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Bilateral Upper-Left Shifts on Double Sequence Spaces

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Abstract. In this paper, we study the bilateral upper-left shifts $\mathcal{B}$ on the weighted double sequence spaces $\mathcal{L}^p(\mathbb{Z}, v)$ and characterize the hypercyclicity and supercyclicity of $\mathcal{B} : \mathcal{L}^p(\mathbb{Z}, v) \to \mathcal{L}^p(\mathbb{Z}, v)$ based on Salas's previous results about the bilateral weighted backward shifts acting on $l^2(\mathbb{Z})$. Furthermore, we construct a special weight sequence such that $\mathcal{B}$ and the weighted bilateral upper-left shifts $\mathcal{B}_w$ are conjugate, and we generalize the results to $\mathcal{B}_w : \mathcal{L}^p(\mathbb{Z}) \to \mathcal{L}^p(\mathbb{Z})$ via this conjugacy. Finally, we investigate the chaoticity of the bilateral upper-left shifts.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ be the sets of nonnegative integers, all integers, respectively. Let X be an infinite-dimensional separable Banach space and $\mathbb{C}$ the complex plane. We denote by $\mathfrak{L}(X)$ the set of all continuous linear operators on X. The operator $T \in \mathfrak{L}(X)$ is said to be hypercyclic if there is some vector $x \in X$ such that the orbit $Orb(x, T) = \{T^n x;\ n \in \mathbb{N}\}$ is dense in X. Such a vector x is said to be hypercyclic for T, and the set of all hypercyclic vectors for T is denoted by $HC(T)$. A vector $x \in X$ is called supercyclic for T if its projective orbit $\mathbb{K} \cdot Orb(x, T) := \{\lambda T^n(x);\ n \in \mathbb{N},\ \lambda \in \mathbb{K}\}$ is dense in X, and the set of all supercyclic vectors for T is denoted by $SC(T)$. It is well known that $HC(T)$ and $SC(T)$ are dense in X. A periodic point for T is a vector $x \in X$ such that $T^n x = x$ for some $n \in \mathbb{N}$. T is said to be chaotic in the sense of Devaney if T is hypercyclic and there exists a dense set of periodic points for T. For more details in this direction, see for instance the excellent books [3] and [6].

The classical examples of hypercyclic operator are Birkhoff's translation operator acting on the space $H(\mathbb{C})$ of entire functions [4], MacLane's differentiation operator on $H(\mathbb{C})$ [8]. The first example of a hypercyclic backward weighted shift was produced by Rolewicz [10]: on the space $X = l^p$, $1 \le p < \infty$, or $X = c_0$, the multiple of the backward shift λB is hypercyclic whenever $|\lambda| > 1$, where B is the (unilateral) backward shift

$$B(x_0, x_1, x_2, \cdots) = (x_1, x_2, x_3, \cdots).$$

In the present paper, we study the bilateral shifts on the double sequence spaces $\mathcal{L}^p(\mathbb{Z})$, which was introduced by Başar and Sever [2]. Let Ω be the set of all real or complex valued double sequences, $\mathcal{L}^p(\mathbb{Z})$

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is defined as

$$\mathcal{L}^p(\mathbb{Z}) := \Big\{x = (x_{i,j})_{i,j\in\mathbb{Z}} \in \Omega; \sum_{i,j\in\mathbb{Z}} |x_{i,j}|^p < \infty\Big\}, \quad 1 \le p < \infty,$$

normed by

$$\|x\| = \Big(\sum_{i,j\in\mathbb{Z}} |x_{i,j}|^p\Big)^{\frac{1}{p}}, \quad x = (x_{i,j}) \in \mathcal{L}^p(\mathbb{Z}),$$

corresponding to the well-known space $l^p(\mathbb{Z})$ of absolutely p-summable bilateral single sequences. The reader can refer to the recent monographs [1] and [9] on the sequence spaces and summability theory, and the papers [14], [15] and [16] on the domain of certain four dimensional triangle matrices in the spaces of double sequences. By the same technique in [13, Theorem 2.1], it can be shown that $\mathcal{L}^p(\mathbb{Z})$ is a separable Banach space. For more knowledge about double sequence spaces one can consult Hardy [7], Bromwich [5], Başar and Sever [2], Başar [1].

