Filomat 36:8 (2022), 2519–2529
https://doi.org/10.2298/FIL2208519A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Spectral Problems for Sturm–Liouville Operator with Eigenparameter Boundary Conditions on Time Scales

İbrahim Adalar[a]

[a]*Zara Veysel Dursun Colleges of Applied Sciences, Sivas Cumhuriyet University Zara/Sivas, Turkey*

Abstract. In this paper, we consider the inverse problem for Sturm-Liouville operators with eigenparameter dependent boundary conditions on time scales. We give new uniqueness theorems and investigate its some special cases.

1. Introduction

Inverse spectral problems consist in recovering the coefficients of an operator from their spectral characteristics. Inverse Sturm-Liouville problems which appear in mathematical physics, mechanics, electronics, geophysics and other branches of natural sciences (see [7, 17, 21]). The first results on inverse theory of classical Sturm-Liouville operator were given by Ambarzumyan and Borg [4, 15].

The half inverse Sturm–Liouville problem which is one of the important subjects of the inverse spectral theory has been studied firstly by Hochstadt and Lieberman [25]. Since then, this result has been generalized to various versions. Some new uniqueness results in inverse spectral analysis with partial information on the potential for some classes of differential equations have been given (see [16, 22, 26, 35]). On the other hand, the inverse problem for interior spectral data of the differential operator consists in reconstruction of this operator from the given eigenvalues and some information on eigenfunctions at an internal point. This kind of problems for the Sturm Liouville operator were studied firstly by Mochizuki and Trooshin [29]. Similar results to Mochizuki and Trooshin have been studied in various papers until today [22, 38, 39]. In classical analysis, Sturm-Liouville problems with boundary conditions which depend on the parameter were studied extensively (see [9, 10, 33, 36, 37]). These kinds of problems appear in physics, mechanics and engineering. For Sturm-Liouville problem with eigenparameter-dependent-boundary conditions on arbitrary time scale we refer to the study [3, 30, 34] and the references therein.

Although the literature for inverse Sturm–Liouville problems on a continuous interval is vast, there is only a few studies about this subject on time scales [1, 14, 28, 31, 32]. Such problems is useful in many applied problems, for example in string theory, in dynamics of population, in spatial networks problems etc.

In the present paper, Sturm–Liouville dynamic equation with boundary conditions depending on the spectral parameter on a time scale is studied. We define an operator which is appropriate to this boundary value problem. We consider a half inverse Sturm–Liouville problem on a time scale and gave a Hochstadt

2020 *Mathematics Subject Classification*. Primary 31B20; Secondary 39A12, 34B24, 34N05
Keywords. Inverse problem; Sturm-Liouville operator; Time scale
Received: 09 December 2020; Accepted: 12 February 2022
Communicated by Dragan S. Djordjević
Email address: iadalar@cumhuriyet.edu.tr (İbrahim Adalar)

and Lieberman-type theorem. We also investigate some special cases of our results and give a Mochizuki and Trooshin-type theorem.

Time scale theory was introduced by Hilger in order to unify continuous and discrete analysis [24]. From then on this approach has received a lot of attention and has applied quickly to various area in mathematics. Sturm-Liouville theory on time scales was studied first by Erbe and Hilger [20] in 1993. Some important results on the properties of eigenvalues and eigenfunctions of a Sturm-Liouville problem on time scales were given in various publications (see e.g. [2, 5, 19, 23, 27] and the references therein).

Before presenting our first result, we recall the some important concepts of the time scale theory.

If $\mathbb{T}$ is a closed subset of $\mathbb{R}$ it called as a time scale. The jump operators σ, ρ and graininess operator on $\mathbb{T}$ are defined as follows:

$$\begin{aligned}
&\sigma : \mathbb{T} \to \mathbb{T},\ \sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ if } t \neq \sup\mathbb{T},\\
&\rho : \mathbb{T} \to \mathbb{T},\ \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ if } t \neq \inf\mathbb{T},\\
&\sigma(\sup\mathbb{T}) = \sup\mathbb{T},\ \rho(\inf\mathbb{T}) = \inf\mathbb{T},\\
&\mu : \mathbb{T} \to [0,\infty)\ \mu(t) = \sigma(t) - t.
\end{aligned}$$

A point of $\mathbb{T}$ is called as left-dense, left-scattered, right-dense, right-scattered and isolated if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$ and $\rho(t) < t < \sigma(t)$, respectively.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous on $\mathbb{T}$ if it is continuous at all right-dense points and has left-sided limits at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$ or C_{rd}. If f is continuous on $\mathbb{T}$ it is also rd-continuous.

Put $\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} - \{\sup\mathbb{T}\}, & \sup\mathbb{T} \text{ is left-scattered} \\ \mathbb{T}, & \text{the other cases} \end{cases}$, $\mathbb{T}^{\kappa^2} := (\mathbb{T}^{\kappa})^{\kappa}$.

Let $t \in \mathbb{T}^{\kappa}$. Suppose that for given any $\varepsilon > 0$, there exists a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon\,|\sigma(t) - s|$$

for all $s \in U$, then f is called delta-differentiable at $t \in \mathbb{T}^{\kappa}$. We call $f^{\Delta}(t)$ the delta derivative of f at t. The set

$$C^n_{rd}(\mathbb{T}) := \{f : f \text{ rd-continuously } n\text{-order delta-differentiable on } \mathbb{T}\}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ defined as $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$ is called an antiderivative of f on $\mathbb{T}$. In this case the Δ-integral of f is defined by

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Put $\mathbb{T}_{\kappa} := \begin{cases} \mathbb{T} - \{\inf\mathbb{T}\}, & \inf\mathbb{T} \text{ is right-scattered} \\ \mathbb{T}, & \text{the other cases} \end{cases}$, $\mathbb{T}_{\kappa^2} := (\mathbb{T}_{\kappa})_{\kappa}$.

