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Fractional Ostrowski Type Inequalities for Interval Valued Functions

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Abstract. In this paper, we establish some generalization of Ostrowski type inequalities for interval valued functions by using the definitions of the gH-derivatives. At the end, a briefly conclusion is given as well.

1. Introduction and Preliminaries

In [15], the following Ostrowski classical integral inequality associated with the differentiable mappings was given.

Theorem 1.1. *Let $\phi : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a differentiable mapping on $\varkappa_1, \varkappa_2)$ whose derivative $\phi' : (\varkappa_1, \varkappa_2) \to \mathbb{R}$ is bounded on $(\varkappa_1, \varkappa_2)$, i.e., $\left\|\phi'\right\|_\infty = \sup\limits_{\tau \in (\varkappa_1, \varkappa_2)} \left|\phi'(t)\right| < \infty$. Then, the inequality holds:*

$$\left|\phi(\xi) - \frac{1}{b - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \phi(\tau) d\tau\right| \leq \left[\frac{1}{4} + \frac{\left(\xi - \frac{\varkappa_1 + \varkappa_2}{2}\right)^2}{(\varkappa_2 - \varkappa_1)^2}\right] (\varkappa_2 - \varkappa_1) \left\|\phi'\right\|_\infty \tag{1}$$

for all $\xi \in [\varkappa_1, \varkappa_2]$. The constant $\frac{1}{4}$ is the best possible.

In recent years, various generalizations, extensions and variants of inequality (1) have been obtained, see [1, 2, 4, 5, 7, 9, 11, 14, 18–21, 24, 25].

We recall now some basic definitions that will be used in sequel.

Let R be the one-dimensional Euclidean space. Let $\mathcal{K}_C$ denote the family of all bounded closed intervals of R, that is,

$$\mathcal{K}_C = \left\{[\varkappa_1, \varkappa_2] \,\middle|\, \varkappa_1, \varkappa_2 \in R \text{ and } \varkappa_1 \leq \varkappa_2\right\}.$$

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The space $(\mathcal{K}_C, d_H)$, where d_H is the Pompei–Hausdorff metric is given by

$$d_H\left([\varkappa_1, \varkappa_2], \left[\varkappa_1', \varkappa_2'\right]\right) = \max\left\{|\varkappa_1 - \varkappa_2|, \left|\varkappa_1' - \varkappa_2'\right|\right\}$$

for all $[\varkappa_1, \varkappa_2], \left[\varkappa_1', \varkappa_2'\right] \in K_C$, is a complete metric space. A defined by $\|A\|= d_H(A, [0, 0])$ for all $A \in \mathcal{K}_C$ is quasinorm $\|.\|$ in $\mathcal{K}_C$. The equality $\left\|A^2\right\| = \|A\|^2$ holds for all $A \in \mathcal{K}_C$. Stefani and Bede introduced the concept of generalized Hukuhara difference of two sets $A, B \in \mathcal{K}_C$ (gH difference for short) as follows:

$$A \ominus_{gH} B = C \Longleftrightarrow \left\{ \begin{array}{c} \text{(a) } A = B + C \\ \text{or (b) } B = A + (-1)C \end{array} \right\}.$$

In case (a), the gH difference coincide with the H difference. Thus, the gH difference is a generalization of the H difference. On the other hand, gH difference exists for any two intervals $A = \left[\underline{\varkappa_1}, \overline{\varkappa_1}\right]$, $B = \left[\underline{\varkappa_2}, \overline{\varkappa_2}\right]$ $\in \mathcal{K}_C$ and

$$A \ominus_{gH} B = \left[\min\left\{\underline{\varkappa_1} - \underline{\varkappa_2}\right\}, \max\left\{\overline{\varkappa_1} - \overline{\varkappa_2}\right\}\right].$$

Using the gH difference, Stefanini and Bede introduced a differentiability concept for interval valued functions, which is more suitable than the H-differentiability.
The following definitions and theorems with respect to H derivative and gH derivative were referred in [22].

Definition 1.2. *$F : T \subseteq \mathbb{R} \to \mathcal{K}_C$ given by $\Phi(x) = \left[\underline{\Phi}(x), \overline{\Phi}(x)\right]$ for all $x \in X$, where $\underline{\Phi}, \overline{\Phi} : T \to \mathbb{R}$ are real valued functions, with $\underline{\Phi}(\xi) \leq \overline{\Phi}(\xi)$ for all $\xi \in T$, it is called an interval function.*

The functions $\underline{\Phi}$ and $\overline{\Phi}$ are called the lower and the upper functions of Φ, respectively.

Definition 1.3. *Let $\Phi : T \to \mathcal{K}_C$ be an interval valued function. $L \in \mathcal{K}_C$ is called a limit of F at $\xi_0 \in T$ if for every $\epsilon > 0$ there exists $\delta(\epsilon, \xi_0) = \delta > 0$ such that $H(\Phi(\xi), L) < \epsilon$ for all $\xi \in T$ with $0 < |\xi - \xi_0| < \delta$. This is denoted by $\lim_{\xi \to \xi_0} \Phi(\xi) = L$.*

