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On Hom-Leibniz Superalgebras

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Abstract. In this paper, we define a new type of cohomology for Hom-Leibniz superalgebras which controls deformations of Hom-Leibniz superalgebras. The cohomology and the associated deformation theory for Hom-Leibniz superalgebras as developed here are also extended to equivariant context, under the presence of finite group actions on Hom-Leibniz superalgebras.

1. Introduction

A Hom-Leibniz algebra is a triple $(L, [,], \alpha)$, where α is a linear self-map, in which the bilinear bracket satisfies a α-twisted variant of the Leibniz identity, called the Hom-Leibniz identity. When α is the identity map, the Hom-Leibniz identity reduces to the usual Leibniz identity, and L is a Leibniz algebra which are been introduced by Loday [1]. Leibniz algebras were studied in [11]. The notion of Hom-Leibniz algebras was introduced by Makhlouf and Silvestrov [8]. Hom-Leibniz algebras were studied extensively in [3, 6, 12, 21]. Furthermore, Hom-Leibniz algebras were generalized to Hom-Leibniz superalgebras by literature [9, 10, 14, 15] as a noncommutative generalization of Hom-Lie superalgebras. When α is the identity map, the Hom-Leibniz superalgebras reduces to the usual Leibniz superalgebras. Leibniz superalgebras were studied in [5, 7, 13].

Nowadays, much researches on certain algebraic objects are concerned with their formal deformations theories. The deformation theory was introduced by Gerstenhaber for rings and algebras in a series of papers [16–19]. Recently, the deformation theory of other algebras has been studied by several authors [4–6, 13, 20]. Equivariant deformation theory of various algebras has been studied in [21–23].

Purpose of this paper is to introduce α-type cohomology, equivariant deformation cohomology and equivariant formal deformation theory of Hom-Leibniz superalgebras based on some work in [9, 21]. This paper is organized as follows. In Section 2, we introduce the Hom-Leibniz superalgebras related known concepts. In Section 3, we introduce the α-type cohomology for Hom-Leibniz superalgebras. In Section 4, we introduce the Deformation theory of Hom-Leibniz superalgebras. In Section 5, we introduce Group action and equivariant cohomology of Hom-Leibniz superalgebras. In Section 6, we introduce Equivariant formal deformation of Hom-Leibniz superalgebras.

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2. Preliminaries

Our aim in this section is to recall some definitions and results related to Hom-Leibniz superalgebras. Let $\mathbb{K}$ be a field of characteristic zero. A vector space L is said to be a $\mathbb{Z}_2$-graded if we are given a family $(L_{\bar{i}})_{\bar{i}\in\mathbb{Z}_2}$ of vector subspace L such that $L = L_{\bar{0}} \oplus L_{\bar{1}}$. We will denote by $\mathcal{H}(L)$ the set of all homogeneous elements of L. The symbol $|x|$ always implies that x is a $\mathbb{Z}_2$-homogeneous element and $|x|$ is the $\mathbb{Z}_2$-degree. Moreover, the $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space $\mathrm{End}_{\mathbb{K}}(L)$ has a natural direct sum decomposition $\mathrm{End}_{\mathbb{K}}(L) = \mathrm{End}_{\mathbb{K}}(L)_{\bar{0}} \oplus \mathrm{End}_{\mathbb{K}}(L)_{\bar{1}}$, where $\mathrm{End}_{\mathbb{K}}(L)_{\bar{j}} = \{\phi \in \mathrm{End}_{\mathbb{K}}(L) | \phi(L_{\bar{i}}) \subseteq L_{\bar{i}+\bar{j}}\}$, $j = 0, 1$. Elements of $\mathrm{End}_{\mathbb{K}}(L)_{\bar{j}}$ are homogeneous of degree j.

Definition 2.1. *[9] A Hom-Leibniz superalgebra is a triple $(L, [,], \alpha)$ consisting of a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$, a $\mathbb{K}$-bilinear map $[,] : L \otimes L \to L$ and $\alpha \in \mathrm{End}_{\mathbb{K}}(L)_{\bar{0}}$ satisfying*

$$[L_{\bar{i}}, L_{\bar{j}}] \subseteq L_{\bar{i}+\bar{j}}, \bar{i}, \bar{j} \in \mathbb{Z}_2,$$
$$\alpha([x, y]) = [\alpha(x), \alpha(y)],$$
$$[[x, y], \alpha(z)] = [\alpha(x), [y, z]] - (-1)^{|x||y|}[\alpha(y), [x, z]],$$

for all $x, y, z \in \mathcal{H}(L)$. Furthermore, if α is an algebra automorphism, we call it a regular Hom-Leibniz superalgebra.

This is in fact a right Hom-Leibniz superalgebra. The dual notion of left Hom-Leibniz superalgebra is made out of the dual relation $[\alpha(x), [y, z]] = [[x, y], \alpha(z)] - (-1)^{|y||z|}[[x, z], \alpha(y)]$. In this paper, we are considering only right Hom-Leibniz superalgebras.

Example 2.2. *Clearly, every Hom-Lie superalgebra is a Hom-Leibniz superalgebra. A Hom-Leibniz superalgebra L is a Hom-Lie superalgebra if $[x, y] = -(-1)^{|x||y|}[y, x]$ for all $x, y \in \mathcal{H}(L)$.*

Example 2.3. *[10] Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a $\mathbb{Z}_2$-graded $\mathbb{K}$-vector space, where $L_{\bar{0}}$ is one dimensional subspace of L generated by $\{x\}$ and $L_{\bar{1}}$ is two-dimensional generated by $\{y, z\}$ and the nonzero product is given by $[y, x] = y$. For $a, b \in \mathbb{K}$, we consider $\alpha \in \mathrm{End}_{\mathbb{K}}(L)_{\bar{0}}$ defined $\alpha(x) = ax, \alpha(y) = ay, \alpha(z) = bz$. for any $a \in \mathbb{K}$, there is the corresponding Hom-Leibniz superalgebra $(L, [,]_\alpha = \alpha \circ [,], \alpha)$ with the nonzero product $[y, x]_\alpha = ay$. It is not a Leibniz superalgebra when $a \neq 0, 1$.*

A morphism between Hom-Leibniz superalgebras $(L_1, [,]_1, \alpha_1)$ and $(L_2, [,]_2, \alpha_2)$ is an even linear map $\varphi : L_1 \to L_2$ which satisfies for all $x, y \in \mathcal{H}(L_1)$,

$$\varphi([x, y]_1) = [\varphi(x), \varphi(y)]_2, \quad \varphi \circ \alpha_1 = \alpha_2 \circ \varphi.$$

Definition 2.4. *A Hom-supermodule is a $\mathbb{Z}_2$-graded vector space M together with an even linear map $\beta : M \to M$ such that $\mathbb{Z}_2$-graded vector space operations are compatible with β. We write a Hom-supermodule as (M, β).*

Definition 2.5. *[10] Let $(L, [,], \alpha)$ be a Hom-Leibniz superalgebra. A representation (L-bimodule) is a Hom-supermodule (M, β) together with two L-actions (left and right multiplications), $\mu_l : L \otimes M \to M$ and $\mu_r : M \otimes L \to M$ satisfying the following conditions:*

$$\begin{aligned}
\mu_l(L_{\bar{i}}, M_{\bar{j}}) &\subseteq M_{\bar{i}+\bar{j}},\\
\mu_r(M_{\bar{i}}, L_{\bar{j}}) &\subseteq M_{\bar{i}+\bar{j}},\\
\beta(\mu_l(x, m)) &= \mu_l(\alpha(x), \beta(m)),\\
\beta(\mu_r(m, x)) &= \mu_r(\beta(m), \alpha(x)),\\
\mu_l([x, y], \beta(m)) &= \mu_l(\alpha(x), \mu_l(y, m)) - (-1)^{|x||y|}\mu_l(\alpha(y), \mu_l(x, m)),\\
\mu_r(\mu_l(x, m), \alpha(y)) &= \mu_l(\alpha(x), \mu_r(m, y)) - (-1)^{|x||m|}\mu_r(\beta(m), [x, y]),\\
\mu_r(\mu_r(m, x), \alpha(y)) &= \mu_r(\beta(m), [x, y]) - (-1)^{|m||x|}\mu_l(\alpha(x), \mu_r(m, y)),
\end{aligned}$$

for all $\bar{i}, \bar{j} \in \mathbb{Z}_2, x, y \in \mathcal{H}(L)$ and $m \in \mathcal{H}(M)$.