Let $t \in \mathbb{T}_{\kappa}$. Suppose that for given any $\varepsilon > 0$, there exists a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ such that

$$\left|[f(\rho(t)) - f(s)] - f^{\nabla}(t)[\rho(t) - s]\right| \leq \varepsilon\left|\rho(t) - s\right|$$

for all $s \in U$, then f is called nabla-differentiable at $t \in \mathbb{T}_{\kappa}$. We call $f^{\nabla}(t)$ the nabla derivative of f at t [6]. Similarly, for $F^{\nabla}(t) = f(t)$, the ∇-integral of f is defined [11] by

$$\int_a^b f(t)\,\nabla t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

For basic concepts of the time scale theory we refer to the textbooks [11–13].
We collect some necessary relations in the following lemma. Their proofs can be found in [11, Chapter1.].

Lemma 1.1. *Let $f : \mathbb{T} \to \mathbb{R}$, $g : \mathbb{T} \to \mathbb{R}$ be two functions and $t \in \mathbb{T}^k$.*

i) If $f^{\Delta}(t)$ exists, then f is continuous at t;

ii) if t is right-scattered and f is continuous at t, then f is Δ-differentiable at t and $f^{\Delta}(t) = \dfrac{f^{\sigma}(t) - f(t)}{\sigma(t) - t}$, where $f^{\sigma}(t) = f(\sigma(t))$;

iii) if $f^{\Delta}(t)$ exists, then $f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t)$; if $f^{\Delta} \equiv 0$, then f is constant.

iv) if $f^{\Delta}(t)$ and $g^{\Delta}(t)$ exist, then $(f \pm g)^{\Delta}(t) = f^{\Delta}(t) \pm g^{\Delta}(t)$, $(fg)^{\Delta}(t) = \left(f^{\Delta} g + f^{\sigma} g^{\Delta}\right)(t)$ and if $(gg^{\sigma})(t) \neq 0$, then $\left(\dfrac{f}{g}\right)^{\Delta}(t) = \left(\dfrac{f^{\Delta} g - f g^{\Delta}}{gg^{\sigma}}\right)(t)$;

v) if $f \in C_{rd}(\mathbb{T})$, then it has an antiderivative on $\mathbb{T}$.

Now, let us recall the notations of the spaces L_p and H_1. For detailed knowledge related to Lebesque measure, Lebesque integration and generalized derivation on the time scale we refer to [19].
$L_p(\mathbb{T}) = \{f : |f|^p$ is integrable on $\mathbb{T}$ in Lebesque sense$\}$ is a Banach space with the norm $\|f\|_{L_p} = \left(\int_a^b |f(t)|^p \Delta t\right)^{\frac{1}{p}}$.
Moreover the set $C^1_{rd}(\mathbb{T}) = \{f : f$ Δ-differentiable on $\mathbb{T}$ and $f^{\Delta} \in C_{rd}(\mathbb{T}^k)\}$ is a normed space with the norm $\|f\|_1 := \|f\|_{L_2} + \|f^{\Delta}\|_{L_2}$. Finally, the sobolev space H_1 is defined to be the completion of $C^1_{rd}(\mathbb{T})$ with respect to the norm $\|.\|_1$. It is proven in [19] that if $f \in H_1$, then there exist the generalized derivative f^{Δ_g} of f in $L_2(\mathbb{T})$ and the following properties.
i) If $f \in H_1$, then the function f^{Δ_g} is unique in Lebesque sense.

ii) If $f \in C^1_{rd}(\mathbb{T})$, then $f^{\Delta_g} = f^{\Delta}$.

iii) (ii)-(iv) in Lemma 1.1 are valid with f^{Δ_g} instead of f^{Δ}.

iv) $\int_a^b f^{\Delta_g}(t) \Delta t = f(b) - f(a)$ for $a, b \in \mathbb{T}$.

2. Half-Inverse Problem

We consider following boundary value problem L on $\mathbb{T} = [0, m_1] \cup \mathbb{T}_1$

$$\ell y := -y^{\Delta\Delta}(t) + q(t) y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in \mathbb{T}^{\kappa^2} \tag{1}$$

$$U(y) := (a_1\lambda + b_1)\, y(0) - (c_1\lambda + d_1)\, y^{\Delta}(0) = 0 \tag{2}$$

$$V(y) := (a_2\lambda + b_2)\, y(l) - (c_2\lambda + d_2)\, y^{\Delta}(l) = 0 \tag{3}$$

where $y^{\Delta\Delta} = \left(y^{\Delta}\right)^{\Delta_g}$, $q(t)$ is real valued continuous function on $\mathbb{T}$, $a_i, b_i, c_i, d_i \in \mathbb{R}$, $i = 1, 2$, $\mathbb{T}_1$ is a bounded time scale with $m_2 := \inf \mathbb{T}_1 \geq m_1$ and $l := \rho(\sup \mathbb{T}_1)$; and λ is the spectral parameter. Additionally we assume that $c_1 c_2 \neq 0$, $\delta_1 := b_1 c_1 - a_1 d_1 > 0$ and $\delta_2 := a_2 d_2 - b_2 c_2 > 0$.
Together with L, we consider a boundary value problem

$$\widetilde{L} = L(\widetilde{q}(t), \widetilde{a_1}, \widetilde{b_1}, \widetilde{c_1}, \widetilde{d_1}, a_2, b_2, c_2, d_2)$$

of the same form but with different coefficients $\widetilde{q}(t), \widetilde{a_1}, \widetilde{b_1}, \widetilde{c_1}$ and $\widetilde{d_1}$. We assume that if a certain symbol s denotes an object related to L , then $\widetilde{s}$ will denote an analogous object related to $\widetilde{L}$.

A function y, defined on $\mathbb{T}$, is called a solution of equation (1) if $y \in C^2_{rd}(\mathbb{T})$ and y satisfies (1) for all $t \in \mathbb{T}$. The values of the λ parameter for which (1)-(3) has nonzero solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions [2]. It is proven in [30] that the problem (1)-(3) has countable many eigenvalues which are real, algebraiclly simple and two eigenfunctions corresponding to the different eigenvalues are orthogonal.

