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Additive Properties of Central Drazin Invertibility of Elements in a Ring

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Abstract. For two central Drazin invertible elements a and b of a ring, we first prove that $a+b$ is central Drazin invertible under the condition $ab = 0$. Then we establish the relation between central Drazin invertibility of $a+b$ and $1 + a^c b$, when $a^2 b = aba$ and $b^2 a = bab$ hold, and also when $ab = ba$ is valid. When a and b are two central group invertible elements, additive properties of central group inverses are studied under the condition $ab = ba$ and also $abb^{©} = baa^{©}$.

1. Introduction

Throughout this paper, R will denote an associative ring with unity 1. Let us recall that the center of R is defined as $C(R) = \{x \in R : ax = xa \textit{ for all } a \in R\}$ and the commutant of an element $a \in R$ is defined as $comm(a) = \{x \in R : ax = xa\}$. Recall that an element $a \in R$ is Drazin invertible [10] if there exists $x \in R$ such that $ax = xa, xax = x, a^{k+1}x = a^k$ for some nonnegative integer k. The element $x \in R$ is unique if it exists and denoted by a^D. The such smallest nonnegative integer k satisfying the above equations is called the Drazin index of a, and denoted by $\mathrm{ind}(a)$. If $k = 1$, then x is called the group inverse of a, and denoted by $x = a^{\#}$. In [10], Drazin proved that if $a \in R$ is Drazin invertible, then for any $b \in R$, $ab = ba$ implies $a^D b = ba^D$.

In 2019, in order to study the commutating properties of Drazin inverses (see [11, Example 2.8]), Wu and Zhao [19] introduced and studied a new class of Drazin inverses in a ring R, which were called central Drazin inverses.

Definition 1.1. [19] *An element $a \in R$ is said to be central Drazin invertible if there exists $x \in R$ such that $xa \in C(R), xax = x, a^{k+1}x = a^k$ for some nonnegative integer k.*

According to [19], the central Drazin inverse is unique if it exists. Any x satisfying the above equations is called the central Drazin inverse of a, and denoted by $x = a^c$. The smallest nonnegative integer k satisfying the above equations is still called the Drazin index of a. If $k = 1$, then x is called the central group inverse of a and denoted by $a^{©}$. They also proved that if $a \in R$ is central Drazin invertible, then a is Drazin invertible and $a^c = a^D$.

The topic concerning additive properties of generalized invertibility of elements is of great interest and many authors have investigated this subject. The problem of Drazin invertibilty of the sum of two

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Drazin invertible elements was first considered by Drazin in his celebrated paper [10]. He presented that $(a+b)^D = a^D + b^D$ for two Drazin invertible elements $a, b \in R$ satisfying the condition $ab = ba = 0$. In 2001, Hartwig et al.[12] gave the expression of $(A+B)^D$ in the case of the one-sided condition $AB = 0$ in complex matrices. Afterwards, the result was extended to the bounded linear operators on an arbitrary complex Banach space by Djordjević and Wei[9] in 2002, and was extended to morphisms in arbitrary additive categories by Chen et al.[2] in 2009. More relevant results on Drazin inverses can be found in [1, 3–8, 15–17].

The relation between Drazin invertibility of $a+b$ and $1+a^Db$ for two Drazin invertible elements a and b was studied widely. In 2011, Wei and Deng[18] considered the relations between Drazin inverses of $A+B$ and $I+A^DB$ for two commutative complex matrices A and B. In 2012, Zhuang et al.[22] extended the result from complex matrices to rings. Under the conditions $P^2Q = PQP$ and $Q^2P = QPQ$, Liu et al.[14] characterized the relation between Drazin inverses of $P+Q$ and $I+P^DQ$ for complex matrices P and Q by using the methods of splitting complex matrices into blocks. The results in [22] and [14] were extended to the condition of $a^2b = aba$ and $b^2a = bab$ in an associative ring by Zhu and Chen[21] in 2017.

The combinations of two group invertible elements were investigated by many scholars. For example, in 2011, Liu et al.[13] investigated the group invertibility of linear combinations of two group invertible matrices $P, Q \in \mathbb{C}^{n\times n}$ under the following conditions: $PQQ^\# = QPP^\#$ or $QQ^\#P = PP^\#Q$ or $QP^\#P = P$. In 2020, Zhou et al.[20] extended the results in [13] to Dedekind-finite ring, and gave the representations of $(a+b)^\#$ and $(a-b)^\#$.