With the above notations, we recall the cohomology of Hom-Leibniz superalgebras $(L, [,], \alpha)$ defined in [9]. Let

$$C^n_{\alpha,\beta}(L, M) = \{\phi : L^{\otimes n} \to M | \beta \circ \phi = \phi \circ \alpha^{\otimes n}\}.$$

Clearly, $C^n_{\alpha,\beta}(L, M) = C^n_{\alpha,\beta}(L, M)_{\bar{0}} \oplus C^n_{\alpha,\beta}(L, M)_{\bar{1}}$, where $C^n_{\alpha,\beta}(L, M)_{\bar{0}}$ and $C^n_{\alpha,\beta}(L, M)_{\bar{1}}$ are submodules containing elements of degree 0 and 1, respectively. For $n \geq 1$, $\delta^n : C^n_{\alpha,\beta}(L, M) \to C^{n+1}_{\alpha,\beta}(L, M)$ is defined as follows:

$$\begin{aligned}
&\delta^n(\phi)(x_1, x_2, \cdots, x_{n+1}) \\
= &\sum_{i=1}^{n} (-1)^{i+1+|x_i|(|\phi|+|x_1|+\cdots+|x_{i-1}|)} [\alpha^{n-1}(x_i), \phi(x_1, \cdots, \widehat{x_i} \cdots, x_{n+1})] \\
&+ \sum_{1\leq i<j\leq n+1} (-1)^{i+|x_i|(|x_{i+1}|+\cdots+|x_{j-1}|)} \phi(\alpha(x_1), \cdots, \widehat{\alpha(x_i)}, \cdots, [x_i, x_j], \alpha(x_{j+1}), \cdots, \alpha(x_{n+1})) \\
&+(-1)^{n+1}[\phi(x_1, \cdots, x_n), \alpha^{n-1}(x_{n+1})],
\end{aligned}$$

for all $x_1, \cdots, x_{n+1} \in \mathcal{H}(L)$, where $\widehat{x_i}$ means that x_i is omitted. Clearly, for all $n \geq 1$, $\delta \circ \delta = 0$, $(C^*_{\alpha,\beta}(L, M), \delta)$ is a cochain complex. The space of n-cocycles, n-coboundaries, and n-th cohomology are defined as:

(1) $Z^n_{\alpha,\beta}(L, M) = \mathrm{Ker}(\delta^n)$, $Z^n_{\alpha,\beta}(L, M)_{\bar{j}} = C^n_{\alpha,\beta}(L, M)_{\bar{j}} \cap Z^n_{\alpha,\beta}(L, M)$, $j = 0, 1$,

(2) $B^n_{\alpha,\beta}(L, M) = \mathrm{Im}(\delta^{n-1})$, $B^n_{\alpha,\beta}(L, M)_{\bar{j}} = C^n_{\alpha,\beta}(L, M)_{\bar{j}} \cap B^n_{\alpha,\beta}(L, M)$, $j = 0, 1$,

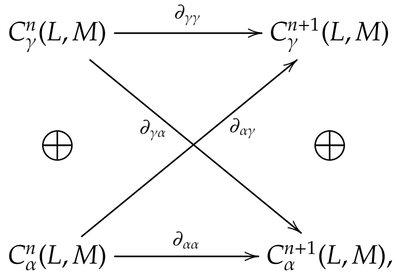
(3) $H^n_{\alpha,\beta}(L, M) = Z^n_{\alpha,\beta}(L, M)/B^n_{\alpha,\beta}(L, M) = H^n_{\alpha,\beta}(L, M)_{\bar{0}} \oplus H^n_{\alpha,\beta}(L, M)_{\bar{1}}$,
where $H^n_{\alpha,\beta}(L, M)_{\bar{j}} = Z^n_{\alpha,\beta}(L, M)_{\bar{j}}/B^n_{\alpha,\beta}(L, M)_{\bar{j}}$, $j = 0, 1$.

3. α-type cohomology for Hom-Leibniz superalgebras

Similarly to [21], we define a cohomology for Hom-Leibniz superalgebras. Let $\gamma(x, y) = [x, y]$ and we define the complex for the cohomology of (L, γ, α) with values in (M, β) is given by

$$\begin{aligned}
\widetilde{C}^n(L, M) &= C^n_\gamma(L, M) \oplus C^n_\alpha(L, M) \\
&= \mathrm{Hom}_{\mathbb{K}}(L^{\otimes n}, M) \oplus \mathrm{Hom}_{\mathbb{K}}(L^{\otimes n-1}, M), \quad for\ all\ n \geq 2. \\
\widetilde{C}^1(L, M) &= C^1_\gamma(L, M) \oplus C^1_\alpha(L, M) = \mathrm{Hom}_{\mathbb{K}}(L, M) \oplus \{0\}. \\
\widetilde{C}^n(L, M) &= \{0\} \quad for\ all\ \ n \leq 0.
\end{aligned}$$

Here $\mathrm{Hom}_{\mathbb{K}}(L^0, M) = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, M) = \{0\}$, instead of $\mathbb{K}$ as usual, otherwise α^{-1} would be needed in the definition of the differential. Clearly, $\widetilde{C}^n(L, M) = \widetilde{C}^n(L, M)_{\bar{0}} \oplus \widetilde{C}^n(L, M)_{\bar{1}}$. We write (ϕ, φ) or $\phi + \varphi$ with $\phi \in C^n_\gamma(L, M)$ and $\varphi \in C^n_\alpha(L, M)$ for an element in $\widetilde{C}^n(L, M)$. We define four maps with domain and range given in the following diagram:



$$
\begin{aligned}
&(\partial_{\gamma\gamma}\phi)(x_1,\cdots,x_{n+1}) \qquad (3.1)\\
=\ & \sum_{i=1}^{n}(-1)^{i+1+|x_i|(|\phi|+|x_1|+\cdots+|x_{i-1}|)}[\alpha^{n-1}(x_i),\phi(x_1,\cdots,\widehat{x_i}\cdots,x_{n+1})]\\
&+\sum_{1\le i<j\le n+1}(-1)^{i+|x_i|(|x_{i+1}|+\cdots+|x_{j-1}|)}\phi(\alpha(x_1),\cdots,\widehat{\alpha(x_i)},\cdots,[x_i,x_j],\alpha(x_{j+1}),\cdots,\alpha(x_{n+1}))\\
&+(-1)^{n+1}[\phi(x_1,\cdots,x_n),\alpha^{n-1}(x_{n+1})],
\end{aligned}
$$