We state the first result of this article.

Let $\Lambda := \{\lambda_n\}_{n\geq 1}$ and $\widetilde{\Lambda} := \left\{\widetilde{\lambda}_n\right\}_{n\geq 1}$ be the eigenvalues sets of L and $\widetilde{L}$, respectively. Consider domain $B_n := \{\lambda : |\lambda - \lambda_n| < \epsilon \}$ for each fixed n and sufficiently small $\epsilon > 0$. Put $B := \mathbb{C} - \bigcup_{n=1}^{\infty} B_n$. The function $\Delta(\lambda)$ is the characteristic function of L.

Theorem 2.1. *Assume that for given any $\epsilon > 0$, there exist some $C_\epsilon > 0$ such that*

$$|\Delta(\lambda)| \geq C_\delta |\lambda|^\alpha \exp|\tau| 2m_1,\ \lambda \in B,\ |\lambda| > \mu \tag{4}$$

where $\alpha > \begin{cases} 3, & \text{if } m_1 < m_2 \\ 2, & \text{if } m_1 = m_2 \end{cases}$, μ is a sufficiently large number and $\tau = Im\sqrt{\lambda}$.
If $\Lambda = \widetilde{\Lambda}$, $q(t) = \widetilde{q}(t)$ on $\mathbb{T}_1$ then $\frac{a_1\lambda+b_1}{c_1\lambda+d_1} = \frac{\widetilde{a_1}\lambda+\widetilde{b_1}}{\widetilde{c_1}\lambda+\widetilde{d_1}}$ and $q(t) = \widetilde{q}(t)$ on $\mathbb{T}$.

Prior to calculations,we need some preliminaries.

Let $\varphi(t,\lambda)$ be the solution of (1) under the initial conditions

$$\varphi(0,\lambda) = c_1\lambda + d_1, \varphi^\Delta(0,\lambda) = a_1\lambda + b_1. \tag{5}$$

The zeros of the function $\Delta(\lambda) = (a_2\lambda + b_2)\varphi(l,\lambda) - (c_2\lambda + d_2)\varphi^\Delta(l,\lambda)$ coincide with the eigenvalues of the problem (1)-(3). It is proven in [30] that the functions $\varphi(t,\lambda)$, $\varphi^\Delta(t,\lambda)$ and so $\Delta(\lambda)$ are entire on λ.

It is clear that $\varphi(t,\lambda)$ satisfies the following integral equation on $(0, m_1)$

$$\begin{aligned} \varphi(t,\lambda) &= (c_1\lambda + d_1)\cos\sqrt{\lambda}t + \int_0^t A(t,x)\cos\sqrt{\lambda}x dx \\ &\quad + (a_1\lambda + b_1)\frac{1}{\sqrt{\lambda}}\left[\sin\sqrt{\lambda}t + \int_0^t B(t,x)\sin\sqrt{\lambda}x dx\right] \end{aligned} \tag{6}$$

where the kernels $A(t,x)$ and $B(t,x)$ satisfy

$$\begin{aligned} &\frac{\partial K(t,x)}{\partial t^2} - q(t)K(t,x) = \frac{\partial K(t,x)}{\partial x^2}, \\ &q(t) = 2\frac{d}{dt}A(t,t) = 2\frac{d}{dt}B(t,t),\quad \left.\frac{\partial A(t,x)}{\partial x}\right|_{x=0} = B(t,0) = 0. \end{aligned}$$

On the other hands, $\varphi^\Delta(t,\lambda)$ is continuous at m_1, and so the relation.

$$m\varphi'(m_1 - 0) = \varphi(m_2) - \varphi(m_1) \tag{7}$$

holds, where $m := m_2 - m_1$. Therefore we obtain the next lemma.

Lemma 2.2. *If $m > 0$, then the following asymptotic formula holds for $|\lambda| \to \infty$;*

$$\varphi(m_2, \lambda) = -mc_1\lambda^{3/2}\sin\sqrt{\lambda}m_1 + O\left(|\lambda|\exp|\tau|\, m_1\right),$$

where $\tau := Im\sqrt{\lambda}$.

Now we are ready to prove our first result.

Proof. [Proof of the Theorem 2.1.] We give proof in the case $m > 0$. The other case is similar and easier. From the equation (1) for φ and $\widetilde{\varphi}$, we have

$$\left[\varphi(t,\lambda)\widetilde{\varphi}^{\Delta}(t,\lambda) - \varphi^{\Delta}(t,\lambda)\widetilde{\varphi}(t,\lambda)\right]^{\Delta} = \left[q(t) - \widetilde{q}(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda). \tag{8}$$

By integrating (in the sense of Δ-integral) both sides of this equality on $[0,l] \cap \mathbb{T}$, we obtain

$$\begin{aligned}
\left[\varphi(t,\lambda)\widetilde{\varphi}^{\Delta}(t,\lambda) - \varphi^{\Delta}(t,\lambda)\widetilde{\varphi}(t,\lambda)\right]_0^l &= \int_0^l \left[q(t) - \widetilde{q}(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda)\Delta t \\
&= \int_0^{m_1} \left[q(t) - \widetilde{q}(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt \\
&\quad + \int_{m_1}^{m_2} \left[q(t) - \widetilde{q}(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda)\Delta t \\
&\quad + \int_{m_2}^{l} \left[q(t) - \widetilde{q}(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)\Delta t \\
&= \int_0^{m_1} \left[q(t) - \widetilde{q}(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt \\
&\quad + \left[q(m_1) - \widetilde{q}(m_1)\right]\varphi^{\sigma}(m_1,\lambda)\widetilde{\varphi}^{\sigma}(m_1,\lambda)\left(m_2 - m_1\right),
\end{aligned}$$

by using the assumption $q = \widetilde{q}$ on $\mathbb{T}_1$.