Theorem 1.4. *Let $\Phi : T \to \mathcal{K}_C$ be an interval valued function such that $\Phi(\xi) = \left[\underline{\Phi}(\xi), \overline{\Phi}(\xi)\right]$ for all $\xi \in T$. Then $L = [L_1, L_2] \in \mathcal{K}_C$ is a limit of Φ at $\xi_0 \in T$ if and only if L_i is the limit of ϕ_i at ξ_0, $i \in \{1, 2\}$. Besides, if L is limit of Φ at ξ_0, then*

$$\lim_{\xi \to \xi_0} \Phi(\xi) = \left[\lim_{\xi \to \xi_0} \underline{\Phi}(\xi), \lim_{\xi \to \xi_0} \overline{\Phi}(\xi)\right].$$

Definition 1.5. *Let $\Phi : T \subseteq \mathbb{R} \to \mathcal{K}_C$ be an interval valued function. Φ is said to be continuous at $\xi_0 \in T$, if $\lim_{\xi \to \xi_0} \Phi(\xi) = \Phi(\xi_0)$.*

Theorem 1.6. *Let $\Phi : T \subseteq \mathbb{R} \to \mathcal{K}_C$ be an interval valued function such that $\Phi(\xi) = \left[\underline{\Phi}(\xi), \overline{\Phi}(\xi)\right]$ for all $x \in T$. Then Φ is continuous at $\xi_0 \in T$ if and only if $\underline{\Phi}$ and $\overline{\Phi}$ are continuous at ξ_0. Besides, Φ is continuous at ξ_0, then*

$$\lim_{\xi \to \xi_0} \Phi(\xi) = \left[\underline{\Phi}(\xi_0), \overline{\Phi}(\xi_0)\right].$$

Definition 1.7. *Let $\Phi : T \to \mathcal{K}_C$ be an interval valued function. We say that F is H-differentiable at $\xi_0 \in T$, if there exists an element $F_H'(\xi_0) \in \mathcal{K}_C$ such that the limits*

$$\lim_{h \to 0^+} \frac{\Phi(\xi_0 + h) - \Phi_H(\xi_0)}{h}$$

and

$$\lim_{h\to 0^+}\frac{\Phi(\xi_0)-\Phi_H(\xi_0-h)}{h}$$

exists and are equal to $\Phi_H^{'}(\xi_0)$ *. In this case* $\Phi_H^{'}(\xi_0)$ *is called the* $H-$*derivative of* Φ *at* ξ_0*.*

Definition 1.8. *The gH-derivative of an interval-valued function* $\Phi : T \to K_C$ *at* $\xi_0 \in T$ *is defined as*

$$\Phi_{gH}^{'}(\xi_0)=\lim_{h\to 0}\frac{\Phi(\xi_0+h)\ominus\Phi_{gH}(\xi_0-h)}{h}. \tag{2}$$

If $\Phi_{gH}^{'}(\xi_0)\in K_C$ satisfying is differentiable, then (2) exist and we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at ξ_0.

Theorem 1.9. *Let* $\Phi : T \to \mathcal{K}_C$ *be an interval valued function such that* $\Phi(\xi)=\left[\underline{\Phi}(\xi),\overline{\Phi}(\xi)\right]$ *for all* $\xi\in T$*. Then* Φ *is gH-differentiable at* $\xi_0\in T$ *if and only if one of the following cases holds:*

(i) *f and g are differentiable at* ξ_0 *and*

$$\Phi_{gH}^{'}(\xi_0)=\left[\min\left\{\underline{\Phi}'(\xi_0),\overline{\Phi}'(\xi_0)\right\},\max\left\{\underline{\Phi}'(\xi_0),\overline{\Phi}'(\xi_0)\right\}\right].$$

(ii) $\underline{\Phi}'_-(\xi_0)$, $\overline{\Phi}'_-(\xi_0)$, $\underline{\Phi}'_+(\xi_0)$ *and* $\overline{\Phi}'_+(\xi_0)$ *exists and satisfies* $\underline{\Phi}'_-(\xi_0)=\overline{\Phi}'_+(\xi_0)$ *and* $\overline{\Phi}'_-(\xi_0)=\underline{\Phi}'_+(\xi_0)$ *. Moreover*

$$\begin{aligned}\Phi_{gH}^{'}(\xi_0) &= \left[\min\left\{\underline{\Phi}'_-(\xi_0),\overline{\Phi}'_-(\xi_0)\right\},\max\left\{\underline{\Phi}'_-(\xi_0),\overline{\Phi}'_-(\xi_0)\right\}\right]\\ &= \left[\min\left\{\underline{\Phi}'_+(\xi_0),\overline{\Phi}'_+(\xi_0)\right\},\max\left\{\underline{\Phi}'_+(\xi_0),\overline{\Phi}'_+(\xi_0)\right\}\right].\end{aligned}$$

Theorem 1.10. *Let* $\Phi : [\varkappa_1,\varkappa_2] \to \mathcal{K}_C$ *be a continuous interval valued function with* $\Phi(\xi)=\left[\underline{\Phi}(\xi),\overline{\Phi}(\xi)\right]$ *for all* $\xi\in[\varkappa_1,\varkappa_2]$ *. If* Φ *is piecewise continuously gH differentiable on* $[\varkappa_1,\varkappa_2]$ *and it has (if there exists) a finite number of switching points on* $(\varkappa_1,\varkappa_2)$*, then* $\underline{\Phi}$ *and* $\overline{\Phi}$ *are absolutely continuous on* $[\varkappa_1,\varkappa_2]$ *.*