$$
\begin{aligned}
&(\partial_{\alpha\alpha}\varphi)(x_1,\cdots,x_n) \qquad (3.2)\\
=\ & \sum_{i=1}^{n-1}(-1)^{i+1+|x_i|(|\varphi|+|x_1|+\cdots+|x_{i-1}|)}[\alpha^{n-2}(x_i),\varphi(x_1,\cdots,\widehat{x_i}\cdots,x_n)]\\
&+\sum_{1\le i<j\le n}(-1)^{i+|x_i|(|x_{i+1}|+\cdots+|x_{j-1}|)}\varphi(\alpha(x_1),\cdots,\widehat{\alpha(x_i)},\cdots,[x_i,x_j],\alpha(x_{j+1}),\cdots,\alpha(x_n))\\
&+(-1)^{n}[\varphi(x_1,\cdots,x_{n-1}),\alpha^{n-2}(x_n)],
\end{aligned}
$$

$$
(\partial_{\gamma\alpha}\phi)(x_1,\cdots,x_n)=\beta(\phi(x_1,\cdots,x_n))-\phi(\alpha(x_1),\cdots,\alpha(x_n)), \qquad (3.3)
$$

$$
\begin{aligned}
&(\partial_{\alpha\gamma}\varphi)(x_1,\cdots,x_{n+1}) \qquad (3.4)\\
=\ & \sum_{1\le i<j\le n}(-1)^{j+|x_i|(|\varphi|+|x_1|+\cdots+|x_{i-1}|)+|x_j|(|\varphi|+|x_1|+\cdots+|x_{j-1}|)+|x_i||x_j|}[[\alpha^{n-2}(x_i),\alpha^{n-2}(x_j)],\varphi(x_1,\cdots,x_{i-1},\widehat{x_i},x_{i+1},\cdots,\widehat{x_j},\cdots,x_{n+1})]\\
&+\sum_{i=1}^{n}(-1)^{i+|x_i|(|\varphi|+|x_{i+1}|+\cdots+|x_n|)}[\varphi(x_1,\cdots,\widehat{x_i}\cdots,x_n),[\alpha^{n-2}(x_i),\alpha^{n-2}(x_{n+1})]],
\end{aligned}
$$

where $x_1,\cdots,x_{n+1}\in\mathcal{H}(L)$. We set

$$
\partial(\phi,\varphi) \;=\; (\partial_{\gamma\gamma}+\partial_{\gamma\alpha})(\phi)-(\partial_{\alpha\alpha}+\partial_{\alpha\gamma})(\varphi)=(\partial_{\gamma\gamma}\phi-\partial_{\alpha\gamma}\varphi,\partial_{\gamma\alpha}\phi-\partial_{\alpha\alpha}\varphi). \qquad (3.5)
$$

The following theorem is similar to the Hom-Leibniz algebras case [21] and we overleap its proof.

Theorem 3.1. *Let (L,γ,α) be a Hom-Leibniz superalgebra and (M,β) be an L-bimodule. Further, let $\partial:\widetilde{C}^n(L,M)\to\widetilde{C}^{n+1}(L,M)$ be a map defined by (3.5). Then the pair $(\widetilde{C}^*(L,M),\partial)$ is a cochain complex.*

We denote the cohomology of $(\widetilde{C}^*(L,M),\partial)$ by $(\widetilde{H}^*(L,M),\partial)$ and call it an α-type cohomology of (L,γ,α) with values in (M,β).

Remark 3.2. *(1)Note that α-type cohomology for Hom-Leibniz superalgebras generalizes the cohomology introduced in [9]. To show this we consider only those elements in $\widetilde{C}^n(L,M)$ where second summand is zero, that is, $\widetilde{C}^n_\alpha(L,M)=\{0\}$. Thus, we define a subcomplex of $\widetilde{C}^n(L,M)$ as follows:*

$$
C^n_{\alpha,\beta}(L,M) \;=\; \{(\phi,0)\in\widetilde{C}^n(L,M)|\partial_{\gamma\alpha}\phi=0\}=\{\phi\in C^n_\gamma(L,M)|\beta\circ\phi=\phi\circ\alpha^{\otimes n}\}.
$$

The map $\partial_{\gamma\gamma}$ defines a differential on this complex and this complex is same as the complex defined in [9]. Thus, α-type cohomology generalizes the cohomology developed in [9].

(2) Note that any Hom-Leibniz superalgebra $(L,[,],\alpha)$ can be considered as a bimodule over itself by taking $\mu_l=\mu_r=[,],M=L$ and $\beta=\alpha$. At this point, we denote the n-th cohomology $\widetilde{H}^n(L,L)=\widetilde{H}^n(L,M)$.

4. Deformation theory of Hom-Leibniz superalgebras

In this section, we introduce one-parameter formal deformation theory for Hom-Leibniz superalgebras and discuss how an α-type cohomology controls deformations.

Definition 4.1. *A one-parameter formal deformation of Hom-Leibniz superalgebra $(L, [,], \alpha)$ is given by a $\mathbb{K}[[t]]$-bilinear map $m_t : L[[t]] \times L[[t]] \to L[[t]]$ and a $\mathbb{K}[[t]]$-linear map $\alpha_t : L[[t]] \to L[[t]]$ of the forms*

$$m_t(x, y) = \sum_{i \geq 0} m_i(x, y)t^i \text{ and } \alpha_t(z) = \sum_{i \geq 0} \alpha_i(z)t^i,$$

for all $x, y, z \in \mathcal{H}(L)$, such that,

(1) For all $i \geq 0$, $m_i \in \mathrm{Hom}_{\mathbb{K}}(L \times L, L)_{\bar{0}}$ and $\alpha_i \in \mathrm{Hom}_{\mathbb{K}}(L, L)_{\bar{0}}$.
(2) $m_0(x, y) = [x, y]$, $\alpha_0 = \alpha$.
(3) $|m_t(x, y)| = |x| + |y|$, $|\alpha_t(z)| = |z|$.
(4) $m_t(m_t(x, y), \alpha_t(z)) = m_t(\alpha_t(x), m_t(y, z)) - (-1)^{|x||y|}m_t(\alpha_t(y), m_t(x, z))$.
(5) $\alpha_t(m_t(x, y)) = m_t(\alpha_t(x), \alpha_t(y))$.

Remark 4.2. *Equations (4)(5) are equivalent to $(n = 0, 1, 2, \cdots)$*

$$\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} (m_i(m_k(x, y), \alpha_j(z)) - m_i(\alpha_j(x), m_k(y, z)) + (-1)^{|x||y|}m_i(\alpha_j(y), m_k(x, z))) = 0; \tag{4.1}$$

$$\sum_{\substack{i+j=n \\ i,j \geq 0}} \alpha_i(m_j(x, y)) - \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} m_i(\alpha_j(x), \alpha_k(y)) = 0. \tag{4.2}$$

For a Hom-Leibniz superalgebra $(L, [,], \alpha)$, an α_j-associator is a map,

$$\begin{array}{rcl} \circ_{\alpha_j} : \mathrm{Hom}_{\mathbb{K}}(L \times L, L)_{\bar{0}} \times \mathrm{Hom}_{\mathbb{K}}(L \times L, L)_{\bar{0}} & \to & \mathrm{Hom}_{\mathbb{K}}(L \times L \times L, L)_{\bar{0}}, \\ (m_i, m_k) & \mapsto & m_i \circ_{\alpha_j} m_k, \end{array}$$

defined as

$$(m_i \circ_{\alpha_j} m_k)(x, y, z) = m_i(m_k(x, y), \alpha_j(z)) - m_i(\alpha_j(x), m_k(y, z)) + (-1)^{|x||y|}m_i(\alpha_j(y), m_k(x, z)).$$