Thus,

$$\begin{aligned}
&\varphi^{\Delta}(l,\lambda)\widetilde{\varphi}(l,\lambda) - \varphi(l,\lambda)\widetilde{\varphi}^{\Delta}(l,\lambda) = \int_0^{m_1} \left[q(t) - \widetilde{q}(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt + \\
&\left(a_1\lambda + b_1\right)\left(\widetilde{c_1}\lambda + \widetilde{d_1}\right) - \left(\widetilde{a_1}\lambda + \widetilde{b_1}\right)\left(c_1\lambda + d_1\right) \\
&+ m\left[q(m_1) - \widetilde{q}(m_1)\right]\varphi(m_2)\widetilde{\varphi}(m_2).
\end{aligned}$$

Let

$$\begin{aligned}
H(\lambda) &:= \int_0^{m_1} \left[q(t) - \widetilde{q}(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt + \\
&+ \left(a_1\widetilde{c_1} - \widetilde{a_1}c_1\right)\lambda^2 + \left(a_1\widetilde{d_1} + b_1\widetilde{c_1} - \widetilde{a_1}d_1 - \widetilde{b_1}c_1\right)\lambda + \left(b_1\widetilde{d_1} - \widetilde{b_1}d_1\right) \\
&+ m\left[q(m_1) - \widetilde{q}(m_1)\right]\varphi(m_2)\widetilde{\varphi}(m_2).
\end{aligned} \tag{9}$$

Since $\varphi(l,\lambda_n)\widetilde{\varphi}^{\Delta}(l,\lambda_n)-\varphi^{\Delta}(l,\lambda_n)\widetilde{\varphi}(l,\lambda_n)=0$, $H(\lambda_n)=0$ for all $\lambda_n\in\Lambda$ and so $\chi(\lambda):=\dfrac{H(\lambda)}{\Delta(\lambda)}$ is entire on λ. On the other hand, from (6) and Lemma 2.2, $H(\lambda)=O(\lambda^3\exp 2|\tau|\,m_1)$ for sufficiently large $|\lambda|$. Thus

$$|\chi(\lambda)|\le C\,|\lambda|^{3-\alpha}.$$

From Liouville's Theorem $\chi(\lambda)=0$ for all λ. Hence, $H(\lambda)\equiv 0$.

It follows from Lemma 2.2 that the equalities

$$\varphi(m_2,\lambda)=-mc_1\lambda^{3/2}\sin\sqrt{\lambda}m_1+O(1),$$
$$\widetilde{\varphi}(m_2,\lambda)=-m\widetilde{c_1}\lambda^{3/2}\sin\sqrt{\lambda}m_1+O(1)$$

are valid for sufficiently large λ on the real axis. From (9), the following equality can be written:

$$(a_1\widetilde{c_1}-\widetilde{a_1}c_1)\,\lambda^2+\left(a_1\widetilde{d_1}+b_1\widetilde{c_1}-\widetilde{a_1}d_1-\widetilde{b_1}c_1\right)\lambda+\left(b_1\widetilde{d_1}-\widetilde{b_1}d_1\right)$$
$$+\int_0^{m_1}[q(t)-\widetilde{q}(t)]\,\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt+$$
$$m\,[q(m_1)-\widetilde{q}(m_1)]\left[m^2\lambda^3\sin^2\sqrt{\lambda}m_1+O(\lambda^{5/2})\right]=0,\quad \lambda\to\infty,\ \lambda\in\mathbb{R}. \tag{10}$$

From (6) and Riemann-Lebesque lemma, we obtain

$$\frac{1}{\lambda^2}\int_0^{m_1}[q(t)-\widetilde{q}(t)]\,\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt=\frac{c_1\widetilde{c_1}}{2}\int_0^{m_1}[q(t)-\widetilde{q}(t)]\,dt+o(1),$$

for $\lambda\to\infty$, $\lambda\in\mathbb{R}$. Then, we have

$$m\,[q(m_1)-\widetilde{q}(m_1)]\,m^2\sin^2\sqrt{\lambda}m_1=o(1),\quad \lambda\to\infty,\ \lambda\in\mathbb{R}.$$

Hence, it can be concluded that $q(m_1)=\widetilde{q}(m_1)$ and divided by λ^2 in (10) so

$$(a_1\widetilde{c_1}-\widetilde{a_1}c_1)+\frac{c_1\widetilde{c_1}}{2}\int_0^{a_1}[q(t)-\widetilde{q}(t)]\,dt=0.$$

From 10, the equality

$$(a_1\widetilde{c_1}-\widetilde{a_1}c_1)\,\lambda^2+\left(a_1\widetilde{d_1}+b_1\widetilde{c_1}-\widetilde{a_1}d_1-\widetilde{b_1}c_1\right)\lambda+\left(b_1\widetilde{d_1}-\widetilde{b_1}d_1\right)$$
$$+\int_0^{m_1}[q(t)-\widetilde{q}(t)]\,\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt=0 \tag{11}$$

is valid on the whole λ-plane.

