, T\bar{z}))d(z, T\bar{z}), \beta(d(Tz, \bar{z}))d(Tz, \bar{z})\} \\ &= \beta(d(z, \bar{z}))d(z, \bar{z}). \end{aligned}$$

Using the fact that $\beta(d(z, \bar{z})) < 1$, we have $d(z, \bar{z}) = 0$ and $z = \bar{z}$. This completes the proof. □

The following corollary is an immediate consequence of Theorem 2.6, which is also a new result.

Corollary 2.7. *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping such that for some $\beta \in \mathcal{B}$ and any $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(m(x, y))m(x, y), \tag{17}$$

where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\}$. Then T has a unique fixed point $x^* \in X$.

Proof. For any $x, y \in X$, $m(x, y)$ is always equal to one of the five terms $d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)$. It follows that

$$\begin{aligned} d(Tx, Ty) &\leq \beta(m(x, y))m(x, y) \\ &\leq \max\{\beta(d(x, y))d(x, y), \beta(d(x, Tx))d(x, Tx), \beta(d(y, Ty))d(y, Ty), \\ &\quad \beta(d(x, Ty))d(x, Ty), \beta(d(Tx, y))d(Tx, y)\} \\ &= M(x, y). \end{aligned}$$

All assumptions of Theorem 2.6 are satisfied and we obtain the conclusion. □

The following example shows that Corollary 2.7 is weaker than Theorem 2.6.

Example 2.8. *Let* $X = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\} \cup \{0\}$ *and* d *be the normal metric on* X*. Set* $T : X \to X$ *to be the mapping defined by*

$$Tx = \begin{cases} 0, & x = 0; \\ \dfrac{1}{n+1}, & x = \dfrac{1}{n} \text{ for all } n \in \mathbb{N}^+. \end{cases}$$

Define a function $\beta : [0, +\infty) \to [0, 1)$ *by*

$$\beta(t) = \begin{cases} \dfrac{1}{1+t}, & t = \dfrac{a}{b} \text{ for some } a, b \in \mathbb{N}^+, \text{ a and b are coprime and a is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

Then the following assertions hold:

(1) *All of the conditions in Theorem 2.6 are satisfied with the* $\beta(t)$ *and* T *has a unique fixed point* $x = 0$*;*
(2) *The condition* (17) *in Corollary 2.7 is not satisfied with the* $\beta(t)$*.*

Proof. (1) It is clear that (X, d) is a complete metric space, $\beta(t) \in \mathcal{B}$ and $x = 0$ is the unique fixed point for T. Thus, it is sufficient to prove that (4) holds with $\beta(t)$.

For any $x, y \in X$, if $x = y$, then (4) yields that $d(Tx, Ty) = 0$. Now we suppose that $x \neq y$. Without loss of generality, let $x < y$. Then we consider the following two cases.

Case a. If $x = 0$, then $y > 0$. Let $y = \frac{1}{n}$ for some $n \in \mathbb{N}^+$. Then we have

$$d(Tx, Ty) = \frac{1}{n+1} = \frac{1}{1+\frac{1}{n}}\frac{1}{n} = \beta\left(\frac{1}{n}\right)\frac{1}{n} = \beta(d(x, y))d(x, y) \leq M(x, y).$$

Case b. Suppose that $x > 0$. Then $x = \frac{1}{m}$ and $y = \frac{1}{n}$ for some $m, n \in \mathbb{N}^+$. Since $x < y$, we have $m > n$.