Motivated by above papers, we investigate relevant additive properties for central Drazin inverses in a ring. An outline of this paper is as follows. In Section 2, we consider central Drazin invertibility of the sum of two central Drazin invertible elements. In Section 3, we prove that $a+b$ is central Drazin invertible if and only if $1+a^cb$ is central Drazin invertible under the condition $a^2b = aba$ and $b^2a = bab$ for two central Drazin invertible elements a and b, and their expressions are presented. In Section 4, additive properties of two central group invertible elements are characterized.

2. Central Drazin invertibility of the sum of two central Drazin invertible elements

In [10, Corollary 2.1] and [2, Theorem 2.1], authors investigated Drazin invertibility of the sum of two Drazin invertible elements in rings. In this section, we will study the relevant results for central Drazin inverses in rings. The following lemmas will be used in the sequel.

Lemma 2.1. [19, Theorem 2.3] *If $a \in R$ is central Drazin invertible, then a is Drazin invertible and $a^c = a^D$. In particular, the central Drazin inverse is unique when it exists.*

Next, we consider the central Drazin invertibility of $a+b$ under the condition $ab = 0$.

Lemma 2.2. [2, Theorem 2.1] *Let $a, b \in R$ be Drazin invertible with $ab = 0$. Then $a+b$ is Drazin invertible. In this case,*

$$(a+b)^D = (1-bb^D)\sum_{i=0}^{k_2-1} b^i(a^D)^i a^D + b^D \sum_{j=0}^{k_1-1} (b^D)^j a^j (1-aa^D),$$

where $k_1 = \mathrm{ind}(a)$, $k_2 = \mathrm{ind}(b)$.

Theorem 2.3. *Let $a, b \in R$ be central Drazin invertible with $ab = 0$. Then $a+b$ is also central Drazin invertible. In this case, $(a+b)^c = a^c + b^c$.*

Proof. Since $a, b \in R$ are central Drazin invertible, we know that $a, b \in R$ are Drazin invertible by Lemma 2.1. Following Lemma 2.2, we get that $a+b$ is also Drazin invertible with

$$(a+b)^D = (1-bb^c)\sum_{i=0}^{k_2-1} b^i(a^c)^i a^c + b^c \sum_{j=0}^{k_1-1} (b^c)^j a^j (1-aa^c),$$

where $k_1 = \mathrm{ind}(a)$, $k_2 = \mathrm{ind}(b)$. From $ab = 0$, we have $ba^c = ba^c aa^c = a^c aba^c = 0$. Similarly $b^c a = 0$, $a^c b = 0$ and $b^c a^c = 0$. Then we can obtain

$$\begin{aligned}(a+b)^D &= (1-bb^c)a^c + (1-bb^c)\sum_{i=1}^{k_2-1} b^i (a^c)^i a^c + b^c(1-aa^c) + b^c \sum_{j=1}^{k_1-1} (b^c)^j a^j (1-aa^c)\\ &= a^c + b^c.\end{aligned}$$

Thus $(a+b)^D(a+b) = (a^c+b^c)(a+b) = a^c a + a^c b + b^c a + b^c b = a^c a + b^c b \in C(R)$. Therefore, $a+b$ is central Drazin invertible with $(a+b)^c = a^c + b^c$. □

Proposition 2.4. *Let $a, b \in R$ be central group invertible with $ab = 0$. Then $a+b$ is also central group invertible with $(a+b)^{©} = a^{©} + b^{©}$.*

Proof. According to [2, Corollary 2.2], we get that $a+b$ is group invertible and $(a+b)^\# = (1-bb^{©})a^{©} + b^{©}(1-aa^{©}) = a^{©} + b^{©}$. Similarly to the proof of Theorem 2.3, we can obtain $(a^{©} + b^{©})(a+b) \in C(R)$, and thus $a+b$ is also central group invertible with $(a+b)^{©} = a^{©} + b^{©}$. □

3. The relation between central Drazin invertibility of $a+b$ and $1 + a^c b$

In this section, we will characterize the relation between central Drazin inverses of $a+b$ and $1+a^c b$, when $a^2 b = aba$ and $b^2 a = bab$ hold. As a corollary, the corresponding conclusion is given in the case when $ab = ba$. First let us look at the following lemmas.