From (11), by integrating again both sides of the equality (8) on $(0,m_1)$, we get

$$\varphi'(m_1,\lambda)\widetilde{\varphi}(m_1,\lambda)=\varphi(m_1,\lambda)\widetilde{\varphi}'(m_1,\lambda) \tag{12}$$

Put $\psi(t,\lambda):=\varphi(m_1-t,\lambda)$. It is clear that $\psi(t,\lambda)$ is the solution of the following problem

$$-y''+q(m_1-t)y=\lambda y,\ t\in(0,m_1)$$
$$y(m_1)=c_1\lambda+d_1,\ y'(m_1)=-(a_1\lambda+b_1)$$

It follows from (12) that

$$\psi'(0,\lambda)\widetilde{\psi}(0,\lambda)=\psi(0,\lambda)\widetilde{\psi}'(0,\lambda).$$

Taking into account uniqueness theorem by Weyl function in [18], it is concluded that $q(t) = \widetilde{q}(t)$ on $[0, m_1]$ and $\frac{a_1\lambda+b_1}{c_1\lambda+d_1} = \frac{\widetilde{a_1}\lambda+\widetilde{b_1}}{\widetilde{c_1}\lambda+\widetilde{d_1}}$. This completes the proof. □

Remark 2.3. *Let $\mathbb{T}$ is a union of finite closed intervals such as $\mathbb{T} = \cup_{j=0}^{k}\left[m_{2j}, m_{2j+1}\right]$, $a_i \neq a_j$ for $i \neq j$, $m_0 = 0$ and $\sum_{j=1}^{k}\left(m_{2j+1} - m_{2j}\right) = m_1$. In this case, one can prove by using similar methods that*

$$\Delta(\lambda) = (-1)^k c_1 c_2 \lambda^{(2k+5)/2} \sin\sqrt{\lambda} m_1 \prod_{j=1}^{k}\left(m_{2j} - m_{2j-1}\right)^2 \cos\sqrt{\lambda}(m_{2j+1} - m_{2j}) \tag{13}$$
$$+O(\lambda^{k+2}\exp 2|\tau| a_1).$$

The condition (4) is valid. Hence Theorem 2.1 can be given as follows.

Theorem 2.4. *If $\Lambda = \widetilde{\Lambda}$ and $q(t) = \widetilde{q}(t)$ on $\cup_{j=1}^{k}\left[m_{2j}, m_{2j+1}\right]$ then $\frac{a_1\lambda+b_1}{c_1\lambda+d_1} = \frac{\widetilde{a_1}\lambda+\widetilde{b_1}}{\widetilde{c_1}\lambda+\widetilde{d_1}}$ and $q(t) = \widetilde{q}(t)$ on the whole $\mathbb{T}$.*

Example 2.5. *Consider the problems*

$$L_0 : \begin{cases} -y^{\Delta\Delta}(t) + q_0(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in T = \{[0,1] \cup [2,3]\} \\ y(0) - (\lambda+2)\, y^{\Delta}(0) = 0 \\ (2\lambda+1)\, y(3) - (4\lambda+5)\, y^{\Delta}(3) = 0 \end{cases}$$

and

$$L_1 : \begin{cases} -y^{\Delta\Delta}(t) + q_1(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in [0,1] \cup [2,3] \\ y(0) - (c_1\lambda + d_1)\, y^{\Delta}(0) = 0 \\ (2\lambda+1)\, y(3) - (4\lambda+5)\, y^{\Delta}(3) = 0. \end{cases}$$

Let Λ_0 and Λ_1 be the eigenvalues sets of L_0 and L_1, respectively. From (13), since $\Delta_0(\lambda) = -2\lambda^{7/2}\sin 2\sqrt{\lambda} + O(\lambda^3 \exp 2|\tau|)$ the condition (4) is valid. According to Theorem 2.1, if $\Lambda_1 = \Lambda_0$ and $q_0(t) = q_1(t)$ on $[2,3]$, then $c_1 = 1$, $d_1 = 2$ and $q_0(t) = \widetilde{q}_1(t)$ on T.

3. The Case $a_1 = c_1 = 0$

In the case $c_2 \neq 0$, $a_1 = c_1 = 0$, (2) replaced by

$$U(y) := \cos\alpha y(0) + \sin\alpha y^{\Delta}(0) = 0, \alpha \in [0, \pi) \tag{14}$$

We denote the Sturm-Liouville problem (1), (14), (3) by $L_0(q(t), \alpha, a_2, b_2, c_2, d_2)$ on $\mathbb{T} = [0, m_1] \cup \mathbb{T}_1$.

Let $\omega(t, \lambda)$ be the solution of (1) under the initial conditions

$$\omega(0, \lambda) = \sin\alpha, \omega^{\Delta}(0, \lambda) = -\cos\alpha.$$

It is shown [21] that $\omega(t, \lambda)$ satisfies the following asymptotic relations on $(0, m_1)$,

$$\omega(t, \lambda) = \sin\alpha\cos(\sqrt{\lambda}t) + O(\frac{1}{\sqrt{\lambda}}\exp|\tau| t), \quad \sin\alpha \neq 0,$$
$$\omega(t, \lambda) = -\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} + O(\frac{1}{\lambda}\exp|\tau| t), \quad \sin\alpha = 0$$

where $\tau := Im\sqrt{\lambda}$. Similar to Lemma 2.2, if $m > 0$, then the following asymptotic formula holds for $|\lambda| \to \infty$;

$$\omega(m_2,\lambda) = -m\sin\alpha\sqrt{\lambda}\sin\sqrt{\lambda}m_1 + O\left(\exp|\tau|\,m_1\right), \quad \sin\alpha \neq 0,$$
$$\omega(m_2,\lambda) = -m\cos\sqrt{\lambda}m_1 + O\left(\frac{1}{\sqrt{\lambda}}\exp|\tau|\,m_1\right), \quad \sin\alpha = 0.$$

$\Delta_0(\lambda)$ is the characteristic function of L_0. Consider a second operator $\widetilde{L_0} = L(\widetilde{q}(t), \widetilde{\alpha}, a_2, b_2, c_2, d_2)$. Hence Theorem 2.1 can be given as follows.

Theorem 3.1. *Assume that for given any $\epsilon > 0$, there exist some $C_\epsilon > 0$ such that*

$$|\Delta_0(\lambda)| \geq C_\delta\,|\lambda|^\alpha \exp|\tau|\,2m_1,\ \lambda \in B,\ |\lambda| > \mu$$

where $\alpha > \begin{cases} 1, & \text{if } m_1 < m_2 \text{ and } \sin\alpha \neq 0 \\ 0, & \text{if } m_1 = m_2 \text{ and } \sin\alpha \neq 0 \\ 0, & \text{if } m_1 < m_2 \text{ and } \sin\alpha = 0 \\ -1, & \text{if } m_1 = m_2 \text{ and } \sin\alpha = 0 \end{cases}$ *, μ is a sufficiently large number and $\tau = Im\lambda$.*
If $\Lambda = \widetilde{\Lambda}$, $q(t) = \widetilde{q}(t)$ on $\mathbb{T}_1$ then $\alpha = \widetilde{\alpha}$ and $q(t) = \widetilde{q}(t)$ on $\mathbb{T}$.