- If $m - n$ is odd integer, then we get

$$\beta(d(x, y)) = \beta\left(\frac{1}{n} - \frac{1}{m}\right) = \beta\left(\frac{m-n}{nm}\right) = \frac{1}{1+\frac{m-n}{nm}} = \frac{nm}{m-n+nm}$$

and

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{n+1} - \frac{1}{m+1} \\ &= \frac{m-n}{m+n+nm+1} \\ &\leq \frac{nm}{m-n+nm} \cdot \frac{m-n}{nm} \\ &= \beta\left(d(x, y)\right)d(x, y) \leq M(x, y). \end{aligned}$$

- If $m-n$ is even integer, then $m+1-n$ is odd integer. Following the above arguments, we can see that

$$\begin{aligned}\beta\left(d(y,Tx)\right) &= \beta\left(\frac{1}{n}-\frac{1}{m+1}\right)=\beta\left(\frac{m+1-n}{n(m+1)}\right)\\ &= \frac{1}{1+\frac{m+1-n}{n(m+1)}}=\frac{n(m+1)}{nm+m+1}>\frac{1}{1+\frac{1}{n}}=\beta\left(\frac{1}{n}\right)\end{aligned}$$

and

$$\begin{aligned}d(Tx,Ty) &= \frac{1}{n+1}-\frac{1}{m+1}\\ &< \frac{1}{1+\frac{1}{n}}\cdot\frac{1}{n}-\frac{1}{1+\frac{1}{n}}\cdot\frac{1}{m+1}\\ &= \beta\left(\frac{1}{n}\right)\frac{1}{n}-\beta\left(\frac{1}{n}\right)\frac{1}{m+1}\\ &< \beta(d(Tx,y))\left(\frac{1}{n}-\frac{1}{m+1}\right)\\ &= \beta(d(Tx,y))d(Tx,y)\leq M(x,y).\end{aligned}$$

From Cases a and b, we show that (4) holds for all $x,y\in X$.

(2) We show that (17) is not satisfied with $\beta(t)$. Let $x_k=\frac{1}{2k+2}$ and $y_k=\frac{1}{2k+1}$ for $k\in\omega$. Then we can see that

$$m(x_k,y_k)=d(Tx_k,y_k)=\frac{1}{2k+1}-\frac{1}{2k+3}=\frac{2}{(2k+1)(2k+3)}.$$

Note that $\beta\left(m(x_k,y_k)\right)=\beta\left(\frac{2}{(2k+1)(2k+3)}\right)=0$. Then we have

$$d(Tx_k,Ty_k)=\frac{1}{2k+2}-\frac{1}{2k+3}>0=\beta(m(x_k,y_k))m(x_k,y_k).$$

That completes the proof. □

Remark 2.9. *Note that $\beta(d(x,y))d(x,y)\leq M(x,y)$, and $\beta(t)\equiv\lambda\in\mathcal{B}$ for constant $\lambda\in[0,1)$. Then, Theorem 1.1 and 1.2 are all special cases of Theorem 2.6. For illustration, we provide at the end of this section a concrete example for which either Geraghty type or Ćirić type fixed point theorem is not applicable. However, Theorem 2.6 can be used to conclude the existence of fixed point of mapping.*

Example 2.10. *Let $X=\{0\}\cup\{a_n\}$, where*

$$a_n=\begin{cases}\dfrac{1}{2^k}, & n=2k \text{ for some } k\in\omega;\\[2mm] \dfrac{5}{5\cdot 2^k+1}, & n=2k+1 \text{ for some } k\in\omega.\end{cases}$$

for $n\in\omega$. Let $d(x,y)=|x-y|$ for all $x,y\in X$. Define a map $T:X\to X$ by

$$Tx=\begin{cases}a_{n+1}, & x=a_n \text{ for all } n\in\omega;\\ 0, & x=0.\end{cases}$$

Then the following assertions hold:

1. *All of the conditions in Theorem 2.6 are satisfied and T has a unique fixed point $x=0$;*
2. *The map T is not a Geraghty type contraction;*
3. *The map T is not a Ćirić type contraction.*

Proof. From the definitions of $\{a_n\}$ and T, we can deduce that $\{a_n\}$ is a decreasing sequence, T is nondecreasing and for each $k \in \omega$,

$$a_{2k+1} = \frac{5}{5 + a_{2k}} a_{2k}. \tag{18}$$

1. Let $\beta(t) = \frac{5}{5+t}$ $(t > 0)$ and $\beta(0) = 0$. Clearly, $\beta \in \mathcal{B}$. It is obviously that (X, d) is a complete metric space, and $x = 0$ is the unique fixed point for T. Therefore, it is sufficient to prove that (4) holds with the $\beta(t)$ above.