The weighted double sequence spaces $\mathcal{L}^p(\mathbb{Z}, v)$ arose in [13, Section 5]:

$$\mathcal{L}^p(\mathbb{Z}, v) := \Big\{x = (x_{i,j})_{i,j\in\mathbb{Z}} \in \Omega; \sum_{i,j\in\mathbb{Z}} |x_{i,j}|^p v_{i,j}^p < \infty\Big\}, \quad 1 \le p < \infty,$$

where $v = (v_{i,j})_{i,j\in\mathbb{Z}}$ is a positive double sequence. Similarly, $\mathcal{L}^p(\mathbb{Z}, v)$ is a separable Banach space endowed with the norm

$$\|x\| = \Big(\sum_{i,j\in\mathbb{Z}} |x_{i,j}|^p v_{i,j}^p\Big)^{\frac{1}{p}}, \quad x = (x_{i,j}) \in \mathcal{L}^p(\mathbb{Z}, v).$$

We consider the bilateral upper-left shift $\mathcal{B}$ acting on $\mathcal{L}^p(\mathbb{Z}, v)$:

$$\mathcal{B}\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & x_{-1,-1} & x_{-1,0} & x_{-1,1} & \cdots \\ \cdots & x_{0,-1} & x_{0,0} & x_{0,1} & \cdots \\ \cdots & x_{1,-1} & x_{1,0} & x_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} = \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ \cdots & x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ \cdots & x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let $(e_{i,j})_{i,j\in\mathbb{Z}}$ denote the canonical basis of $\mathcal{L}^p(\mathbb{Z}, v)$, i.e., the element of $(e_{i,j})$ in the ith row, jth column is 1, and the other elements are zero. Then $\mathcal{B}$ can also be presented as $\mathcal{B}(e_{i,j}) = (e_{i-1,j-1})$. $\mathcal{B}$ is an continuous operator on $\mathcal{L}^p(\mathbb{Z}, v)$ if and only if there is an $M > 0$ such that for all $x = (x_{i,j}) \in \mathcal{L}^p(\mathbb{Z}, v)$,

$$\Big(\sum_{i,j\in\mathbb{Z}} v_{i,j}^p |x_{i+1,j+1}|^p\Big)^{\frac{1}{p}} \le M\Big(\sum_{i,j\in\mathbb{Z}} v_{i,j}^p |x_{i,j}|^p\Big)^{\frac{1}{p}},$$

which is equivalent to $\sup_{i,j\in\mathbb{Z}} \frac{v_{i,j}}{v_{i+1,j+1}} < \infty$.

In [11] and [12], Salas studied the hypercyclicity and supercyclicity of bilateral backward shifts $B : (\cdots, x_{-1}, x_0, x_1, \cdots) \to (\cdots, x_0, x_1, x_2, \cdots)$ acting on weighted sequence spaces

$$l^2(\mathbb{Z}, \omega) = \Big\{x \in \mathbb{C}^{\mathbb{Z}}; \|x\|^2 := \sum_{n\in\mathbb{Z}} \omega_n^2 |x_n|^2 < \infty\Big\}.$$

Inspired by the Salas's results, it is interesting to explore the hypercyclicity and supercyclicity of the bilateral upper-left shifts $\mathcal{B}$ acting on the weighted double sequence spaces $\mathcal{L}^p(\mathbb{Z}, v)$, and how about the chaoticity? In this paper, we study these problems.

The organization of this paper is as follows. In Section 2 we present some criterions that supply sufficient conditions for an operator to be hypercyclic, supercyclic and mixing, respectively. We also construct a special weight sequence such that $\mathcal{B}$ and the weighted bilateral upper-left shifts $\mathcal{B}_w$ are conjugate. In Section 3 we characterize the hypercyclicity, supercyclicity and mixing property of $\mathcal{B} : \mathcal{L}^p(\mathbb{Z}, v) \to \mathcal{L}^p(\mathbb{Z}, v)$. Finally, we investigate the chaoticity of the bilateral upper-left shifts.