4. Interior-Inverse Problem

In this section, we consider an interior inverse Sturm–Liouville problem on a time scale and give a Mochizuki and Trooshin-type theorem. The techniques we use in proof of the following theorem are valid if only the time scale is union of two closed intervals with same lenghts. In case $m_1 + m_2 \neq l$, it may be needed the knowledge of the second spectrum, as in [29]. Thus, we introduce our uniqueness theorem under the condition $m_1 + m_2 = l$. In addition, in case $a_1 = a_2 = c_1 = c_2 = 0$, $m_1 = m_2$ and so $\mathbb{T} = [0, l]$, the following theorem coincides with Mochizuki-Trooshin theorem in [29].

Let $\Lambda := \{\lambda_n\}_{n\geq 1}$ and $\widetilde{\Lambda} := \left\{\widetilde{\lambda}_n\right\}_{n\geq 1}$ be the eigenvalues sets of L and $\widetilde{L}$, $y_n(t)$ and $\widetilde{y_n}(t)$ are eigenfunctions related to this eigenvalues, respectively.

Theorem 4.1. *If $\Lambda = \widetilde{\Lambda}$, $m_1 + m_2 = l$ and for any $n \in \mathbb{N}$,*

$$\frac{y_n^\Delta(m_1)}{y_n(m_1)} = \frac{\widetilde{y}_n^\Delta(m_1)}{\widetilde{y}_n(m_1)} \tag{15}$$

then $\frac{a_1\lambda + b_1}{c_1\lambda + d_1} = \frac{\widetilde{a_1}\lambda + \widetilde{b_1}}{\widetilde{c_1}\lambda + \widetilde{d_1}}$ and $q(t) = \widetilde{q}(t)$ on $\mathbb{T}$.

Proof. Let us consider the problem (1)-(3) on the time scale $\mathbb{T} = [0, m_1] \cup [m_2, l]$, where $m_1 < m_2$ and $l - m_2 = m_1$. Since $\varphi(t,\lambda)$ satisfies the equation (1) for $t = m_1$, it follows that

$$(m^2\lambda + k)\varphi(m_2) + a\varphi'(m_2 + 0) + \varphi(m_1) = 0, \tag{16}$$

where $k := -m^2 q(m_1) - 1$. It can be calculated from (5) and (16) that the following asymptotic formula holds for $|\lambda| \to \infty$:

$$\varphi(t,\lambda) = \begin{cases} c_1\lambda\cos\sqrt{\lambda}t + O\left(\sqrt{\lambda}\exp|\tau|\,t\right), & t \in [0, m_1], \\ c_1 m^2\lambda^2 \sin\sqrt{\lambda}m_1 \sin\sqrt{\lambda}(t - m_2) + O\left(\lambda^{3/2}\exp|\tau|\,(t - m_2 + m_1)\right), & t \in [m_2, l], \end{cases} \tag{17}$$

$$\varphi^\Delta(t,\lambda) = \begin{cases} -c_1\lambda^{3/2}\sin\sqrt{\lambda}t + +O\left(\lambda\exp|\tau|\,t\right), & t \in [0, m_1), \\ c_1 m^2\lambda^{5/2}\sin\sqrt{\lambda}m_1\cos\sqrt{\lambda}(t - m_2) + O\left(\lambda^2\exp|\tau|\,(t - m_2 + m_1)\right), & t \in [m_2, l], \end{cases} \tag{18}$$

It follows from (17) and (18) that the asymptotic relation

$$\Delta(\lambda) = -c_1 c_2 m^2 \lambda^{7/2} \sin \sqrt{\lambda} m_1 \cos \sqrt{\lambda}(l - m_2) + O\left(\lambda^3 \exp|\tau|(l - m_2 + m_1)\right) \tag{19}$$

is valid for $|\lambda| \to \infty$.

Similar to the proof of Theorem 2.1, we have

$$\frac{a_1\lambda + b_1}{c_1\lambda + d_1} = \frac{\widetilde{a_1}\lambda + \widetilde{b_1}}{\widetilde{c_1}\lambda + \widetilde{d_1}} \text{ and } q(t) = \widetilde{q}(t) \text{ on } [0, m_1]. \tag{20}$$

To prove that $q(t) = \widetilde{q}(t)$ on $[m_2, l]$, we will consider the supplementary problem L_1 :

$$\begin{aligned} &-y^{\nabla\nabla} + q_1(t) y^{\rho} = \lambda y^{\rho},\ t \in \mathbb{T}, \\ &(a_2\lambda + b_2)\, y(0) + (c_2\lambda + d_2)\, y^{\nabla}(0) = 0, \\ &(a_1\lambda + b_1)\, y(l) + (c_1\lambda + d_1)\, y^{\nabla}(l) = 0, \end{aligned}$$

where $q_1(t) = q(l-t)$. By using chain rule for nabla deriative in [8], we have that $\varphi_1(t,\lambda) = \varphi(l-t,\lambda)$ satisfies the equation

$$-\varphi_1^{\nabla\nabla} + q_1(t)\varphi_1^{\rho} = \lambda\varphi_1^{\rho}$$

and the conditions

$$\varphi_1(l,\lambda) = c_1\lambda + d_1, \varphi_1^{\nabla}(l,\lambda) = -(a_1\lambda + b_1).$$