By using α_j-associator, the equation(4.1) may be written as

$$\sum_{i,j,k \geq 0} (m_i \circ_{\alpha_j} m_k)t^{i+j+k} = 0; \quad \sum_{n \geq 0} (\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} (m_i \circ_{\alpha_j} m_k))t^n = 0.$$

Thus, for $n = 0, 1, 2, \cdots$, we have the following infinite equations:

$$\sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} (m_i \circ_{\alpha_j} m_k) = 0. \tag{4.3}$$

We can rewrite the Equation(4.3) as follows:

$$(\partial_{\gamma\gamma} m_n - \partial_{\alpha\gamma} \alpha_n)(x, y, z) = - \sum_{\substack{i+j+k=n \\ i,j,k > 0}} (m_i \circ_{\alpha_j} m_k)(x, y, z), \tag{4.4}$$

where

$$\begin{aligned} \partial_{\gamma\gamma} m_n(x, y, z) = {} & [m_n(x, y), \alpha(z)] - [\alpha(x), m_n(y, z)] + (-1)^{|x||y|}[\alpha(y), m_n(x, z)] \\ & -m_n(\alpha(x), [y, z]) + m_n([x, y], \alpha(z)) + (-1)^{|x||y|}m_n(\alpha(y), [x, z]), \end{aligned} \tag{4.5}$$

and

$$\partial_{\alpha\gamma}\alpha_n(x,y,z) = -[[x,y],\alpha_n(z)] + [\alpha_n(x),[y,z]] - (-1)^{|x||y|}[\alpha_n(y),[x,z]]. \quad (4.6)$$

From the multiplicativity of α_t, we have

$$\sum_{\substack{i+j=n\\ i,j\geq 0}} \alpha_i(m_j(x,y)) - \sum_{\substack{i+j+k=n\\ i,j,k\geq 0}} m_i(\alpha_j(x),\alpha_k(y)) = 0. \quad (4.7)$$

We can rewrite the Equation (4.17) as follows:

$$(\partial_{\alpha\alpha}\alpha_n - \partial_{\gamma\alpha}m_n)(x,y) = -\sum_{\substack{i+j+k=n\\ i,j,k>0}} m_i(\alpha_j(x),\alpha_k(y)) + \sum_{\substack{i+j=n\\ i,j>0}} \alpha_i(m_j(x,y)), \quad (4.8)$$

where

$$\begin{aligned} \partial_{\alpha\alpha}\alpha_n(x,y) &= [\alpha(x),\alpha_n(y)] + [\alpha_n(x),\alpha(y)] - \alpha_n([x,y]), & (4.9)\\ \partial_{\gamma\alpha}m_n(x,y) &= \alpha m_n(x,y) - m_n(\alpha(x),\alpha(y)). & (4.10)\end{aligned}$$

For $n = 0$,

$$m_0 \circ_{\alpha_0} m_0 = 0, \quad [[x,y],\alpha(z)] - [\alpha(x),[y,z]] + (-1)^{|x||y|}[\alpha(y),[x,z]] = 0.$$

From the Equation (4.7) we have

$$\alpha([x,y]) = [\alpha(x),\alpha(y)].$$

This just shows α is multiplicative.

For $n = 1$, from the Equation (4.3) we have

$$\begin{aligned} & m_0 \circ_{\alpha_0} m_1 + m_1 \circ_{\alpha_0} m_0 + m_0 \circ_{\alpha_1} m_0 = 0,\\ & [m_1(x,y),\alpha(z)] - [\alpha(x),m_1(y,z)] + (-1)^{|x||y|}[\alpha(y),m_1(x,z)]\\ & +m_1([x,y],\alpha(z)) - m_1(\alpha(x),[y,z]) + (-1)^{|x||y|}m_1(\alpha(y),[x,z])\\ & +[[x,y],\alpha_1(z)] - [\alpha_1(x),[y,z]] + (-1)^{|x||y|}[\alpha_1(y),[x,z]] = 0.\end{aligned}$$

This is same as

$$\partial_{\gamma\gamma}m_1(x,y,z) - \partial_{\alpha\gamma}\alpha_1(x,y,z) = 0.$$

Now from the multiplicative part of the deformation, we have

$$[\alpha(x),\alpha_1(y)] + [\alpha_1(x),\alpha(y)] + m_1(\alpha(x),\alpha(y)) - \alpha(m_1(x,y)) - \alpha_1([x,y]) = 0.$$

This is same as

$$\partial_{\alpha\alpha}\alpha_1(x,y) - \partial_{\gamma\alpha}m_1(x,y) = 0.$$

Thus, we have

$$\partial(m_1,\alpha_1) = 0.$$

Definition 4.3. *The infinitesimal of the deformation (m_t,α_t) is the pair (m_1,α_1). Suppose more generally that (m_n,α_n) is the first non-zero term of (m_t,α_t) after (m_0,α_0), such (m_n,α_n) is called a n-infinitesimal of the deformation.*

Therefore, we have the following theorem.

Theorem 4.4. *Let $(L,[,],\alpha)$ be a Hom-Leibniz superalgebra, and (L_t,m_t,α_t) be its one-parameter deformation then the infinitesimal of the deformation is a 2-cocycle of the α-type cohomology.*

Now we discuss obstructions of deformations for Hom-Leibniz superalgebras from the cohomological point of view.

Definition 4.5. *A n-deformation of a Hom-Leibniz superalgebra is a formal deformation of the forms*

$$m_t = \sum_{i=0}^{n} m_i t^i \text{ and } \alpha_t = \sum_{i=0}^{n} \alpha_i t^i,$$

for all $x, y, z \in \mathcal{H}(L)$, such that,

(1) For all $0 \le i \le n$, $m_i \in \mathrm{Hom}_{\mathbb{K}}(L \times L, L)_{\bar{0}}$ and $\alpha_i \in \mathrm{End}_{\mathbb{K}}(L)_{\bar{0}}$.
(2) $m_0(x, y) = [x, y]$, $\alpha_0 = \alpha$.
(3) $|m_t(x, y)| = |x| + |y|$, $|\alpha_t(z)| = |z|$.
(4) $m_t(m_t(x, y), \alpha_t(z)) = m_t(\alpha_t(x), m_t(y, z)) - (-1)^{|x||y|} m_t(\alpha_t(y), m_t(x, z))$.
(5) $\alpha_t(m_t(x, y)) = m_t(\alpha_t(x), \alpha_t(y))$.