2. Preliminaries

Recall that a continuous linear operator $T : X \to X$ is said to be (topologically) mixing if the following property holds: For any pair (U, V) of nonempty open subsets of X, one can find an $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. $T : X \to X$ is called weakly mixing if $T \times T$ is hypercyclic. The following observation is well known:

mixing $\Rightarrow$ weaking mixing $\Rightarrow$ hypercyclic $\Rightarrow$ supercyclic.

The following three criterions are used frequently to show that an operator is hypercyclic, supercyclic and mixing, respectively.

Proposition 2.1. [6, Theorem 3.12] (Hypercyclicity Criterion). *Let X be a Banach space and $T \in \mathfrak{L}(X)$. If there are dense subsets $\mathcal{D}_1, \mathcal{D}_2 \subset X$, an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, and maps $S_{n_k} : \mathcal{D}_2 \to X$ such that, for any $x \in \mathcal{D}_1$, $y \in \mathcal{D}_2$,*

(i) $T^{n_k}x \to 0$,

(ii) $S_{n_k}y \to 0$,

(iii) $T^{n_k}S_{n_k}y \to y$,

then T is weakly mixing, and in particular hypercyclic.

Proposition 2.2. [3, Theorem 1.14] (Supercyclicity Criterion). *Let X be a Banach space and $T \in \mathfrak{L}(X)$. If there are dense subsets $\mathcal{D}_1, \mathcal{D}_2 \subset X$, an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, and maps $S_{n_k} : \mathcal{D}_2 \to X$ such that, for any $x \in \mathcal{D}_1$, $y \in \mathcal{D}_2$,*

(i) $\|T^{n_k}x\|\|S_{n_k}y\| \to 0$,

(ii) $T^{n_k}S_{n_k}y \to y$,

then T is supercyclic.

Proposition 2.3. [6, Theorem 3.4] (Kitai's Criterion). *Let X be a Banach space and $T \in \mathfrak{L}(X)$. If there are dense subsets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a map $S : \mathcal{D}_2 \to \mathcal{D}_2$ such that, for any $x \in \mathcal{D}_1$, $y \in \mathcal{D}_2$,*

(i) $T^n x \to 0$,

(ii) $S^n y \to 0$,

(iii) $TSy = y$,

then T is mixing.

Let $T : X \to X$ and $S : Y \to Y$ be two continuous maps acting on topological spaces X and Y. Then T is called quasiconjugate to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$, that is, the diagram

$$\begin{array}{ccc} Y & \xrightarrow[S]{} & Y \\ \phi\big\downarrow & & \phi\big\downarrow \\ X & \xrightarrow[T]{} & X \end{array}$$

commutes. If ϕ can be chosen to be a homeomorphism then T and S are called conjugate. It is well known that all the dynamical properties are preserved under conjugacy. Using a suitable conjugacy, we can also consider the weighted bilateral upper-left shifts $B_w(e_{i,j}) = (w_{i,j}e_{i-1,j-1})$ acting on $\mathcal{L}^p(\mathbb{Z})$, where $w = (w_{i,j})_{i,j\in\mathbb{Z}}$ is a bounded sequence of positive real numbers. We define a special weight sequence $(v_{m,n})_{m,n\in\mathbb{Z}}$ by

$$v_{m,n} = \begin{cases} \left(w_{m,n}w_{m-1,n-1}\cdots w_{m-\xi,n-\xi}\right)^{-1}, & m \geq 1, n \geq 1, \\ 1, & m = 0, n \geq 0, \text{ or } n = 0, m \geq 0, \\ w_{m+1,n+1}w_{m+2,n+2}\cdots w_{m+\eta,n+\eta}, & m < 0, \text{ or } n < 0, \end{cases}$$

where both ξ and η are nonnegative integers such that $\min\{m-\xi, n-\xi\} = 1$ and $\min\{m+\eta, n+\eta\} = 0$. Consider the associated weighted double sequence spaces $\mathcal{L}^p(\mathbb{Z}, v)$, and the map

$$\begin{aligned} \phi_v : \mathcal{L}^p(\mathbb{Z}, v) &\to \mathcal{L}^p(\mathbb{Z}) \\ (x_{m,n}) &\mapsto (x_{m,n}v_{m,n}). \end{aligned}$$