Lemma 3.1. *Let $a, b \in R$ be central Drazin invertible with $a^2 b = aba$ and $b^2 a = bab$. Then*

(i) $\{ab,\ a^c b,\ ab^c,\ a^c b^c\} \subseteq comm(a)$.

(ii) $\{ba,\ b^c a,\ ba^c,\ b^c a^c\} \subseteq comm(b)$.

Proof. (i) Obviously $ab \in comm(a)$. Since $aa^c b = (a^c)^2 a^2 b = (a^c)^2 aba = a^c ba$, we have $a^c b \in comm(a)$. From $ba \in comm(b)$, then $ba \in comm(b^c)$. Thus, $ab^c a = a(b^c)^2 ba = aba(b^c)^2 = a^2 b(b^c)^2 = a^2 b^c$, that is $ab^c \in comm(a)$. Note that $ab^c a = a^2 b^c$, then we can obtain $aa^c b^c = (a^c)^2 a^2 b^c = (a^c)^2 ab^c a = a^c b^c a$, thus $a^c b^c \in comm(a)$.

(ii) It is available directly from symmetry of a and b. □

Lemma 3.2. *Let $a, b \in R$ be central Drazin invertible with $a^2 b = aba$ and $b^2 a = bab$. Then $a^c b = ba^c$ and $ab^c = b^c a$.*

Proof. By Lemma 3.1, we have $a^c b = a^c aa^c b = a^c baa^c = aa^c ba^c = baa^c a^c = ba^c$. Similarly we can also obtain $ab^c = b^c a$. □

Lemma 3.3. *Let $a, b \in R$ be central Drazin invertible with $a^2 b = aba$ and $b^2 a = bab$. Then $\{a,\ a^c,\ ab,\ a^c b,\ ab^c,\ a^c b^c\} \subseteq comm(1 + a^c b)$.*

Proof. Since $a^c ba = aa^c b$, we have $(1+a^c b)a = a(1+a^c b)$, that is $a \in comm(1+a^c b)$, then we obtain $a^c \in comm(1+a^c b)$. From $(a^c b)ab = a^c b^2 a = b^2 a^c a = a^c(ab)b = aba^c b$, we get $ab \in comm(1+a^c b)$. Obviously $a^c b \in comm(1+a^c b)$. Since $ab^c a^c b = aa^c bb^c = a^c bab^c$, it follows that $ab^c \in comm(1+a^c b)$. Finally from $a^c ba^c b^c = a^c b^c ba^c = a^c b^c a^c b$, we can obtain $a^c b^c \in comm(1+a^c b)$. □

Lemma 3.4. [21, Theorem 3.1] *Let $a, b \in R$ be Drazin invertible with $a^2 b = aba$ and $b^2 a = bab$. Then ab is Drazin invertible, and $(ab)^D = a^D b^D$.*

Proposition 3.5. *Let $a, b \in R$ be central Drazin invertible with $a^2 b = aba$ and $b^2 a = bab$. Then ab is central Drazin invertible, and $(ab)^c = a^c b^c = b^c a^c$.*

Proof. It is easy to know that we only need to prove $(ab)^D ab \in C(R)$. In fact, since $(ab)^D ab = a^c b^c ab = aa^c bb^c \in C(R)$, it follows that ab is central Drazin invertible with $(ab)^c = a^c b^c$. Moreover $a^c b^c = a^c b^c bb^c = b^c ba^c b^c = b^c b^c ba^c = b^c a^c$. □

Lemma 3.6. [21, Theorem 3.3] *Let $a, b \in R$ be Drazin invertible with $a^2 b = aba$, $b^2 a = bab$ and $\mathrm{ind}(a) = k$. Then $a + b$ is Drazin invertible if and only if $1 + a^D b$ is Drazin invertible. In this case,*

$$\begin{aligned}(a+b)^D &= a^D(1+a^Db)^D + a^\pi b[a^D(1+a^Db)^D]^2 + \sum_{i=0}^{k-1}(b^D)^{i+1}(-a)^i a^\pi \\ &\quad + b^\pi a\sum_{i=0}^{k-2}(i+1)(b^D)^{i+2}(-a)^i a^\pi\end{aligned}$$

and $(1 + a^D b)^D = a^\pi + a^2 a^D (a + b)^D$, where $a^\pi = 1 - aa^D$, $b^\pi = 1 - bb^D$.