Definition 1.11. *A partition of* $[\varkappa_1,\varkappa_2]$ *is any finite ordered subset P having the form*

$$P:\varkappa_1=\tau_0<\tau_1<\ldots<\tau_n=\varkappa_2.$$

The mesh of a partition P is defined by

$$mesh(P)=\max\{\tau_i-\tau_{i-1}: i=1,2,\ldots,n\}.$$

We denote by $P([\varkappa_1,\varkappa_2])$ the set of all partition of $[\varkappa_1,\varkappa_2]$. Let $P(\delta,[\varkappa_1,\varkappa_2])$ be the set of all $P\in P([\varkappa_1,\varkappa_2])$ such that $mesh(P)<\delta$. Choosing an arbitrary point ξ_i in interval $[\tau_{i-1},\tau_i]$, for all $i=1,2,\ldots,n$, we define the sum

$$S(\Phi,P,\delta)=\sum_{i=1}^{n}\Phi(\xi_i)[\tau_i-\tau_{i-1}],$$

where $\Phi : [\varkappa_1,\varkappa_2]\to\mathbb{R}_{\mathcal{I}}$. We call $S(\Phi,P,\delta)$ a Riemann sum of Φ corresponding to $P\in P(\delta,[\varkappa_1,\varkappa_2])$.

Definition 1.12. *[6, 16, 17] A function* $\Phi : [\varkappa_1,\varkappa_2]\to\mathbb{R}_{\mathcal{I}}$ *is called interval Riemann integrable (IR-integrable) on* $[\varkappa_1,\varkappa_2]$ *, if there exists* $A\in\mathbb{R}_I$ *such that, for each* $\varepsilon>0$*, there exists* $\delta>0$*, where*

$$d(S(\Phi,P,\delta),A)<\varepsilon$$

for every Riemann sum S of Φ corresponding to each $P \in P(\delta, [\varkappa_1, \varkappa_2])$ and independent of choice $\xi_i \in [\tau_{i-1}, \tau_i]$ for $1 \leq i \leq n$. In this case, A is called the IR-integral of Φ on $[\varkappa_1, \varkappa_2]$ and is denoted by

$$A = (IR) \int_{\varkappa_1}^{\varkappa_1} \Phi(\tau) d\tau.$$

The collection of all functions that are IR-integrable on $[\varkappa_1, \varkappa_2]$ will be denote by $\mathcal{IR}_{([\varkappa_1, \varkappa_2])}$.

The following theorem gives relation between *IR*-integrable and Riemann integrable (*R*-integrable).

Theorem 1.13. *Let $\Phi : [\varkappa_1, \varkappa_2] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $\Phi(\tau = \left[\underline{\Phi}(\tau), \overline{\Phi}(\tau)\right]$. $\Phi \in \mathcal{IR}_{([\varkappa_1, \varkappa_2])}$ if and only if $\underline{\Phi}(\tau), \overline{\Phi}(\tau) \in \mathcal{R}_{([\varkappa_1, \varkappa_2])}$ and*

$$(IR) \int_{\varkappa_1}^{\varkappa_2} \Phi(\tau) d\tau = \left[(R) \int_{\varkappa_1}^{\varkappa_2} \underline{\Phi}(\tau) d\tau, \, (R) \int_{\varkappa_1}^{\varkappa_2} \overline{\Phi}(\tau) d\tau \right],$$

where $\mathcal{R}_{([\varkappa_1, \varkappa_2])}$ denotes the set of all R-integrable functions.

Now, we recall the Riemann–Lioville integrals as follows:

Definition 1.14. *[10] Let $\phi \in L_1[\varkappa_1, \varkappa_2]$. The Riemann–Liouville integrals $J^{\alpha}_{\varkappa_1+}\phi$ and $J^{\alpha}_{\varkappa_2-}\phi$ of order $\alpha > 0$ with $\varkappa_1 \geq 0$ are defined by*

$$\phi^{\alpha}_{\varkappa_1+}(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa_1}^{\xi} (\xi - \tau)^{\alpha-1} \phi(\tau) d\tau, \quad \xi > \varkappa_1$$

and

$$J^{\alpha}_{\varkappa_2-}\phi(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\xi}^{\varkappa_2} (\tau - \xi)^{\alpha-1} \phi(\tau) d\tau, \quad \xi < \varkappa_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{\varkappa_1+}\phi(\xi) = J^0_{\varkappa_2-}\phi(\xi) = \phi(\xi)$.

For more information about Riemann–Liouville integrals, see [8, 10, 13].

By considering Riemann–Liouville integral for real valued functions, Lupulescu in [12], defined the following interval-valued left-sided Riemann–Liouville fractional integral:

Definition 1.15. *Let $\Phi : [\varkappa_1, \varkappa_2] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $\Phi(\tau) = \left[\underline{\Phi}(\tau), \overline{\Phi}(\tau)\right]$ and let $\alpha > 0$. The interval-valued left-sided Riemann–Liouville fractional integral of function F is defined by*

$$\mathcal{J}^{\alpha}_{\varkappa_1+}\Phi(\xi) = \frac{1}{\Gamma(\alpha)} (IR) \int_{\varkappa_1}^{\xi} (\xi - \tau)^{\alpha-1} \Phi(\tau) d\tau, \quad \xi > \varkappa_1,$$

where Γ is Euler Gamma function.