We have that ϕ_v is a vector space isomorphism and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}^p(\mathbb{Z}, v) & \xrightarrow[B]{} & \mathcal{L}^p(\mathbb{Z}, v) \\ \phi_v \big\downarrow & & \phi_v \big\downarrow \\ \mathcal{L}^p(\mathbb{Z}) & \xrightarrow[B_w]{} & \mathcal{L}^p(\mathbb{Z}). \end{array}$$

In fact, one can easily verify that

$$\frac{v_{m,n}}{v_{m+1,n+1}} = w_{m+1,n+1}.$$

Then for any $(x_{m,n}) \in \mathcal{L}^p(\mathbb{Z}, v)$,

$$\phi_v \circ B_w(x_{m,n}) = \phi_v(x_{m-1,n-1}) = x_{m-1,n-1}v_{m-1,n-1},$$

and

$$\begin{aligned} B_w \circ \phi_v(x_{m,n}) &= B_w(x_{m,n}v_{m,n}) = w_{m,n}x_{m-1,n-1}v_{m,n} \\ &= \frac{v_{m-1,n-1}}{v_{m,n}}x_{m-1,n-1}v_{m,n} = x_{m-1,n-1}v_{m-1,n-1}. \end{aligned}$$

Thus we have $B_w \circ \phi_v = \phi_v \circ B$.

3. Hypercyclicity and supercyclicity

In this section, we discuss the hypercyclicity and supercyclicity of $\mathcal{B} : \mathcal{L}^p(\mathbb{Z}, v) \to \mathcal{L}^p(\mathbb{Z}, v)$, whose proofs are inspired by that of Salas [11, 12] (see also [3, pp. 18–19]). Meanwhile, we generalize the results to $\mathcal{B}_w : \mathcal{L}^p(\mathbb{Z}) \to \mathcal{L}^p(\mathbb{Z})$ via the conjugacy as in Section 2. The following theorem supplies the equivalent conditions for $\mathcal{B} : \mathcal{L}^p(\mathbb{Z}, v) \to \mathcal{L}^p(\mathbb{Z}, v)$ to be hypercyclic.

Theorem 3.1. *Let $v = (v_{m,n})_{m,n\in\mathbb{Z}}$ be a sequence of positive numbers such that $\sup_{m,n} \frac{v_{m,n}}{v_{m+1,n+1}} < \infty$, and let $\mathcal{B}$ be the bilateral upper-left shift acting on $\mathcal{L}^p(\mathbb{Z}, v)$, $1 \leq p < \infty$. Then the following statements are equivalent:*

(i) $\mathcal{B}$ is hypercyclic;

(ii) $\mathcal{B}$ is weakly mixing;

(iii) There is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that, for any $i, j \in \mathbb{Z}$,

$$\lim_{k\to\infty} v_{n_k+i,n_k+j} = 0 \quad \textit{and} \quad \lim_{k\to\infty} v_{-n_k+i,-n_k+j} = 0. \tag{1}$$

Proof. (i)⇒(iii). Suppose that $\mathcal{B}$ is hypercyclic and fix $i, j \in \mathbb{Z}$. Since the hypercyclic vectors are dense in $\mathcal{L}^p(\mathbb{Z}, v)$. For any $\epsilon \in (0, 1)$, assume without loss of generality that $\epsilon < v_{i,j}$, one may find $x \in \mathcal{L}^p(\mathbb{Z}, v)$ and an integer n_k such that

$$\|x - e_{i,j}\| < \epsilon \quad \text{and} \quad \|\mathcal{B}^{n_k}x - e_{i,j}\| < \epsilon.$$