Theorem 3.7. *Let $a, b \in R$ be central Drazin invertible with $a^2 b = aba$, $b^2 a = bab$ and $\mathrm{ind}(a) = k$. Then $a + b$ is central Drazin invertible if and only if $1 + a^c b$ is central Drazin invertible. In this case,*

$$(a+b)^c = a^c(1+a^cb)^c + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi$$

and $(1 + a^c b)^c = a^\pi + a^2 a^c (a + b)^c$, where $a^\pi = 1 - aa^c$.

Proof. (⇒): Obviously a, b and $a + b$ are Drazin invertible by Lemma 2.1, then from Lemma 3.6, we know that $1 + a^c b$ is Drazin invertible, and $(1 + a^c b)^D = a^\pi + a^2 a^c (a + b)^c$. By Lemma 3.2, we have

$$\begin{aligned}(1+a^cb)^D(1+a^cb) &= [a^\pi + a^2a^c(a+b)^c](1+a^cb) \\ &= a^\pi + a^\pi a^c b + a^2a^c(a+b)^c + a^2a^c(a+b)^c a^c b \\ &= a^\pi + a(a+b)^c aa^c + a(a+b)^c ba^c \\ &= a^\pi + a(a+b)^c(a+b)a^c \\ &= a^\pi + (a+b)^c(a+b)aa^c \in C(R).\end{aligned}$$

Thus $1 + a^c b$ is central Drazin invertible. In this case, $(1 + a^c b)^c = a^\pi + a^2 a^c (a + b)^c$.

(⇐): By Lemma 3.6, $a + b$ is Drazin invertible, and

$$(a+b)^D = a^c(1+a^cb)^c + a^\pi b[a^c(1+a^cb)^c]^2 + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi + b^\pi a\sum_{i=0}^{k-2}(i+1)(b^c)^{i+2}(-a)^i a^\pi.$$

Also since

$$a^\pi b[a^c(1+a^cb)^c]^2 = ba^\pi a^c(1+a^cb)^c a^c(1+a^cb)^c = 0,$$

and

$$b^\pi a\sum_{i=0}^{k-2}(i+1)(b^c)^{i+2}(-a)^i a^\pi = a\sum_{i=0}^{k-2}(i+1)b^\pi b^c(b^c)^{i+1}(-a)^i a^\pi = 0,$$

we can obtain

$$(a+b)^D = a^c(1+a^cb)^c + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi.$$

Now it suffices to prove that $(a+b)^D(a+b) \in C(R)$. In fact, by Lemma 3.3,

$$
\begin{aligned}
&(a+b)^D(a+b)\\
&= [a^c(1+a^cb)^c + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi](a+b)\\
&= a^c(1+a^cb)^c a + a^c(1+a^cb)^c b + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi a + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi b\\
&= (1+a^cb)^c a^c a + (1+a^cb)^c a^c b + \sum_{i=0}^{k-1}(-1)^i(b^c)^{i+1}a^{i+1}a^\pi + \sum_{i=0}^{k-1}(-1)^i(b^c)^{i+1}a^i a^\pi b\\
&= (1+a^cb)^c a^c a + (1+a^cb)^c a^c b + \sum_{i=1}^{k}(-1)^{i-1}(b^c)^i a^i a^\pi + \sum_{i=1}^{k-1}(-1)^i(b^c)^i a^i a^\pi b^c b + a^\pi b^c b\\
&= (1+a^cb)^c a^c a + (1+a^cb)^c a^c b + \sum_{i=1}^{k-1}(-1)^{i-1}(b^c)^i a^i a^\pi + \sum_{i=1}^{k-1}(-1)^i(b^c)^i a^i a^\pi b^c b + a^\pi b^c b\\
&= (1+a^cb)^c a^c a + (1+a^cb)^c(1+a^cb-1) + \sum_{i=1}^{k-1}(-1)^{i-1}(b^c)^i a^i a^\pi b^\pi + a^\pi b^c b\\
&= (1+a^cb)^c(1+a^cb) - (1+a^cb)^c a^\pi + a^\pi b^c b.
\end{aligned}
$$

For arbitrary $x \in R$, since $a^\pi \in C(R)$, we have $(1+a^cb)xa^\pi = xa^\pi(1+a^cb)$, then $(1+a^cb)^c xa^\pi = xa^\pi(1+a^cb)^c$. Hence $(1+a^cb)^c a^\pi x = x(1+a^cb)^c a^\pi$, that is $(1+a^cb)^c a^\pi \in C(R)$, then $(a+b)^D(a+b) \in C(R)$. Therefore $a+b$ is central Drazin invertible with $(a+b)^c = a^c(1+a^cb)^c + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi$. □

Remark 3.8. *Let $a, b \in R$ be central Drazin invertible. Then $ab = ba$ implies $a^2b = aba$ and $b^2a = bab$. However, the reverse is not true in general.*

For example, let $a, b \in \mathbb{C}^{4\times 4}$, and take

$$
a = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

Then we can obtain $a^2 = 0, a^c = 0, b^2 = 0$ and $b^c = 0$. So after calculation, we have $a^2b = aba, b^2a = bab$, but $ab \neq ba$.