Based on the definition of Lupulescu, Budak et al. in [3], define the corresponding interval-valued right-sided Riemann–Liouville fractional integral of function F by

$$\mathcal{J}^{\alpha}_{\varkappa_2-}\Phi(\xi) = \frac{1}{\Gamma(\alpha)} (IR) \int_{\xi}^{\varkappa_2} (\tau - \xi)^{\alpha-1} \Phi(\tau) d\tau, \quad \xi < \varkappa_2.$$

Theorem 1.16. *If $\Phi : [\varkappa_1, \varkappa_2] \to \mathbb{R}_I$ is an interval-valued function such that $\Phi(\tau) = \left[\underline{\Phi}(\tau), \overline{\Phi}(\tau)\right]$, then we have*

$$\mathcal{J}^{\alpha}_{\varkappa_1+}\Phi(\xi) = \left[I^{\alpha}_{\varkappa_1+}\underline{\Phi}(\xi), I^{\alpha}_{\varkappa_1+}\overline{\Phi}(\xi)\right]$$

and

$$\mathcal{J}^{\alpha}_{\varkappa_2-}\Phi(\xi) = \left[I^{\alpha}_{\varkappa_2-}\underline{\Phi}(\xi), I^{\alpha}_{\varkappa_2-}\overline{\Phi}(\xi)\right].$$

On the other hand Tunç in [23], define the following generalized fractional integrals of interval-valued function:

Definition 1.17. *Let $p : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be an increasing and positive function on $(\varkappa_1, \varkappa_2]$, having a continuous derivative $p'(x)$ on $(\varkappa_1, \varkappa_2)$ and $\Phi \in \mathcal{IR}_{([\varkappa_1,\varkappa_2])}$. The interval- valued left-sided and right-sided fractional integrals of F with respect to the function p on $[\varkappa_1, \varkappa_2]$ of order $\alpha > 0$ are defined by*

$$\mathcal{J}^{\alpha}_{\varkappa_1+;p}\Phi(\xi) = \frac{1}{\Gamma(\alpha)}(IR)\int_{\varkappa_1}^{\xi} \frac{p'(\tau)}{[p(\xi) - p(\tau)]^{1-\alpha}}\Phi(\tau)d\tau, \quad \xi > \varkappa_1$$

and

$$\mathcal{J}^{\alpha}_{\varkappa_2-;p}\Phi(\xi) = \frac{1}{\Gamma(\alpha)}(IR)\int_{\xi}^{\varkappa_2} \frac{p'(\tau)}{[p(\tau) - p(\xi)]^{1-\alpha}}\Phi(\tau)d\tau, \quad \xi < \varkappa_2,$$

respectively.

Corollary 1.18. *If $\Phi : [\varkappa_1, \varkappa_2] \to \mathbb{R}_I$ is an interval-valued function such that $\Phi(\tau) = \left[\underline{\Phi}(\tau), \overline{\Phi}(\tau)\right]$ with $\underline{\Phi}(\tau), \overline{\Phi}(\tau) \in \mathcal{R}_{([\varkappa_1,\varkappa_2])}$ and $p : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be an increasing and positive function on $(\varkappa_1, \varkappa_2]$, having a continuous derivative $p'(\xi)$ on $(\varkappa_1, \varkappa_2)$, then we obtain the following relations*

$$\mathcal{J}^{\alpha}_{\varkappa_1+;p}\Phi(\xi) = \left[\mathcal{I}^{\alpha}_{\varkappa_1+;p}\underline{\Phi}(\xi), \mathcal{I}^{\alpha}_{\varkappa_1+;p}\overline{\Phi}(\xi)\right]$$

and

$$\mathcal{J}^{\alpha}_{\varkappa_2-;p}\Phi(\xi) = \left[\mathcal{I}^{\alpha}_{\varkappa_2-;p}\underline{\Phi}(\xi), \mathcal{I}^{\alpha}_{\varkappa_2-;p}\overline{\Phi}(\xi)\right].$$

In [10], the generalized Riemann–Liouville fractional operators, $\mathcal{I}^{\alpha}_{\varkappa_1+;p}\phi$ and $\mathcal{I}^{\alpha}_{\varkappa_2-;p}\phi$ are defined by

$$\mathcal{I}^{\alpha}_{\varkappa_1+;p}\phi(\xi) = \frac{1}{\Gamma(\alpha)}(R)\int_{\varkappa_1}^{\xi} \frac{p'(\tau)}{[p(\xi) - p(\tau)]^{1-\alpha}}\phi(\tau)d\tau, \quad \xi > \varkappa_1$$

and

$$\mathcal{I}^{\alpha}_{\varkappa_2-;p}\phi(\xi) = \frac{1}{\Gamma(\alpha)}(R)\int_{\xi}^{\varkappa_2} \frac{p'(\tau)}{[p(\tau) - p(\xi)]^{1-\alpha}}\phi(\tau)d\tau, \quad \xi < \varkappa_2,$$

respectively.

Motivated by above notions and results, we will establish in the next section some generalization of Ostrowski type inequalities for interval valued functions by using the definitions of the *gH*-derivatives.