Looking at the (i, j)th and the $(n_k + i, n_k + j)$th coordinate in the first inequality we have

$$|v_{i,j}(x_{i,j} - 1)| < \epsilon, \tag{2}$$

and

$$|v_{n_k+i,n_k+j}x_{n_k+i,n_k+j}| < \epsilon. \tag{3}$$

Likewise, looking at the (i, j)th and the $(-n_k + i, -n_k + j)$th coordinate in the second inequality, we can obtain

$$|v_{i,j}(x_{n_k+i,n_k+j} - 1)| < \epsilon, \tag{4}$$

and

$$|v_{-n_k+i,-n_k+j}x_{i,j}| < \epsilon. \tag{5}$$

From (4) we have that $|x_{n_k+i,n_k+j}| > \frac{v_{i,j}-\epsilon}{v_{i,j}}$. Applying (3) yields

$$v_{n_k+i,n_k+j} < \frac{v_{i,j}\epsilon}{v_{i,j} - \epsilon}.$$

Similarly, combining (2) with (5) we obtain

$$v_{-n_k+i,-n_k+j} < \frac{v_{i,j}\epsilon}{v_{i,j} - \epsilon}.$$

Since ϵ is arbitrary, these give (1).

(iii)⇒(ii). Suppose that (1)) holds, and let us show that $\mathcal{B}$ satisfies the Hypercyclicity Criterion. Let $\mathcal{D}_1 = \mathcal{D}_2 = \text{span}\{e_{i,j} : i, j \in \mathbb{Z}\}$. For S_{n_k} we take the n_kth iterates of the low-righ shift

$$\mathcal{F}\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & x_{-1,-1} & x_{-1,0} & x_{-1,1} & \cdots \\ \cdots & x_{0,-1} & x_{0,0} & x_{0,1} & \cdots \\ \cdots & x_{1,-1} & x_{1,0} & x_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} = \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & x_{-2,-2} & x_{-2,-1} & x_{-2,0} & \cdots \\ \cdots & x_{-1,-2} & x_{-1,-1} & x_{-1,0} & \cdots \\ \cdots & x_{0,-2} & x_{0,-1} & x_{0,0} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix},$$

that is, $S_{n_k} = \mathcal{F}^{n_k} : \mathcal{D}_2 \to \mathcal{L}^p(\mathbb{Z}, v)$. Since $\mathcal{BF} = I$ on $\mathcal{D}_2$, we only need to show that $\mathcal{B}^{n_k}e_{i,j}$ and $S_{n_k}e_{i,j}$ both tend to zero for any $i, j \in \mathbb{Z}$ (then we conclude by using linearity). But this is clear since

$$\|\mathcal{B}^{n_k}e_{i,j}\| = v_{-n_k+i,-n_k+j} \quad \text{and} \quad \|S_{n_k}e_{i,j}\| = v_{n_k+i,n_k+j}.$$

Thus $\mathcal{B}$ is weakly mixing.

(ii)⇒(i). This is trivial. □

For the bilateral weighted upper-left shift $\mathcal{B}_w$ on the unweighted double sequence spaces $\mathcal{L}^p(\mathbb{Z})$, $(1 \leq p < \infty)$. Using the weight sequence $(v_{m,n})_{m,n\in\mathbb{Z}}$ in Section 2, we can easily get the following corollary.

Corollary 3.2. *Let $\mathcal{B}_w$ be a bilateral weighted upper-left shift on $\mathcal{L}^p(\mathbb{Z})$, $1 \leq p < \infty$, with weighted sequence $w = (w_{i,j})_{i,j\in\mathbb{Z}}$. Then the following statements are equivalent:*

(i) $\mathcal{B}_w$ is hypercyclic;

(ii) $\mathcal{B}_w$ is weakly mixing;

(iii) There is an increasing sequence $(n_k)_{k\in\mathbb{Z}}$ of positive integers such that, for all $i, j \in \mathbb{Z}$,

$$\lim_{k\to\infty} \frac{1}{w_{n_k+i,n_k+j}w_{n_k+i-1,n_k+j-1}\cdots w_{n_k+i-\xi,n_k+j-\xi}} = 0,$$

and

$$\lim_{k\to\infty} w_{-n_k+i+1,-n_k+j+1}w_{-n_k+i+2,-n_k+j+2}\cdots w_{-n_k+i+\eta,-n_k+j+\eta} = 0,$$

where ξ, η are nonnegative integers such that $\min\{n_k + i - \xi, n_k + j - \xi\} = 1$ and $\min\{-n_k + i + \eta, -n_k + j + \eta\} = 0$.