Since $y_n^{\Delta}(m_1) = -y_n^{\nabla}(m_2)$, the condition (15) can be replaced by

$$\frac{\varphi_1^{\nabla}(m_2,\lambda_n)}{\varphi_1(m_2,\lambda_n)} = \frac{\tilde{\varphi}_1^{\nabla}(m_2,\lambda_n)}{\tilde{\varphi}_1(m_2,\lambda_n)}.$$

Furthermore, the assumption (15) holds. We replace equation (8) by

$$\left[\varphi_1(t,\lambda)\widetilde{\varphi_1}^{\nabla}(t,\lambda) - \varphi_1^{\nabla}(t,\lambda)\widetilde{\varphi_1}(t,\lambda)\right]^{\nabla} = \left[q_1(t) - \widetilde{q_1}(t)\right]\varphi_1^{\rho}(t,\lambda)\widetilde{\varphi}_1^{\rho}(t,\lambda).$$

We obtain

$$\begin{aligned} \left[\varphi_1(t,\lambda)\widetilde{\varphi}_1^{\nabla}(t,\lambda) - \varphi_1^{\nabla}(t,\lambda)\widetilde{\varphi}_1(t,\lambda)\right]_0^{m_2} &= \int_0^{m_2} \left[q_1(t) - \widetilde{q_1}(t)\right]\varphi_1^{\rho}(t,\lambda)\widetilde{\varphi}_1^{\rho}(t,\lambda)\nabla t \\ &= \int_0^{m_1} \left[q_1(t) - \widetilde{q_1}(t)\right]\varphi_1(t,\lambda)\widetilde{\varphi}_1(t,\lambda)dt \\ &\quad + \int_{m_1}^{m_2} \left[q_1(t) - \widetilde{q_1}(t)\right]\varphi_1^{\rho}(t,\lambda)\widetilde{\varphi}_1^{\rho}(t,\lambda)\nabla t \\ &= \int_0^{m_1} \left[q_1(t) - \widetilde{q_1}(t)\right]\varphi_1(t,\lambda)\widetilde{\varphi}_1(t,\lambda)dt \\ &\quad + \left[q(m_2) - \widetilde{q}(m_2)\right]\varphi^{\rho}(m_2,\lambda)\widetilde{\varphi}^{\rho}(m_2,\lambda)(m_2 - m_1). \end{aligned}$$

Since $q(m_1) = \widetilde{q}(m_1)$, then $q_1(m_2) = \widetilde{q}_1(m_2)$ and

$$\int_{m_1}^{m_2} [q(t) - \widetilde{q}(t)]\, \varphi^{\rho}(t, \lambda)\widetilde{\varphi}^{\rho}(t, \lambda)\nabla t = 0.$$

Therefore, we have

$$\varphi_1^{\nabla}(m_2, \lambda)\widetilde{\varphi}_1(m_2, \lambda) - \varphi_1(m_2, \lambda)\widetilde{\varphi}_1^{\nabla}(m_2, \lambda) = \int_0^{m_1} [q_1(t) - \widetilde{q}_1(t)]\, \varphi_1(t, \lambda)\widetilde{\varphi}_1(t, \lambda)dt. \tag{21}$$

If we repeat the arguments in the proof of Theorem 2.1 then it is concluded that $q_1(t) = \widetilde{q}_1(t)$ on $[0, m_1]$, that is $q(t) = \widetilde{q}(t)$ on $[m_2, l]$. This completes the proof. □

Example 4.2. *Consider the following problems on* $T = [0, 2] \cup [3, 5]$*:*

$$L_0 : \begin{cases} -y^{\Delta\Delta}(t) = \lambda y^{\sigma}(t), \quad t \in [0, 2] \cup [3, 5] \\ y(0) - (\lambda + 2)\, y^{\Delta}(0) = 0 \\ (2\lambda + 1)\, y(5) - (4\lambda + 5)\, y^{\Delta}(5) = 0 \end{cases}$$

and

$$\widetilde{L}_0 : \begin{cases} -y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in [0, 2] \cup [3, 5] \\ y(0) - (c_1\lambda + d_1)\, y^{\Delta}(0) = 0 \\ (2\lambda + 1)\, y(5) - (4\lambda + 5)\, y^{\Delta}(5) = 0 \end{cases}$$

Let $\Lambda_0 := \{\lambda_n\}_{n\geq 1}$ *and* $\widetilde{\Lambda}_0 := \left\{\widetilde{\lambda}_n\right\}_{n\geq 1}$ *be the eigenvalues sets of L and* $\widetilde{L}_0$*,* $y_n(t)$ *and* $\widetilde{y}_n(t)$ *are eigenfunctions related to this eigenvalues, respectively. According to Theorem 4.1, if* $\Lambda_0 = \widetilde{\Lambda}_0$ *and for any* $n \in \mathbb{N}$*,*

$$\frac{y_n^{\Delta}(2)}{y_n(2)} = \frac{\widetilde{y}_n^{\Delta}(2)}{\widetilde{y}_n(2)}$$

then $c_1 = 1$*,* $d_1 = 2$ *and* $q(t) \equiv 0$*.*

Acknowledgments

The author thanks to the reviewers for constructive comments and recommendations which help to improve the readability and quality of the paper.

References

[1] I. Adalar, A. S. Ozkan, An interior inverse Sturm–Liouville problem on a time scale, Analysis and Mathematical Physics, 10(4) (2020) 1-10.
[2] R. P. Agarwal, M. Bohner, P. J. Y. Wong, Sturm-Liouville eigenvalue problems on time scales, Appl. Math. Comput. 99(2-3) (1999) 153–166.
[3] B. P. Allahverdiev, A. Eryilmaz, H. Tuna, Dissipative Sturm-Liouville operators with a spectral parameter in the boundary condition on bounded time scales, Electronic Journal of Differential Equations, 95 (2017) 1–13.
[4] V. A. Ambarzumyan, Über eine Frage der Eigenwerttheorie. Z. Phys. 53 (1929) 690–695.
[5] P. Amster, P. De Na´poli, J. P. Pinasco, Eigenvalue distribution of second-order dynamic equations on time scales considered as fractals, J. Math. Anal. Appl. 343(1) (2008) 573–584.
[6] F. M. Atıcı, G. Guseinov, On Green's functions and positive solutions for boundary value problems on time scales, Journal of Computational and Applied Mathematics 141(1-2) (2002)75-99.
[7] F. Atkinson, Discrete and Continuous Boundary Problems. Academic Press, New York, 1964.
[8] Z. Bartosiewicz, E. Piotrowska, Lyapunov functions in stability of nonlinear systems on time scales, J. Differ. Equ. Appl. 17(3) (2011) 309–325.