2. Fractional Ostrowski Type Inequalities For Interval-Valued Functions

Sarikaya et al. in [19], obtain the following Ostrowski inequality for fractional integrals:

Theorem 2.1. *Let $\phi : [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a differentiable mapping on $(\varkappa_1, \varkappa_2)$ with $\varkappa_1 < \varkappa_2$ and $\phi' \in L[\varkappa_1, \varkappa_2]$. If $\phi' : (\varkappa_1, \varkappa_2) \to \mathbb{R}$ is bounded on $(\varkappa_1, \varkappa_2)$, i.e., $\left\|\phi'\right\|_\infty = \sup_{\tau \in (\varkappa_1, \varkappa_2)} \left|\phi'(t)\right| < \infty$, then for $\alpha > 0$, the following inequality for fractional integrals holds:*

$$\left|\left(\frac{(\xi - \varkappa_1)^\alpha + (\varkappa_2 - \xi)^\alpha}{\varkappa_2 - \varkappa_1}\right)\phi(\xi) - \frac{\Gamma(\alpha+1)}{\varkappa_2 - \varkappa_1}\left[I^\alpha_{\xi+}\phi(\varkappa_2) + I^\alpha_{\xi-}\phi(\varkappa_1)\right]\right| \tag{3}$$

$$\leq \frac{\left\|\phi'\right\|_\infty}{\varkappa_2 - \varkappa_1}\left[\frac{(\xi - \varkappa_1)^{\alpha+1} + (\varkappa_2 - \xi)^{\alpha+1}}{\alpha+1}\right].$$

Farid in [7], obtain the following Ostrowski inequality for fractional integrals:

Theorem 2.2. *Assume that the conditions of the Theorem 2.1 are satisfied. Then for $\alpha, \beta > 0$, the following inequality for fractional integrals holds:*

$$\left|\left((\xi - \varkappa_1)^\alpha + (\varkappa_2 - \xi)^\beta\right)\phi(\xi) - \left[\Gamma(\beta+1) I^\beta_{\varkappa_2-}\phi(\xi) + \Gamma(\alpha+1) I^\alpha_{\varkappa_1+}\phi(\xi)\right]\right| \tag{4}$$

$$\leq \left\|\phi'\right\|_\infty\left[\frac{\beta}{\beta+1}(\varkappa_2 - \xi)^{\beta+1} + \frac{\alpha}{\alpha+1}(\xi - \varkappa_1)^{\alpha+1}\right].$$

Basci and Baleanu in [2], proved the following Ostrowski type inequalities for generalized Riemann–Liouville fractional integrals:

Theorem 2.3. *Assume that the conditions of the Theorem 2.1 are satisfied. Also, suppose that the function $p \in C^1([\varkappa_1, \varkappa_2])$ is increasing and positive and $p'(x) \geq 1$ for all $x \in (\varkappa_1, \varkappa_2)$. Then for $\alpha, \beta > 0$, the following inequality for fractional integrals holds:*

$$\left|\left((p(\xi) - p(\varkappa_1))^\alpha + (p(\varkappa_2) - p(\xi))^\beta\right)\phi(\xi)\right. \tag{5}$$

$$\left. - \left[\Gamma(\beta+1)\mathcal{I}^\beta_{b-;p}\phi(\xi) + \Gamma(\alpha+1)\mathcal{I}^\alpha_{\varkappa_1+;p}\phi(\xi)\right]\right|$$

$$\leq \left\|\phi'\right\|_\infty\left[\frac{\beta}{\beta+1}(p(\varkappa_2) - p(\xi))^{\beta+1} + \frac{\alpha}{\alpha+1}(p(\xi) - p(\varkappa_1))^{\alpha+1}\right].$$

Now, using Theorems 2.1, 2.2 and 2.3, we prove some fractional Ostrowski type inequalities for interval-valued functions.

Theorem 2.4. *Let $\Phi : [a, b] \to \mathcal{K}_C$ be an interval-valued function such that $\underline{\Phi}$, $\overline{\Phi}$ are continuously differentiable functions. Then, Φ is continuously gH-differentiable and we have the following fractional Ostrowski type inequality for interval-valued functions:*

$$H\left(\frac{\Gamma(\alpha+1)}{(\varkappa_2 - \varkappa_1)^\alpha}\left[\mathcal{J}^\alpha_{\xi+}\Phi(\varkappa_2) + \mathcal{J}^\alpha_{\xi-}\Phi(\varkappa_1)\right], \left(\frac{(\xi - \varkappa_1)^\alpha + (\varkappa_2 - \xi)^\alpha}{\varkappa_2 - \varkappa_1}\right)\Phi(\xi)\right)$$

$$\leq \|\Phi'\|_\infty\left[\frac{(\xi - \varkappa_1)^{\alpha+1} + (\varkappa_2 - \xi)^{\alpha+1}}{(\alpha+1)(\varkappa_2 - \varkappa_1)}\right] \tag{6}$$

for all $\xi \in [\varkappa_1, \varkappa_2]$ and $\alpha > 0$.