We say a n-deformation (m_t, α_t) of a Hom-Leibniz superalgebra is extendable to a $(n + 1)$-deformation if there is an element $m_{n+1} \in C^2_\gamma(L, L)$ and $\alpha_{n+1} \in C^2_\alpha(L, L)$ such that

$$\overline{m_t} = m_t + m_{n+1} t^{n+1}, \qquad \overline{\alpha_t} = \alpha_t + \alpha_{n+1} t^{n+1},$$

and $(\overline{m_t}, \overline{\alpha_t})$ satisfies (1)-(5). The $(n + 1)$-deformation $(\overline{m_t}, \overline{\alpha_t})$ gives us the following equations.

$$\sum_{\substack{i+j+k=n+1 \\ i,j,k \ge 0}} (m_i(m_k(x, y), \alpha_j(z)) - m_i(\alpha_j(x), m_k(y, z)) + (-1)^{|x||y|} m_i(\alpha_j(y), m_k(x, z))) = 0; \tag{4.11}$$

$$\sum_{\substack{i+j+k=n+1 \\ i,j,k \ge 0}} \alpha_i(m_j(x, y)) - \sum_{\substack{i+j+k=n+1 \\ i,j,k \ge 0}} m_i(\alpha_j(x), \alpha_k(y)) = 0. \tag{4.12}$$

This is same as the following equations

$$\begin{aligned} & (\partial_{\gamma\gamma} m_{n+1} - \partial_{\alpha\gamma} \alpha_{n+1})(x, y, z) \\ = & - \sum_{\substack{i+j+k=n+1 \\ i,j,k > 0}} (m_i(m_k(x, y), \alpha_j(z)) - m_i(\alpha_j(x), m_k(y, z)) + (-1)^{|x||y|} m_i(\alpha_j(y), m_k(x, z))) \\ = & - \sum_{\substack{i+j+k=n+1 \\ i,j,k > 0}} (m_i \circ_{\alpha_j} m_k)(x, y, z). \\ & (\partial_{\alpha\alpha} \alpha_{n+1} - \partial_{\gamma\alpha} m_{n+1})(x, y) = - \sum_{\substack{i+j+k=n+1 \\ i,j,k > 0}} m_i(\alpha_j(x), \alpha_k(y)) + \sum_{\substack{i+j=n+1 \\ i,j > 0}} \alpha_i(m_j(x, y)). \end{aligned}$$

We define the n-th obstruction to extend a deformation of Hom-Leibniz superalgebra of order n to order $n + 1$ as $\mathrm{Obs}^n = (\mathrm{Obs}^n_\gamma, \mathrm{Obs}^n_\alpha)$, where

$$\mathrm{Obs}^n_\gamma(x, y, z) := - \sum_{\substack{i+j+k=n+1 \\ i,j,k > 0}} (m_i \circ_{\alpha_j} m_k)(x, y, z) = (\partial_{\gamma\gamma} m_{n+1} - \partial_{\alpha\gamma} \alpha_{n+1})(x, y, z), \tag{4.13}$$

$$\begin{aligned} \mathrm{Obs}^n_\alpha(x, y) : &= - \sum_{\substack{i+j+k=n+1 \\ i,j,k > 0}} m_i(\alpha_j(x), \alpha_k(y)) + \sum_{\substack{i+j=n+1 \\ i,j > 0}} \alpha_i(m_j(x, y)) \\ &= (\partial_{\alpha\alpha} \alpha_{n+1} - \partial_{\gamma\alpha} m_{n+1})(x, y). \end{aligned} \tag{4.14}$$

Thus, $(\mathrm{Obs}^n_\gamma, \mathrm{Obs}^n_\alpha) \in \widetilde{C}^3(L, L)$ and $(\mathrm{Obs}^n_\gamma, \mathrm{Obs}^n_\alpha) \in \partial(m_{n+1}, \alpha_{n+1})$.

Theorem 4.6. *A deformation of order n extends to a deformation of order $n + 1$ if and only if cohomology class of Obs^n vanishes.*

The proof of the above theorem is similar to the Hom-Leibniz algebra case [21].

Proposition 4.7. *If $\widetilde{H}^3(L,L)=0$ then any 2-cocycle gives a one-parameter formal deformation of $(L,[,],\alpha)$.*

Definition 4.8. *Suppose $L_t=(L,m_t,\alpha_t)$ and $L'_t=(L,m'_t,\alpha'_t)$ be two one-parameter Hom-Leibniz superalgebra deformations of $(L,[,],\alpha)$, where $m_t=\sum_{i\geq 0}m_it^i,\alpha_t=\sum_{i\geq 0}\alpha_it^i$ and $m'_t=\sum_{i\geq 0}m'_it^i,\alpha'_t=\sum_{i\geq 0}\alpha'_it^i$. Two deformations L_t and L'_t are said to be equivalent if there exists a $\mathbb{K}[[t]]$-linear isomorphism $\Psi_t:L[[t]]\to L[[t]]$ of the form $\Psi_t=\sum_{i\geq 0}\psi_it^i$, where $\psi_0=\mathrm{Id}$ and $\psi_i:L\to L$ are $\mathbb{K}$-linear maps such that the following relations holds:*

$$|\Psi_t(x)|=|x|,$$

$$\Psi_t(m'_t(x,y))=m_t(\Psi_t(x),\Psi_t(y)), \tag{4.15}$$

$$\alpha_t(\Psi_t(x))=\Psi_t(\alpha'_t(x)), \tag{4.16}$$

for all $x,y\in\mathcal{H}(L)$.

The above equations (4.15) and (4.16) are equivalent to the following equations:

$$\sum_{i\geq 0}\psi_i(\sum_{j\geq 0}m'_j(x,y)t^j)t^i=\sum_{i\geq 0}m_i(\sum_{j\geq 0}\psi_j(x)t^j,\sum_{k\geq 0}\psi_k(y)t^k)t^i, \tag{4.17}$$

$$\sum_{i\geq 0}\alpha_i(\sum_{j\geq 0}\psi_j(x)t^j)t^i=\sum_{i\geq 0}\psi_i(\sum_{j\geq 0}\alpha'_j(x)t^j)t^i. \tag{4.18}$$

Comparing constant terms on both sides of the above equations, we have

$$m'_0(x,y)=m_0(x,y)=[x,y],\ \alpha_0(x)=\alpha'_0(x)=\alpha(x).$$

Now comparing coefficients of t, we have

$$m'_1(x,y)+\psi_1(m'_0(x,y))=m_1(x,y)+m_0(\psi_1(x),y)+m_0(x,\psi(y)), \tag{4.19}$$

$$\alpha_1(x)+\alpha_0(\psi_1(x))=\alpha'_1(x)+\psi_1(\alpha'_0(x)). \tag{4.20}$$

The Equations (4.19) and (4.20) are same as

$$m'_1(x,y)-m_1(x,y)=[\psi_1(x),y]+[x,\psi_1(y)]-\psi_1([x,y])=\partial_{\gamma\gamma}\psi_1(x,y).$$

$$\alpha'_1(x)-\alpha_1(x)=\alpha(\psi_1(x))-\psi_1(\alpha(x))=\partial_{\gamma\alpha}\psi_1(x).$$

Thus, we have the following proposition.

Proposition 4.9. *Two equivalent deformations have cohomologous infinitesimals.*

The proof of the above proposition is similar to the Hom-Leibniz algebra case [21].

Definition 4.10. *A deformation L_t of a Hom-Leibniz superalgebra L is called trivial if L_t is equivalent to L. A Hom-Leibniz superalgebra L is called rigid if it has only trivial deformation upto equivalence.*

Proposition 4.11. *A non-trivial deformation of a Hom-Leibniz superalgebra is equivalent to a deformation whose infinitesimal is not a coboundary.*

Proposition 4.12. *Let $(L,[,],\alpha)$ be a Hom-Leibniz superalgebra. If $\widetilde{H}^2(L,L)=0$ then L is rigid.*

5. Group action and equivariant cohomology

Definition 5.1. *Let G be a finite group and $(L,[,],\alpha)$ be a Hom-Leibniz superalgebra. We say group G acts on the Hom-Leibniz superalgebra L from the left if there is a funtion*

$$\Phi:G\times L\to L,$$

satisfying

(1)For each $g\in G,x\in\mathcal{H}(L)$, the map $\Phi_g=\Phi(g,):L\to L,x\mapsto gx$ is a $\mathbb{K}$-linear map and $|\Phi_g(x)|=|x|$, that is $\Phi_g\in\mathrm{End}_{\mathbb{K}}(L)_{\bar{0}}$.