Next we consider the supercyclicity of the bilateral upper-left shift.

Theorem 3.3. *Let $v = (v_{m,n})_{m,n\in\mathbb{Z}}$ be a sequence of positive numbers such that $\sup_{m,n} \frac{v_{m,n}}{v_{m+1,n+1}} < \infty$, and let $\mathcal{B}$ be the bilateral upper-left shift acting on $\mathcal{L}^p(\mathbb{Z}, v)$, $1 \leq p < \infty$. Then $\mathcal{B}$ is supercyclic if and only if there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that for any $i, j \in \mathbb{Z}$,*

$$\lim_{k\to\infty} v_{n_k+i,n_k+j}v_{-n_k+i,-n_k+j} = 0. \tag{6}$$

Proof. Suppose that $\mathcal{B}$ is supercyclic and fix $i, j \in \mathbb{Z}$. Since the supercyclic vectors are dense in $\mathcal{L}^p(\mathbb{Z}, v)$, for any $\epsilon \in (0, 1)$, one may find $x = (x_{i,j}) \in \mathcal{L}^p(\mathbb{Z}, v)$, $\lambda \in \mathbb{K}\backslash\{0\}$ and an integer n_k such that

$$\|x - e_{i,j}\| < \epsilon \quad \text{and} \quad \|\lambda\mathcal{B}^{n_k}x - e_{i,j}\| < \epsilon.$$

As in Theorem 3.1, we can get

$$|v_{i,j}(x_{i,j} - 1)| < \epsilon, \quad |v_{n_k+i,n_k+j}x_{n_k+i,n_k+j}| < \epsilon,$$

from the first inequality, and

$$|v_{i,j}(x_{i,j} - 1)| < \epsilon, \quad |\lambda v_{-n_k+i,-n_k+j}x_{i,j}| < \epsilon,$$

from the second inequality. Putting all this together, using the triangle inequality, we can conclude that

$$v_{n_k+i,n_k+j}v_{-n_k+i,-n_k+j} < \frac{\epsilon^2}{|\lambda||x_{i,j}||x_{n_k+i,n_k+j}|} < \left(\frac{v_{i,j}\epsilon}{v_{i,j} - \epsilon}\right)^2,$$

and (6) follows.

Conversely, if (6) holds, it is clear that the Supercyclicity Criterion is satisfied for $\mathcal{D}_1 = \mathcal{D}_2 = \text{span}\{e_{i,j} : i, j \in \mathbb{Z}\}$ and the nth iterate of the low-right shift, i.e., $S_{n_k} := F^{n_k}$, since

$$\|\mathcal{B}^{n_k}(e_{i,j})\|\|S_{n_k}(e_{i,j})\| = v_{-n_k+i,-n_k+j}v_{n_k+i,n_k+j}.$$

□

Corollary 3.4. *Let $\mathcal{B}_w$ be a bilateral weighted upper-left shift on $\mathcal{L}^p(\mathbb{Z})$, $1 \le p < \infty$, with weight sequence $w = (w_n)_{n\in\mathbb{Z}}$. Then $\mathcal{B}_w$ is supercyclic if and only if there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that for any $i, j \in \mathbb{Z}$,*

$$\liminf_{k\to\infty} \frac{w_{-n_k+i+1,-n_k+j+1} w_{-n_k+i+2,-n_k+j+2} \cdots w_{-n_k+i+\eta,-n_k+j+\eta}}{w_{n_k+i,n_k+j} w_{n_k+i-1,n_k+j-1} \cdots w_{n_k+i-\xi,n_k+j-\xi}} = 0,$$

where ξ, η are nonnegative integers such that $\min\{n_k + i - \xi, n_k + j - \xi\} = 1$ *and* $\min\{-n_k + i + \eta, -n_k + j + \eta\} = 0$.

By the same arguments using in the proof of Theorem 3.1, but employing Kitai's Criterion instead of the Hypercyclicity Criterion, we obtain a characterization of the mixing property for $\mathcal{B}$.