Remark 3.9. *In general, $\mathrm{ind}(a+b) \neq \mathrm{ind}(1+a^cb)$. For example, we take a, b as Remark 3.8. Then $a+b$ is nilpotent and the nilpotent index is equal to 3, it follows that $\mathrm{ind}(a+b) = 3$. However, $1+a^cb$ is invertible with $\mathrm{ind}(1+a^cb) = 0$.*

Proposition 3.10. *Let $a, b \in R$ be central Drazin invertible with $ab = ba$. Then $a+b$ is central Drazin invertible if and only if $1+a^cb$ is central Drazin invertible. In this case,*

$$
\begin{aligned}
(a+b)^c &= (1+a^cb)^c a^c + b^c(1+aa^\pi b^c)^{-1}a^\pi && (3.1)\\
&= (1+a^cb)^c a^c + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi && (3.2)\\
&= a^c(1+a^cb)^c bb^c + b^\pi(1+bb^\pi a^c)^{-1}a^c + b^c(1+aa^\pi b^c)^{-1}a^\pi && (3.3)\\
&= a^c(1+a^cb)^c bb^c + \sum_{i=0}^{l-1}(-b)^i(a^c)^{i+1}b^\pi + \sum_{i=0}^{k-1}(b^c)^{i+1}(-a)^i a^\pi && (3.4)
\end{aligned}
$$

and

$$(1 + a^c b)^c = a^\pi + a^2 a^\pi (a + b)^c,$$

where $a^\pi = 1 - aa^c$, $b^\pi = 1 - bb^c$, $\mathrm{ind}(a) = k$, $\mathrm{ind}(b) = l$.

Proof. Since $\mathrm{ind}(a) = k$, it follows that $(aa^\pi b^c)^k = 0$, then $1 + aa^\pi b^c$ is invertible and

$$\begin{aligned}(1 + aa^\pi b^c)^{-1} &= 1 + (-aa^\pi b^c) + (-aa^\pi b^c)^2 + \cdots + (-aa^\pi b^c)^{k-1}\\ &= \sum_{i=0}^{k-1} (-aa^\pi b^c)^i.\end{aligned}$$

Then from Theorem 3.7, we have

$$(a + b)^c = (1 + a^c b)^c a^c + b^c (1 + aa^\pi b^c)^{-1} a^\pi.$$

Let $\xi = 1 + a^c b$. Then it suffices to prove

$$\xi^c a^c = a^c \xi^c bb^c + b^\pi (1 + bb^\pi a^c)^{-1} a^c.$$

In fact, note that $a^c b^\pi (1 + bb^\pi a^c) = a^c b^\pi + a^c b^\pi bb^\pi a^c = \xi a^c b^\pi$. Then we have

$$[1 - (1 + bb^\pi a^c)^{-1} \xi \xi^\pi] \xi^\pi a^c b^\pi = 0.$$

Since $(1 + bb^\pi a^c)^{-1}\xi\xi^\pi$ is nilpotent, we have $1 - (1 + bb^\pi a^c)^{-1}\xi\xi^\pi$ is invertible, thus $\xi^\pi a^c b^\pi = 0$. Hence we can obtain

$$a^c b^\pi = \xi^c \xi a^c b^\pi = \xi^c a^c b^\pi (1 + bb^\pi a^c),$$

then $\xi^c a^c b^\pi = a^c b^\pi (1 + bb^\pi a^c)^{-1}$. Therefore, $\xi^c a^c = a^c \xi^c bb^c + b^\pi (1 + bb^\pi a^c)^{-1} a^c$ is proved.