(2) $ex=x$ for all $x\in\mathcal{H}(L)$, where e denotes identity element of the group G.

(3) $(g_1g_2)x=g_1(g_2(x))$ for all $g_1,g_2\in G$ and $x\in\mathcal{H}(L)$.

(4) For all $g\in G$ and $x,y\in\mathcal{H}(L)$, $g[x,y]=[gx,gy]$ and $\alpha(gx)=g\alpha(x)$.

We denote an action as above by $(G, L, [,], \alpha)$.

Proposition 5.2. *Let G be a finite group and $(L, [,], \alpha)$ be a Hom-Leibniz superalgebra. The group G acts on L from the left if and only if there is a group homomorphism*

$$\Psi : G \to Iso_{\mathrm{HLs}}(L), g \mapsto \Phi_g,$$

where $Iso_{\mathrm{HLs}}(L)$ denotes group of ismorphisms of Hom-Leibniz superalgebras from L to L.

Proof. For an action $(G, L, [,], \alpha)$, we define a map $\Psi : G \to Iso_{\mathrm{HLs}}(L)$ by $\Psi(g) = \Phi_g$. One can verify easily that Ψ is a group homomorphism. Now, let $\Psi : G \to Iso_{\mathrm{HLs}}(L)$ be a group homomorphism. Define a map $\Phi : G \times L \to L$, by $(g, x) \mapsto \Psi(g)(x)$. It can be easily seen that this is an action of G on the Hom-Leibniz superalgebra L. □

Let M, M' be Hom-Leibniz superalgebras equipped with actions of group G. We say an even $\mathbb{K}$-linear map $f : M \to M'$ is equivariant if for all $g \in G$ and $x \in \mathcal{H}(M)$, $f(gx) = gf(x)$. We write the set of all equivariant maps from M to M' as $\mathrm{Hom}^G_{\mathbb{K}}(M, M')$.

Definition 5.3. *A G-Hom-supermodule is a Hom-supermodule (M, β) together with an action of G on M, and $\beta : M \to M$ is an equivariant map. We denote an equivariant Hom-supermodule as triple (G, M, β).*

Definition 5.4. *Let $(G, L, [,], \alpha)$ be a Hom-Leibniz superalgebra equipped with an action of a finite group G. A G-bimodule over L is a G-Hom-supermodule (G, M, β) together with two L-actions (left and right multiplications), $\mu_l : L \otimes M \to M$ and $\mu_l : M \otimes L \to M$ such that μ_l, μ_r satisfying the following conditions:*

$$\begin{aligned}
\mu_l(gL_{\bar{i}}, gM_{\bar{j}}) &\subseteq gM_{\bar{i}+\bar{j}},\\
\mu_r(gM_{\bar{i}}, gL_{\bar{j}}) &\subseteq gM_{\bar{i}+\bar{j}},\\
\mu_l(gx, gm) &= g\mu_l(x, m),\\
\mu_r(gm, gx) &= g\mu_r(m, x),\\
\beta(\mu_l(x, m)) &= \mu_l(\alpha(x), \beta(m)),\\
\beta(\mu_r(m, x)) &= \mu_r(\beta(m), \alpha(x)),\\
\mu_l([x, y], \beta(m)) &= \mu_l(\alpha(x), \mu_l(y, m)) - (-1)^{|x||y|}\mu_l(\alpha(y), \mu_l(x, m)),\\
\mu_r(\mu_l(x, m), \alpha(y)) &= \mu_l(\alpha(x), \mu_r(m, y)) - (-1)^{|x||m|}\mu_r(\beta(m), [x, y]),\\
\mu_r(\mu_r(m, x), \alpha(y)) &= \mu_r(\beta(m), [x, y]) - (-1)^{|m||x|}\mu_l(\alpha(x), \mu_r(m, y)).
\end{aligned}$$

for all $\bar{i}, \bar{j} \in \mathbb{Z}_2, x, y \in \mathcal{H}(L), m \in \mathcal{H}(M)$ and $g \in G$.

We now introduce an equivariant cohomology of Hom-Leibniz superalgebras L equipped with an action of a finite group G.

Set

$$\widetilde{C}^n_G(L, M) : \;=\; \{(c_\gamma, c_\alpha) \in \widetilde{C}^n(L, M) | c_\gamma(gx_1, \cdots, gx_n) = gc_\gamma(x_1, \cdots, x_n), c_\alpha(gx_1, \cdots, gx_{n-1}) = gc_\alpha(x_1, \cdots, x_{n-1})\},$$

for all $x_1, \cdots, x_n \in \mathcal{H}(L)$. Here $\widetilde{C}^n(L, M)$ is n-cochain group of the Hom-Leibniz superalgebra $(L, [,], \alpha)$ and $\widetilde{C}^n_G(L, M)$ consists of all n-cochains which are equivariant. Clearly, $\widetilde{C}^n_G(L, M) = \widetilde{C}^n_G(L, M)_{\bar{0}} \oplus \widetilde{C}^n_G(L, M)_{\bar{1}}$, $\widetilde{C}^n_G(L, M)$ is a submodule of $\widetilde{C}^n(L, M)$ and $(c_\gamma, c_\alpha) \in \widetilde{C}^n_G(L, M)$ is called an invariant n-cochain.

Lemma 5.5. *If an n-cochain (c_γ, c_α) is invariant then $\partial(c_\gamma, c_\alpha)$ is also an invariant $(n + 1)$-cochain.*

Proof. Let $(c_\gamma, c_\alpha) \in \widetilde{C}^n_G(L, M)$ and $g \in G, x_1, \cdots, x_n \in \mathcal{H}(L)$. By definition, we have

$$c_\gamma(gx_1, \cdots, gx_n) \;=\; gc_\gamma(x_1, \cdots, x_n), \quad c_\alpha(gx_1, \cdots, gx_{n-1}) = gc_\alpha(x_1, \cdots, x_{n-1}).$$

It is enough to show that the four differentials $\partial_{\gamma\gamma}, \partial_{\gamma\alpha}, \partial_{\alpha\alpha}, \partial_{\alpha\gamma}$ respect the group action. Observe that

$$\begin{aligned}
&\partial_{\gamma\gamma}(c_\gamma)(gx_1, gx_2, \cdots, gx_{n+1})\\
= &\sum_{i=1}^{n}(-1)^{i+1+|x_i|(|\phi|+|x_1|+\cdots+|x_{i-1}|)}[\alpha^{n-1}(gx_i), c_\gamma(gx_1, \cdots, \widehat{gx_i}\cdots, gx_{n+1})]\\
&+\sum_{1\le i<j\le n+1}(-1)^{i+|x_i|(|x_{i+1}|+\cdots+|x_{j-1}|)}c_\gamma(\alpha(gx_1), \cdots, \widehat{\alpha(gx_i)}, \cdots, [gx_i, gx_j], \alpha(gx_{j+1}), \cdots, \alpha(gx_{n+1}))\\
&+(-1)^{n+1}[c_\gamma(gx_1, \cdots, gx_n), \alpha^{n-1}(gx_{n+1})]\\
= &\sum_{i=1}^{n}(-1)^{i+1+|x_i|(|\phi|+|x_1|+\cdots+|x_{i-1}|)}[g\alpha^{n-1}(x_i), gc_\gamma(x_1, \cdots, \widehat{x_i}\cdots, x_{n+1})]\\
&+\sum_{1\le i<j\le n+1}(-1)^{i+|x_i|(|x_{i+1}|+\cdots+|x_{j-1}|)}gc_\gamma(\alpha(x_1), \cdots, \widehat{\alpha(x_i)}, \cdots, [x_i, x_j], \alpha(x_{j+1}), \cdots, \alpha(x_{n+1}))\\
&+(-1)^{n+1}[gc_\gamma(x_1, \cdots, x_n), g\alpha^{n-1}(x_{n+1})]\\
= &\ g(\partial_{\gamma\gamma}c_\gamma)(x_1, x_2, \cdots, x_{n+1}).
\end{aligned}$$