Theorem 3.5. *Let $v = (v_{m,n})_{m,n\in\mathbb{Z}}$ be a sequence of positive numbers such that $\sup_{m,n} \frac{v_{m,n}}{v_{m+1,n+1}} < \infty$, and let $\mathcal{B}$ be the unweighted upper-left shift acting on $\mathcal{L}^p(\mathbb{Z}, v)$, $1 \le p < \infty$. Then $\mathcal{B}$ is mixing if and only if for any $i, j \in \mathbb{Z}$,*

$$\lim_{n\to\infty} v_{n+i,n+j} = 0 \quad \text{and} \quad \lim_{n\to\infty} v_{-n+i,-n+j} = 0.$$

Corollary 3.6. *Let $\mathcal{B}_w$ be a bilateral weighted upper-left shift on $\mathcal{L}^p(\mathbb{Z})$, $1 \le p < \infty$, with weight sequence $w = (w_n)_{n\in\mathbb{Z}}$. Then $\mathcal{B}_w$ is mixing if and only if for any $i, j \in \mathbb{Z}$,*

$$\lim_{n\to\infty} \frac{1}{w_{n+i,n+j} w_{n+i-1,n+j-1} \cdots w_{n+i-\xi,n+j-\xi}} = 0,$$

and

$$\lim_{n\to\infty} w_{-n+i+1,-n+j+1} w_{-n+i+2,-n+j+2} \cdots w_{-n+i+\eta,-n+j+\eta} = 0,$$

where ξ, η are nonnegative integers such that $\min\{n_k + i - \xi, n_k + j - \xi\} = 1$ *and* $\min\{-n_k + i + \eta, -n_k + j + \eta\} = 0$.

4. Chaoticity

In the last section, we deal with the chaoticity of the bilateral upper-left shift. We give a sufficient condition and a necessary condition of $\mathcal{B} : \mathcal{L}^p(\mathbb{Z}, v) \to \mathcal{L}^p(\mathbb{Z}, v)$ to be chaotic.

Theorem 4.1. *Let $v = (v_{m,n})_{m,n\in\mathbb{Z}}$ be a sequence of positive numbers such that $\sup_{m,n} \frac{v_{m,n}}{v_{m+1,n+1}} < \infty$, and let $\mathcal{B}$ be the upper-left shift acting on $\mathcal{L}^p(\mathbb{Z}, v)$, $1 \le p < \infty$. Then $\mathcal{B}$ is chaotic if $\sum_{i,j\in\mathbb{Z}} v_{i,j}^p < \infty$.*

Proof. Assume that $\sum_{i,j\in\mathbb{Z}} v_{i,j}^p < \infty$, it follows from Theorem 3.1 that $\mathcal{B}$ is hypercyclic. It remains to show that $\mathcal{B}$ has a dense set of periodic points. For any $\epsilon > 0$, there exist $K, L, M, N \in \mathbb{N}$ with $K + L = M + N$ such that

$$\sum_{\substack{i\ge K,\ or\ i\le -L \\ j\ge M,\ or\ j\le -N}} v_{i,j}^p < \epsilon.$$

For $x = (x_{i,j})_{i,j\in\mathbb{Z}} \in \mathcal{L}^p(\mathbb{Z}, v)$, we have $\sum_{i,j\in\mathbb{Z}} v_{i,j}^p |x_{i,j}|^p < \infty$. Set

$$y_{i,j} = \begin{cases} x_{i\mp mt, j\mp mt}, & -L \pm mt \le i \le K + mt,\ -N \pm mt \le j \le M + mt, \\ & m = 0, 1, 2, 3, \cdots,\ t = K + L + 1 = M + N + 1. \\ 0, & \text{otherwise.} \end{cases}$$