Proof. By the fractional Ostrowski inequality (3), we have

$$\begin{aligned}
& H\left(\frac{\Gamma(\alpha+1)}{(\varkappa_2-\varkappa_1)^{\alpha}}\left[\mathcal{J}_{\xi+}^{\alpha}\Phi(\varkappa_2)+\mathcal{J}_{\xi-}^{\alpha}\Phi(\varkappa_1)\right],\left(\frac{(\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\alpha}}{\varkappa_2-\varkappa_1}\right)\Phi(\xi)\right) \\
= & H\left(\frac{\Gamma(\alpha+1)}{(\varkappa_2-\varkappa_1)^{\alpha}}\left[I_{\xi+}^{\alpha}\underline{\Phi}(\varkappa_2)+I_{\xi-}^{\alpha}\underline{\Phi}(\varkappa_1),I_{\xi+}^{\alpha}\overline{\Phi}(\varkappa_2)+I_{\xi-}^{\alpha}\overline{\Phi}(\varkappa_1)\right],\left(\frac{(\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\alpha}}{\varkappa_2-\varkappa_1}\right)\left[\underline{\Phi}(\xi),\overline{\Phi}(\xi)\right]\right) \\
= & \max\left\{\left|\frac{\Gamma(\alpha+1)}{(\varkappa_2-\varkappa_1)^{\alpha}}\left[I_{\xi+}^{\alpha}\underline{\Phi}(\varkappa_2)+I_{\xi-}^{\alpha}\underline{\Phi}(\varkappa_1)\right]-\left(\frac{(\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\alpha}}{\varkappa_2-\varkappa_1}\right)\underline{\Phi}(\xi)\right|,\right. \\
& \left.\left|\frac{\Gamma(\alpha+1)}{(\varkappa_2-\varkappa_1)^{\alpha}}\left[I_{\xi+}^{\alpha}\overline{\Phi}(\varkappa_2)+I_{\xi-}^{\alpha}\overline{\Phi}(\varkappa_1)\right]-\left(\frac{(\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\alpha}}{\varkappa_2-\varkappa_1}\right)\overline{\Phi}(\xi)\right|\right\} \\
\leq & \max\left\{\frac{\left\|\left(\underline{\Phi}\right)'\right\|}{\varkappa_2-\varkappa_1}\left(\frac{(\xi-\varkappa_1)^{\alpha+1}+(\varkappa_2-\xi)^{\alpha+1}}{\alpha+1}\right),\frac{\left\|\left(\overline{\Phi}\right)'\right\|}{\varkappa_2-\varkappa_1}\left(\frac{(\xi-\varkappa_1)^{\alpha+1}+(\varkappa_2-\xi)^{\alpha+1}}{\alpha+1}\right)\right\} \\
= & \left(\frac{(\xi-\varkappa_1)^{\alpha+1}+(\varkappa_2-\xi)^{\alpha+1}}{(\varkappa_2-\varkappa_1)(\alpha+1)}\right)\max\left\{\left\|\left(\underline{\Phi}\right)'\right\|,\left\|\left(\overline{\Phi}\right)'\right\|\right\} \\
= & \frac{\|\Phi'\|_{\infty}}{\varkappa_2-\varkappa_1}\left(\frac{(\xi-\varkappa_1)^{\alpha+1}+(\varkappa_2-\xi)^{\alpha+1}}{\alpha+1}\right).
\end{aligned}$$

The proof of Theorem 2.4 is completed. □

Remark 2.5. *If we choose $\alpha = 1$ in Theorem 2.4, then we have the following inequality*

$$H\left(\frac{1}{\varkappa_2-\varkappa_1}(IR)\int_{\varkappa_1}^{\varkappa_2}\Phi(\tau)d\tau,\ \Phi(\xi)\right)\leq\frac{\|\Phi'\|_{\infty}}{\varkappa_2-\varkappa_1}\left(\frac{(\xi-\varkappa_1)^2+(\varkappa_2-\xi)^2}{2}\right),$$

which is proved by Chalco-Cano et al. in [4].

Theorem 2.6. *Assume that the conditions of the Theorem 2.4 are satisfied. Then we have the following fractional Ostrowski type inequality for interval-valued functions:*

$$\begin{aligned}
& H\left(\Gamma(\beta+1)\mathcal{J}_{\varkappa_2-}^{\beta}\Phi(\xi)+\Gamma(\alpha+1)\mathcal{J}_{\varkappa_1+}^{\alpha}\Phi(\xi),\left((\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\beta}\right)\Phi(\xi)\right) \\
\leq & \|\Phi'\|_{\infty}\left[\frac{\beta}{\beta+1}(\varkappa_2-\xi)^{\beta+1}+\frac{\alpha}{\alpha+1}(\xi-\varkappa_1)^{\alpha+1}\right]
\end{aligned} \tag{7}$$

for all $\xi \in [\varkappa_1, \varkappa_2]$ and $\alpha, \beta > 0$.

Proof. By the fractional Ostrowski inequality (4), we have

$$\begin{aligned}
& H\left(\Gamma(\beta+1)\mathcal{J}_{\varkappa_2-}^{\beta}\Phi(\xi)+\Gamma(\alpha+1)\mathcal{J}_{\varkappa_1+}^{\alpha}\Phi(\xi),\left((\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\beta}\right)\Phi(\xi)\right) \\
= & H\left(\begin{array}{c}\Gamma(\beta+1)\left[I_{\varkappa_2-}^{\beta}\underline{\Phi}(\xi),I_{\varkappa_2-}^{\beta}\overline{\Phi}(\xi)\right]+\Gamma(\alpha+1)\left[I_{\varkappa_1+}^{\alpha}\underline{\Phi}(\xi),I_{\varkappa_1+}^{\alpha}\overline{\Phi}(\xi)\right], \\ \left((\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\beta}\right)\left[\underline{\Phi}(\xi),\overline{\Phi}(\xi)\right]\end{array}\right) \\
= & H\left(\begin{array}{c}\left[\Gamma(\beta+1)I_{\varkappa_2-}^{\beta}\underline{\Phi}(\xi)+\Gamma(\alpha+1)I_{\varkappa_1+}^{\alpha}\underline{\Phi}(\xi),\Gamma(\beta+1)I_{\varkappa_2-}^{\beta}\overline{\Phi}(\xi)+\Gamma(\alpha+1)I_{\varkappa_1+}^{\alpha}\overline{\Phi}(\xi)\right], \\ \left((\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\beta}\right)\left[\underline{\Phi}(\xi),\overline{\Phi}(\xi)\right]\end{array}\right)
\end{aligned}$$