For any $x, y \in X$, without loss of generality, we suppose that $x \leq y$ and $y \neq 0$. Let $y = a_n$ for some $n \in \omega$. Then we consider the following two cases.

Case a. If $n = 2k$ for some $k \in \omega$, then from (18) we get

$$Ty = Ta_{2k} = a_{2k+1} = \frac{5}{5 + a_{2k}} a_{2k} = \beta(a_{2k})a_{2k} = \beta(y)y.$$

Note that $\beta(t)$ is decreasing and $\beta(t) < 1$ $(t > 0)$. So we have

$$\begin{aligned} d(Tx, Ty) &= Ty - Tx = \beta(y)y - Tx \leq \beta(y)y - \beta(y)Tx \leq \beta(y - Tx)(y - Tx) \\ &= \beta(d(y, Tx))d(y, Tx) \leq M(x, y). \end{aligned}$$

Case b. If $n = 2k + 1$ for some $k \in \omega$, then from (18) we deduce

$$Ty = Ta_{2k+1} = a_{2k+2} = \frac{1}{2} a_{2k} = \frac{1}{2} \cdot \frac{5 + a_{2k}}{5} a_{2k+1}.$$

Since a_n is a decreasing sequence and $\beta(t)$ is decreasing, we obtain that $\frac{1}{2} < \left(\frac{5}{5+1}\right)^2 \leq \left(\frac{5}{5+a_{2k}}\right)^2$. Then we have

$$Ty = \frac{1}{2} \cdot \frac{5 + a_{2k}}{5} a_{2k+1} < \frac{5}{5 + a_{2k}} a_{2k+1} \leq \frac{5}{5 + a_{2k+1}} a_{2k+1}.$$

Following a similar argument as in Case a, (4) can be verified.

From Cases a and b, we show that (4) holds for all $x, y \in X$.

2. If there exists a function $\beta \in \mathcal{B}$ such that

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$. Since $\beta(t) < 1$ $(t \geq 0)$, we have $d(Tx, Ty) < d(x, y)$. However, let $x = a_{2k+1}$ and $y = a_{2k}$ for some $k \in \omega$. Note that $\frac{3}{4} < \frac{5}{5+1} \leq \frac{5}{5+a_{2k}}$. Then, by means of (18) we get

$$\begin{aligned} d(Tx, Ty) &= a_{2k+1} - a_{2k+2} = \left(\frac{5}{5 + a_{2k}} - \frac{1}{2}\right) a_{2k} \\ &> \frac{1}{4} a_{2k} > \left(1 - \frac{5}{5 + a_{2k}}\right) a_{2k} \\ &= a_{2k} - a_{2k+1} = d(x, y). \end{aligned}$$

That is a contradiction. Thus, the map T is not a Geraghty type contraction.

3. Let $x_n = 0$ and $y_n = a_{2n}$. Then from (18) we have

$$\frac{d(Tx_n, Ty_n)}{d(x_n, y_n)} = \frac{a_{2n+1}}{a_{2n}} = \frac{5}{5 + a_{2n}} \longrightarrow 1 \quad \text{as } n \to \infty.$$

Hence, there is no constant $\lambda \in [0, 1)$ such that

$$\begin{aligned} d(Tx_n, Ty_n) &\leq \lambda d(x_n, y_n) \\ &= \lambda \max\{d(x_n, y_n), d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, Ty_n), d(Tx_n, y_n)\}. \end{aligned}$$

Therefore, the map T can not be a Ćirić type contraction. □

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