Similarly, it is easy to show that

$$\begin{aligned}
\partial_{\alpha\alpha}(c_\alpha)(gx_1, gx_2, \cdots, gx_n) &= g(\partial_{\alpha\alpha}c_\alpha)(x_1, x_2, \cdots, x_n),\\
\partial_{\gamma\alpha}(c_\gamma)(gx_1, gx_2, \cdots, gx_n) &= g(\partial_{\gamma\alpha}c_\gamma)(x_1, x_2, \cdots, x_n),\\
\partial_{\alpha\gamma}(c_\alpha)(gx_1, gx_2, \cdots, gx_{n+1}) &= g(\partial_{\alpha\gamma}c_\alpha)(x_1, x_2, \cdots, x_{n+1}).
\end{aligned}$$

Therefore, $\partial(c_\gamma, c_\alpha) \in \widetilde{C}^{n+1}_G(L, M)$. □

The cochain complex $(\widetilde{C}^*_G(L, M), \partial)$ is called an equivariant cochain complex of $(G, L, [,], \alpha)$. We define n-th equivariant cohomology group of $(G, L, [,], \alpha)$ with values in (M, β) is given by

$$\widetilde{H}^n_G(L, M) := H^n(\widetilde{C}^*_G(L, M)).$$

Remark 5.6. *Any Hom-Leibniz superalgebra $(G, L, [,], \alpha)$ equipped with an action of a finite group G is a G-bimodule over itself, that is $M = L, \beta = \alpha$. At this point, we denote the cohomology $\widetilde{H}^n_G(L, L) = \widetilde{H}^n_G(L, M)$.*

6. Equivariant formal deformation of Hom-Leibniz superalgebra

Definition 6.1. *An equivariant one-parameter formal deformation of $(G, L, [,], \alpha)$ is given by a $\mathbb{K}[[t]]$-bilinear map $m_t : L[[t]] \times L[[t]] \to L[[t]]$ and a $\mathbb{K}[[t]]$-linear map $\alpha_t : L[[t]] \to L[[t]]$ of the forms*

$$m_t(x, y) = \sum_{i\ge 0} m_i(x, y)t^i \ \text{ and }\ \alpha_t(z) = \sum_{i\ge 0}\alpha_i(z)t^i,$$

for all $x, y, z \in \mathcal{H}(L)$, such that,

(1) For all $i \ge 0$, $m_i \in \mathrm{Hom}^G_{\mathbb{K}}(L\otimes L, L)_{\bar 0}$ and $\alpha_i \in \mathrm{End}^G_{\mathbb{K}}(L)_{\bar 0}$.
(2) $m_0(x, y) = [x, y]$, $\alpha_0 = \alpha$.
(3) $|m_t(x, y)| = |x| + |y|$, $|\alpha_t(z)| = |z|$.
(4) $m_t(m_t(x, y), \alpha_t(z)) = m_t(\alpha_t(x), m_t(y, z)) - (-1)^{|x||y|}m_t(\alpha_t(y), m_t(x, z))$.
(5) $\alpha_t(m_t(x, y)) = m_t(\alpha_t(x), \alpha_t(y))$.

Remark 6.2. *Equations (4)(5) are equivalent to ($n = 0, 1, 2, \cdots$)*

$$\sum_{\substack{i+j+k=n\\ i,j,k\ge 0}}(m_i(m_k(x, y), \alpha_j(z)) - m_i(\alpha_j(x), m_k(y, z)) + (-1)^{|x||y|}m_i(\alpha_j(y), m_k(x, z))) = 0; \tag{6.1}$$

$$\sum_{\substack{i+j+k=n\\ i,j,k\ge 0}} m_i(\alpha_j(x), \alpha_k(y)) - \sum_{\substack{i+j=n\\ i,j\ge 0}}\alpha_i(m_j(x, y)) = 0. \tag{6.2}$$

Definition 6.3. *An equivariant 2-cochain* (m_1, α_1) *is called an equivariant infinitesimal of the equivariant deformation* (m_t, α_t)*. Suppose more generally that* (m_n, α_n) *is the first non-zero term of* (m_t, α_t) *after* (m_0, α_0)*, such* (m_n, α_n) *is called an equivariant n-infinitesimal of the equivariant deformation.*

Proposition 6.4. *Let G be a finite group and* $(L, [,], \alpha)$ *be a Hom-Leibniz superalgebra, suppose* (G, L_t, m_t, α_t) *is its equivariant one-parameter deformation, then the equivariant infinitesimal of an equivariant deformation is a two-cocycle of the equivariant cohomology.*

Proof Let (m_n, α_n) is an equivariant n-infinitesimal of the equivariant deformation. Thus, for all $1 \leq i \leq n-1, x, y, z \in \mathcal{H}(L)$ $m_i(x, y) = 0, \alpha_i(z) = 0$. By the equation (6.1), have

$$\begin{aligned} & m_0(m_n(x,y),\alpha_0(z)) - m_0(\alpha_0(x), m_n(y,z)) + (-1)^{|x||y|} m_0(\alpha_0(y), m_n(x,z)) + \\ & m_n(m_0(x,y),\alpha_0(z)) - m_n(\alpha_0(x), m_0(y,z)) + (-1)^{|x||y|} m_n(\alpha_0(y), m_0(x,z)) + \\ & m_0(m_0(x,y),\alpha_n(z)) - m_0(\alpha_n(x), m_0(y,z)) + (-1)^{|x||y|} m_0(\alpha_n(y), m_0(x,z)) \\ = \ & (\partial_{\gamma\gamma} m_n - \partial_{\alpha\gamma}\alpha_n)(x,y,z) = 0. \end{aligned}$$

By the equation (6.2), have

$$\begin{aligned} & m_0(\alpha_0(x), \alpha_n(y)) + m_0(\alpha_n(x), \alpha_0(y)) + m_n(\alpha_0(x), \alpha_0(y)) - \alpha_0(m_n(x,y)) - \alpha_n(m_0(x,y)) \\ = \ & (\partial_{\alpha\alpha}\alpha_1 - \partial_{\gamma\alpha} m_1)(x,y) = 0. \end{aligned}$$

Therefore, $\partial^2(m_n, \alpha_n) = 0$. □

Definition 6.5. *An equivariant n-deformation of a Hom-Leibniz superalgebra equipped with a finite group action is a formal deformation of the forms*

$$m_t = \sum_{i=0}^{n} m_i t^i \text{ and } \alpha_t = \sum_{i=0}^{n} \alpha_i t^i,$$

for all $x, y, z \in \mathcal{H}(L)$*, such that,*

(1) For all $0 \leq i \leq n$, $m_i \in \mathrm{Hom}^G_{\mathbb{K}}(L \otimes L, L)_{\bar{0}}$ and $\alpha_i \in \mathrm{End}^G_{\mathbb{K}}(L)_{\bar{0}}$.