$$= \max\left\{\begin{array}{c}\left|\Gamma(\beta+1)\,I^{\beta}_{\varkappa_2-}\underline{\Phi}(\xi)+\Gamma(\alpha+1)I^{\alpha}_{\varkappa_1+}\underline{\Phi}(\xi)-\left((\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\beta}\right)\underline{\Phi}(\xi)\right|,\\ \left|\Gamma(\beta+1)\,I^{\beta}_{\varkappa_2-}\overline{F}(\xi)+\Gamma(\alpha+1)I^{\alpha}_{\varkappa_1+}\overline{\Phi}(\xi)-\left((\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\beta}\right)\overline{\Phi}(\xi)\right|\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c}\left\|\left(\underline{\Phi}\right)'\right\|\left[\frac{\beta}{\beta+1}(\varkappa_2-\xi)^{\beta+1}+\frac{\alpha}{\alpha+1}(\xi-\varkappa_1)^{\alpha+1}\right],\\ \left\|\left(\overline{\Phi}\right)'\right\|\left[\frac{\beta}{\beta+1}(\varkappa_2-\xi)^{\beta+1}+\frac{\alpha}{\alpha+1}(\xi-\varkappa_1)^{\alpha+1}\right]\end{array}\right\}$$

$$= \left[\frac{\beta}{\beta+1}(\varkappa_2-\xi)^{\beta+1}+\frac{\alpha}{\alpha+1}(\xi-\varkappa_1)^{\alpha+1}\right]\max\left\{\left\|\left(\underline{\Phi}\right)'\right\|,\left\|\left(\overline{\Phi}\right)'\right\|\right\}$$

$$= \|\Phi'\|_{\infty}\left[\frac{\beta}{\beta+1}(\varkappa_2-\xi)^{\beta+1}+\frac{\alpha}{\alpha+1}(\xi-\varkappa_1)^{\alpha+1}\right].$$

The proof of Theorem 2.6 is completed. □

Corollary 2.7. *If we choose $\alpha=\beta$ in Theorem 2.6, then we have the following fractional Ostrowski type inequalities for interval-valued functions:*

$$H\left(\Gamma(\alpha+1)\left[\mathcal{J}^{\alpha}_{\varkappa_2-}\Phi(\xi)+\mathcal{J}^{\alpha}_{\varkappa_1+}\Phi(\xi)\right],((\xi-\varkappa_1)^{\alpha}+(\varkappa_2-\xi)^{\alpha})\,\Phi(\xi)\right)$$

$$\leq \|\Phi'\|_{\infty}\frac{\alpha}{\alpha+1}\left[(\varkappa_2-\xi)^{\alpha+1}+(\xi-\varkappa_1)^{\alpha+1}\right]$$

for all $x\in[\varkappa_1,\varkappa_2]$ and $\alpha>0$.

Theorem 2.8. *Assume that the conditions of the Theorem 2.4 are satisfied. Also, suppose that the function $p\in C^1([\varkappa_1,\varkappa_2])$ is increasing and positive and $p'(\xi)\geq 1$ for all $x\in(\varkappa_1,\varkappa_2)$. Then we have the following generalized fractional Ostrowski type inequality for interval-valued functions:*

$$H\left(\Gamma(\beta+1)\,\mathcal{J}^{\beta}_{\varkappa_2-;p}\Phi(\xi)+\Gamma(\alpha+1)\,\mathcal{J}^{\alpha}_{\varkappa_1+;p}\Phi(\xi),\left((p(\xi)-p(\varkappa_1))^{\alpha}+(p(\varkappa_2)-p(\xi))^{\beta}\right)\Phi(\xi)\right)$$

$$\leq \|\Phi'\|_{\infty}\left[\frac{\beta}{\beta+1}(p(\varkappa_2)-p(\xi))^{\beta+1}+\frac{\alpha}{\alpha+1}(p(\xi)-p(\varkappa_1))^{\alpha+1}\right]$$

for all $\xi\in[\varkappa_1,\varkappa_2]$ and $\alpha,\beta>0$.

Proof. By the fractional Ostrowski inequality (5) and Corollary 1.18, we have

$$H\left(\Gamma(\beta+1)\,\mathcal{J}^{\beta}_{\varkappa_2-;p}\Phi(\xi)+\Gamma(\alpha+1)\,\mathcal{J}^{\alpha}_{\varkappa_1+;p}\Phi(\xi),\left((p(\xi)-p(\varkappa_1))^{\alpha}+(p(\varkappa_2)-p(\xi))^{\beta}\right)\Phi(\xi)\right)$$