[9] P. A. Binding, P. J. Browne, K. Seddighi, Sturm–Liouville problems with eigenparameter dependent boundary conditions, Proc. Edinburgh Math. Soc., 37(1) (1993) 57–72.
[10] P. A. Binding, P. J. Browne, B. A. Watson, Inverse spectral problems for Sturm–Liouville equations with eigenparameter dependent boundary conditions, Journal of the London Mathematical Society 62(1) (2000) 161-182.
[11] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, Birkha¨user, Boston, MA 2001.
[12] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkha¨user, Boston, MA 2003.
[13] M. Bohner, S. G. Georgiev, Multivariable dynamic calculus on time scales, Cham, Switzerland: Springer, 2016.
[14] M. Bohner, H. Koyunbakan, Inverse problems for Sturm–Liouville difference equations, Filomat 30(5) (2016) 1297-1304.
[15] G. Borg, Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte, Acta Math. 78 (1946) 1–96.
[16] S. Buterin, On half inverse problem for differential pencils with the spectral parameter in boundary conditions, Tamkang journal of mathematics 42(3) (2011) 355-364.
[17] K. Chadan, D. Colton, L. Päivärinta, W. Rundell, An introduction to inverse scattering and inverse spectral problems, Society for Industrial and Applied Mathematics, 1997.
[18] A. Chernozhukova, G. Freiling, A uniqueness theorem for the boundary value problems with non-linear dependence on the spectral parameter in the boundary conditions, Inverse Problems in Science and Engineering 17(6) (2009) 777-785.
[19] F. A. Davidson, B. P. Rynne, Self-adjoint boundary value problems on time scales, Electron. J. Differ. Equ. 175 (2007) 1–10.
[20] L. Erbe, S. Hilger, Sturmian theory on measure chains, Differ. Equ. Dyn. Syst. 1 (1993) 223–244.
[21] G. Freiling, V. A. Yurko, Inverse Sturm–Liouville Problems and their Applications, Nova Science, New York, 2001.
[22] F. Gesztesy, B. Simon, Inverse spectral analysis with partial information on the potential II. The case of discrete spectrum, Trans. Am. Math. Soc. 352(6) (2000) 2765-2787.
[23] G. S. Guseinov, Eigenfunction expansions for a Sturm-Liouville problem on time scales, Int. J. Differ. Equ. 2 (2007) 93–104.
[24] S. Hilger, Analysis on measure chains – a unified approach to continuous and discrete calculus, Results in Math. 18 (1990) 18–56.
[25] H. Hochstadt, B. Lieberman, An inverse Sturm–Liouville problem with mixed given data, SIAM Journal on Applied Mathematics 34(4) (1978) 676-680.
[26] M. Horvath, Inverse spectral problems and closed exponential systems, Ann. of Math. 162 (2005) 885-918.
[27] Q. Kong, Sturm-Liouville problems on time scales with separated boundary conditions, Results Math. 52 (2008) 111–121.
[28] M. A. Kuznetsova, A Uniqueness Theorem on Inverse Spectral Problems for the Sturm–Liouville Differential Operators on Time Scales, Results Math 75(2) (2020) 1-23.
[29] K. Mochizuki, I. Trooshin, Inverse problem for interior spectral data of Sturm-Liouville operator, Journal of Inverse and Ill-posed Problems 9 (2001) 425-433.
[30] A. S. Ozkan, Sturm-Liouville operator with parameter-dependent boundary conditions on time scales, Electron. J. Differential Equations 212 (2017) 1-10.
[31] A. S. Ozkan, Ambarzumyan-type theorems on a time scale, Journal of Inverse and Ill-posed Problems 26(5) (2018) 633–637.
[32] A. S. Ozkan, I. Adalar, Half-inverse Sturm-Liouville problem on a time scale, Inverse Problems 36 (2020) 025015.
[33] A. S. Ozkan, B. Keskin, Spectral problems for Sturm–Liouville operator with boundary and jump conditions linearly dependent on the eigenparameter, Inverse Problems in Science and Engineering 20(6) (2012) 799-808.
[34] A. S. Ozkan, Boundary-value problem for a class of second-order parameter-dependent dynamic equations on a time scale, Mathematical Methods in the Applied Sciences 43(7) (2020) 4353-4359.
[35] L. Sakhnovich, Half inverse problems on the finite interval, Inverse Problems 17 (2001) 527–532.
[36] Y. P. Wang, C. F. Yang, Z. Huang, Half inverse problem for Sturm-Liouville operators with boundary conditions dependent on the spectral parameter, Turkish Journal of Mathematics 37(3) (2013) 445-454.
[37] Y. P. Wang, C. T. Shieh, Inverse Problems for Sturm-Liouville Equations with Boundary Conditions Linearly Dependent on the Spectral Parameter from Partial Information, Results in Mathematics 65 (2014) 105-119.
[38] Y. P. Wang, An interior inverse problem for Sturm-Liouville operators with eigenparameter dependent boundary conditions, Tamkang Journal of Mathematics 42(3) (2011) 395-403.
[39] C. F. Yang, Y. X. Guo, Determination of a differential pencil from interior spectral data, Journal of Mathematical Analysis and Applications 375(1) (2011) 284-293.