Then, $y = (y_{i,j})$ belongs to the space $\mathcal{L}^p(\mathbb{Z}, v)$, since

$$\begin{aligned}\|y\|_p^p = \sum_{i,j\in\mathbb{Z}} v_{i,j}^p |y_{i,j}|^p &\le \sum_{\substack{-L\le i\le K,\\ -N\le j\le M}} v_{i,j}^p |x_{i,j}|^p + \sum_{\substack{i\ge K,\ or\ i\le -L\\ j\ge M,\ or\ j\le -N}} v_{i,j}^p |y_{i,j}|^p \\ &\le \|x\|_p^p + \Big(\sup_{\substack{-L\le i\le K,\\ -N\le j\le M}} |x_{i,j}|^p\Big)\cdot \sum_{\substack{i\ge K,\ or\ i\le -L\\ j\ge M,\ or\ j\le -N}} v_{i,j}^p \\ &\le \|x\|_p^p + \Big(\sup_{\substack{-L\le i\le K,\\ -N\le j\le M}} |x_{i,j}|^p\Big)\epsilon < \infty.\end{aligned}$$

Clearly, $\mathcal{B}^t y = y$. Therefore, we have that y is a periodic point whose period is t. Now we need to show that the set $A := \{y : y = (y_{i,j})\}$ is dense in $\mathcal{L}^p(\mathbb{Z}, v)$,

$$\begin{aligned}\|x-y\|_p^p &= \sum_{\substack{i\ge K,\ or\ i\le -L\\ j\ge M,\ or\ j\le -N}} v_{i,j}^p |x_{i,j} - y_{i,j}|^p \\ &\le \Big(\sup_{\substack{i\ge K,\ or\ i\le -L\\ j\ge M,\ or\ j\le -N}} |x_{i,j} - y_{i,j}|^p\Big) \sum_{\substack{i\ge K,\ or\ i\le -L\\ j\ge M,\ or\ j\le -N}} v_{i,j}^p \\ &\le \Big(\sup_{\substack{i\ge K,\ or\ i\le -L\\ j\ge M,\ or\ j\le -N}} |x_{i,j} - y_{i,j}|^p\Big)\epsilon.\end{aligned}$$

Therefore $\mathcal{B}$ is chaotic. □

Theorem 4.2. *Let $v = (v_{m,n})_{m,n\in\mathbb{Z}}$ be a sequence of positive numbers such that $\sup_{m,n} \frac{v_{m,n}}{v_{m+1,n+1}} < \infty$, and let B be the upper-left shift acting on $\mathcal{L}^p(\mathbb{Z}, v)$, $1 \le p < \infty$. If B is chaotic, then*

$$\sum_{i\in\mathbb{Z}} v_{i,i+m}^p < \infty, \quad m \in \mathbb{Z}.$$

Proof. Assume that $\mathcal{B}$ is chaotic, then $\mathcal{B}$ has a dense set of periodic points. Let $x = (x_{i,j})_{i,j\in\mathbb{Z}}$ be periodic for $\mathcal{B}$ with period t such that $x_{n,n} \neq 0$ for some $-t \le n \le t$, and we have that $x_{n,n} = x_{n+lt,n+lt}$ for $l \in \mathbb{Z}$. Then

$$\sum_{l\in\mathbb{Z}} v_{n+lt,n+lt}^p |x_{n,n}|^p \le \sum_{i,j\in\mathbb{Z}} v_{i,j}^p |x_{i,j}|^p < \infty.$$

It follows that $\sum_{l\in\mathbb{Z}} v_{n+lt,n+lt}^p < \infty$, that is to say

$$\sum_{l\in\mathbb{Z}} e_{n+lt,n+lt} \in \mathcal{L}^p(\mathbb{Z}, v).$$

Applying the upper-left shift $t-1$ times and adding the results we obtain

$$\sum_{i\in\mathbb{Z}} e_{i,i} \in \mathcal{L}^p(\mathbb{Z}, v).$$

In the same way, we can get

$$\sum_{i\in\mathbb{Z}} e_{i,i+m} \in \mathcal{L}^p(\mathbb{Z}, v), \quad m \in \mathbb{Z},$$

that is to say,

$$\sum_{i,j\in\mathbb{Z}} e_{i,j} \in \mathcal{L}^p(\mathbb{Z}, v).$$

Equivalently, $\sum_{i\in\mathbb{Z}} v_{i,j}^p < \infty$. $\square$

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