$$= H\left(\begin{array}{c}\Gamma(\beta+1)\left[\mathcal{I}^{\beta}_{\varkappa_2-;p}\underline{\Phi}(\xi),\mathcal{I}^{\beta}_{\varkappa_2-;p}\overline{\Phi}(\xi)\right]+\Gamma(\alpha+1)\left[\mathcal{I}^{\alpha}_{\varkappa_1+;p}\underline{\Phi}(\xi),\mathcal{I}^{\alpha}_{\varkappa_1+;p}\overline{\Phi}(\xi)\right],\\ \left((p(\xi)-p(\varkappa_1))^{\alpha}+(p(\varkappa_2)-p(\xi))^{\beta}\right)\left[\underline{\Phi}(\xi),\overline{\Phi}(\xi)\right]\end{array}\right)$$

$$= H\left(\begin{array}{c}\left[\Gamma(\beta+1)\,\mathcal{I}^{\beta}_{\varkappa_2-;p}\underline{\Phi}(\xi)+\Gamma(\alpha+1)\mathcal{I}^{\alpha}_{\varkappa_1+;p}\underline{\Phi}(\xi),\Gamma(\beta+1)\,\mathcal{I}^{\beta}_{\varkappa_2-;p}\overline{\Phi}(\xi)+\Gamma(\alpha+1)\mathcal{I}^{\alpha}_{\varkappa_1+;p}\overline{\Phi}(\xi)\right],\\ \left((p(\xi)-p(\varkappa_1))^{\alpha}+(p(\varkappa_2)-p(\xi))^{\beta}\right)\left[\underline{\Phi}(\xi),\overline{\Phi}(\xi)\right]\end{array}\right)$$

$$= \max\left\{\begin{array}{c}\left|\Gamma(\beta+1)\,\mathcal{I}^{\beta}_{\varkappa_2-;p}\underline{\Phi}(\xi)+\Gamma(\alpha+1)\mathcal{I}^{\alpha}_{\varkappa_1+;p}\underline{\Phi}(\xi)-\left((p(\xi)-p(\varkappa_1))^{\alpha}+(p(\varkappa_2)-p(\xi))^{\beta}\right)\underline{\Phi}(\xi)\right|,\\ \left|\Gamma(\beta+1)\,\mathcal{I}^{\beta}_{\varkappa_2-;p}\overline{\Phi}(\xi)+\Gamma(\alpha+1)\mathcal{I}^{\alpha}_{\varkappa_1+;p}\overline{\Phi}(\xi)-\left((p(\xi)-p(\varkappa_1))^{\alpha}+(p(\varkappa_2)-p(\xi))^{\beta}\right)\overline{\Phi}(\xi)\right|\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c}\left\|\left(\underline{\Phi}\right)'\right\|\left[\frac{\beta}{\beta+1}(p(\varkappa_2)-p(\xi))^{\beta+1}+\frac{\alpha}{\alpha+1}(p(\xi)-p(\varkappa_1))^{\alpha+1}\right],\\ \left\|\left(\overline{\Phi}\right)'\right\|\left[\frac{\beta}{\beta+1}(p(\varkappa_2)-p(\xi))^{\beta+1}+\frac{\alpha}{\alpha+1}(p(\xi)-p(\varkappa_1))^{\alpha+1}\right]\end{array}\right\}$$

$$= \left[\frac{\beta}{\beta+1}\left(p(\varkappa_2)-p(\xi)\right)^{\beta+1}+\frac{\alpha}{\alpha+1}\left(p(\xi)-p(\varkappa_1)\right)^{\alpha+1}\right]\max\left\{\left\|\left(\underline{\Phi}\right)'\right\|,\left\|\left(\overline{\Phi}\right)'\right\|\right\}$$

$$= \|\Phi'\|_\infty\left[\frac{\beta}{\beta+1}\left(p(\varkappa_2)-p(\xi)\right)^{\beta+1}+\frac{\alpha}{\alpha+1}\left(p(\xi)-p(\varkappa_1)\right)^{\alpha+1}\right].$$

The proof of Theorem 2.8 is completed. □

Remark 2.9. *If we choose $p(\xi)=\xi$ for all $\xi\in[\varkappa_1,\varkappa_2]$ in Theorem 2.8, then Theorem 2.8 reduces to Theorem 2.6.*

Corollary 2.10. *If we choose $\alpha=\beta$ in Theorem 2.8, then we have the following fractional Ostrowski type inequality for interval-valued functions:*

$$H\left(\Gamma(\alpha+1)\left[\mathcal{J}^{\alpha}_{\varkappa_2-;p}\Phi(\xi)+\mathcal{J}^{\alpha}_{\varkappa_1+;p}\Phi(\xi)\right],\left(\left(p(\xi)-p(\varkappa_1)\right)^{\alpha}+\left(p(\varkappa_2)-p(\xi)\right)^{\alpha}\right)\Phi(x)\right)$$

$$\leq \|\Phi'\|_\infty\frac{\alpha}{\alpha+1}\left[\left(p(\varkappa_2)-p(\xi)\right)^{\alpha+1}+\left(p(\xi)-p(\varkappa_1)\right)^{\alpha+1}\right]$$

for all $\xi\in[\varkappa_1,\varkappa_2]$ and $\alpha>0$.

Remark 2.11. *If we choose $p(\xi)=\xi$ for all $\xi\in[\varkappa_1,\varkappa_2]$ in Corollary 2.10, then Corollary 2.10 reduces to Corollary 2.7.*

3. Conclusion

In this paper, we given some new fractional Ostrowski type inequalities for interval valued functions by using the definitions of the gH-derivatives. Interested readers can establish new inequalities via other generalized operators using our technique. Also, this results can be applied in convex analysis, optimization and different areas of pure and applied sciences.

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