(2) $m_0(x, y) = [x, y]$, $\alpha_0 = \alpha$.

(3) $|m_t(x, y)| = |x| + |y|$, $|\alpha_t(z)| = |z|$.

(4) $m_t(m_t(x, y), \alpha_t(z)) = m_t(\alpha_t(x), m_t(y, z)) - (-1)^{|x||y|} m_t(\alpha_t(y), m_t(x, z))$.

(5) $\alpha_t(m_t(x, y)) = m_t(\alpha_t(x), \alpha_t(y))$.

We say an equivariant n-deformation (m_t, α_t) of a Hom-Leibniz superalgebra $(G, L, [,], \alpha)$ is extendable to an equivariant $(n+1)$-deformation if there is an element $(m_{n+1}, \alpha_{n+1}) \in \widetilde{C}^{n+1}_G(L, L)$ such that

$$\begin{aligned} \overline{m_t} &= m_t + m_{n+1} t^{n+1}, \\ \overline{\alpha_t} &= \alpha_t + \alpha_{n+1} t^{n+1}, \end{aligned}$$

and $(\overline{m_t}, \overline{\alpha_t})$ satisfies (1)-(5). The $(n+1)$-deformation $(\overline{m_t}, \overline{\alpha_t})$ gives us the following equations.

$$\sum_{\substack{i+j+k=n+1 \\ i,j,k \geq 0}} (m_i(m_k(x,y), \alpha_j(z)) - m_i(\alpha_j(x), m_k(y,z)) + (-1)^{|x||y|} m_i(\alpha_j(y), m_k(x,z))) = 0; \tag{6.3}$$

$$\sum_{\substack{i+j+k=n+1 \\ i,j,k \geq 0}} \alpha_i(m_j(x,y)) - \sum_{\substack{i+j+k=n+1 \\ i,j,k \geq 0}} m_i(\alpha_j(x), \alpha_k(y)) = 0. \tag{6.4}$$

This is same as the following equations

$$\begin{aligned} & (\partial_{\gamma\gamma} m_{n+1} - \partial_{\alpha\gamma}\alpha_{n+1})(x,y,z) \\ = & -\sum_{\substack{i+j+k=n+1 \\ i,j,k>0}} (m_i(m_k(x,y),\alpha_j(z)) - m_i(\alpha_j(x), m_k(y,z)) + (-1)^{|x||y|} m_i(\alpha_j(y), m_k(x,z))) \\ = & -\sum_{\substack{i+j+k=n+1 \\ i,j,k>0}} (m_i \circ_{\alpha_j} m_k)(x,y,z). \end{aligned}$$

$$(\partial_{\alpha\alpha}\alpha_{n+1} - \partial_{\gamma\alpha} m_{n+1})(x,y) = -\sum_{\substack{i+j+k=n+1 \\ i,j,k>0}} m_i(\alpha_j(x),\alpha_k(y)) + \sum_{\substack{i+j=n+1 \\ i,j>0}} \alpha_i(m_j(x,y)).$$

We define the n-th obstruction to extend a deformation of Hom-Leibniz superalgebra of order n to order $n+1$ as $\mathrm{Obs}_G^n = (\mathrm{Obs}_{G,\gamma}^n, \mathrm{Obs}_{G,\alpha}^n)$, where

$$\mathrm{Obs}_{G,\gamma}^n(x,y,z) := -\sum_{\substack{i+j+k=n+1 \\ i,j,k>0}} (m_i \circ_{\alpha_j} m_k)(x,y,z) = (\partial_{\gamma\gamma} m_{n+1} - \partial_{\alpha\gamma}\alpha_{n+1})(x,y,z), \tag{6.5}$$

$$\begin{aligned} \mathrm{Obs}_{G,\alpha}^n(x,y) := & -\sum_{\substack{i+j+k=n+1 \\ i,j,k>0}} m_i(\alpha_j(x),\alpha_k(y)) + \sum_{\substack{i+j=n+1 \\ i,j>0}} \alpha_i(m_j(x,y)) \\ = & (\partial_{\alpha\alpha}\alpha_{n+1} - \partial_{\gamma\alpha} m_{n+1})(x,y). \end{aligned} \tag{6.6}$$

Lemma 6.6. *Let (m_t, α_t) is an equivariant n-deformations, then for all $n \geq 1$ $Obs_G^n \in \widetilde{C}_G^3(L,L)$ is a cocycle.*

The proof of the above lemma is similar to the Hom-Leibniz algebra case [21].

Similar to the non-equivariant case, we have the following theorem for equivariant deformations.

Theorem 6.7. *An equivariant deformation of order n extends to an equivariant deformation of order $n+1$ if and only if cohomology class of Obs_G^n vanishes.*

Proposition 6.8. *If $\widetilde{H}_G^3(L,L) = 0$ then any equivariant 2-cocycle gives an equivariant one-parameter formal deformation of $(G, L, [,], \alpha)$.*

Definition 6.9. *Let $L_t^G = (G, L, m_t, \alpha_t)$ and $L_t'^G = (G, L, m_t', \alpha_t')$ be two equivariant one-parameter Hom-Leibniz superalgebra deformations of $(G, L, [,], \alpha)$, where $m_t = \sum_{i\geq 0} m_i t^i, \alpha_t = \sum_{i\geq 0} \alpha_i t^i$ and $m_t' = \sum_{i\geq 0} m_i' t^i, \alpha_t' = \sum_{i\geq 0} \alpha_i' t^i$. Two deformations L_t^G and $L_t'^G$ are said to be equivalent if there exists a $\mathbb{K}[[t]]$-linear isomorphism $\Psi_t : L[[t]] \to L[[t]]$ of the form $\Psi_t = \sum_{i\geq 0} \psi_i t^i$, where $\psi_0 = \mathrm{Id}$ and $\psi_i : L \to L$ are equivariant $\mathbb{K}$-linear maps such that the following relations holds:*

$$|\Psi_t(x)| = |x|,$$

$$\Psi_t(m_t'(x,y)) = m_t(\Psi_t(x), \Psi_t(y)), \tag{6.7}$$

$$\alpha_t(\Psi_t(x)) = \Psi_t(\alpha_t'(x)), \tag{6.8}$$

for all $x, y \in \mathcal{H}(L)$.

The above equations (6.7) and (6.8) are equivalent to the following equations:

$$\sum_{i\geq 0} \psi_i(\sum_{j\geq 0} m_j'(x,y)t^j)t^i = \sum_{i\geq 0} m_i(\sum_{j\geq 0} \psi_j(x)t^j, \sum_{k\geq 0} \psi_k(y)t^k)t^i, \tag{6.9}$$

$$\sum_{i\geq 0} \alpha_i(\sum_{j\geq 0} \psi_j(x)t^j)t^i = \sum_{i\geq 0} \psi_i(\sum_{j\geq 0} \alpha_j'(x)t^j)t^i. \tag{6.10}$$

Comparing coefficients of infinitesimals on both sides of the above equations, we have the following proposition.

Proposition 6.10. *Equivariant infinitesimals of two equivalent equivariant deformations determine the same cohomology class.*

Proposition 6.11. *Let* $(G, L, [,], \alpha)$ *be a Hom-Leibniz superalgebra equipped with an action of finite group G. If* $\widetilde{H}^2_G(L, L) = 0$ *then L is equivariantly rigid.*

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