Similarly, we can obtain $b^\pi (1 + bb^\pi a^c)^{-1} = b^\pi \sum_{i=0}^{l-1} (a^c)^i (-b)^i$. This completes the proof. □

Corollary 3.11. *Let* $a, b \in R$ *be central Drazin invertible with* $\mathrm{ind}(a) = k, \mathrm{ind}(b) = l$ *and* $ab = ba$. *Suppose that* $1 + a^c b$ *is central Drazin invertible. Then the following statements hold.*

(i) *If* $a^c b^c = 0$, *then* $(a + b)^c = \sum_{i=0}^{k-1} (b^c)^{i+1} (-a)^i + \sum_{i=0}^{l-1} (-b)^i (a^c)^{i+1}$.

(ii) *If* $a^c b = 0$, *then* $(a + b)^c = a^c + \sum_{i=0}^{k-1} (b^c)^{i+1} (-a)^i$.

(iii) *If* $\mathrm{ind}(a) = 1$, *then* $(a + b)^c = (1 + a^{©} b)^c a^{©} + (1 - aa^{©}) b^c$.

Proof. (i) Substitute $a^c b^c = 0$ directly into (3.4) of Proposition 3.10, then the expression is obtained.

(ii) From $a^c b = 0$, we have $b^c a = b^c bb^c aaa^c = b^c aa^c bb^c a = 0$. By (3.2) of Proposition 3.10, we can obtain $(a + b)^c = a^c + \sum_{i=0}^{k-1} (b^c)^{i+1} (-a)^i$.

(iii) Since $\mathrm{ind}(a) = 1$ implies $aa^\pi = 0$, it follows that by (3.1) of Proposition 3.10 we have $(a + b)^c = (1 + a^{©} b)^c a^{©} + (1 - aa^{©}) b^c$. □

Remark 3.12. *In the (iii) of Corollary 3.11,* $a + b$ *may be not central group invertible in general.*

For example, let $R = \mathbb{Z}_2 S_3$. *We take* $a = (1) + (123) + (132)$, $b = (13) + (123) + (132)$. *It is easy to check that* $ab = ba$, a *is central group invertible with* $a^{©} = (1) + (123) + (132)$, b *is central Drazin invertible and* $b^c = (12) + (13) + (23)$, *which is not central group invertible. By calculation, we get that* $1 + a^{©} b = (1) + (12) + (13) + (23)$ *is central Drazin invertible but not central group invertible, and* $(1 + a^{©} b)^c = (123) + (132)$. *Then by Corollary 3.11,* $a + b = (1) + (13)$ *is central Drazin invertible with* $(a + b)^c = 0$, *but it is not central group invertible.*

4. Additive properties of central group invertibility

In this section, we prove that $a+b$ is central group invertible if and only if $1+a^{©}b$ is central group invertible, for two central group invertible elements a and b, under the condition $ab=ba$. Then, the explicit representations of central group inverses of $a+b$ and of $a-b$ are given, under the condition $abb^{©}=baa^{©}$.

Lemma 4.1. *Let $a,b\in R$ be group invertible with $ab=ba$. Then the following statements hold.*

(i) *a, b, $a^{\#}$ and $b^{\#}$ commute.*

(ii) *If $1+a^{\#}b$ is group invertible, then $\{a,\ b,\ a^{\#},\ b^{\#}\}\subseteq comm((1+a^{\#}b)^{\#})$.*

Proof. (i) Note that $a^{\#}\in comm^2(a)$ and $b^{\#}\in comm^2(b)$. Since $ab=ba$, we have $a^{\#}b=ba^{\#}$ and $b^{\#}a=ab^{\#}$, then $b^{\#}a^{\#}=a^{\#}b^{\#}$.

(ii) From (i), we have $\{a,b\}\subseteq comm(1+a^{\#}b)$, then the result is directly obtained. □

Lemma 4.2. *Let $a,b\in R$ be group invertible with $ab=ba$. Then $a+b$ is group invertible if and only if $1+a^{\#}b$ is group invertible. In this case,*

$$\begin{aligned}(a+b)^{\#}&=(1+a^{\#}b)^{\#}a^{\#}+b^{\#}a^{\pi}\\&=a^{\#}(1+a^{\#}b)^{\#}bb^{\#}+a^{\#}b^{\pi}+b^{\#}a^{\pi}\end{aligned}$$

and $(1+a^{\#}b)^{\#}=a^{\pi}+a(a+b)^{\#}$, where $a^{\pi}=1-aa^{\#}$, $b^{\pi}=1-bb^{\#}$.

Proof. The proof is similar to [22, Theorem 3]. □

Theorem 4.3. *Let $a,b\in R$ be central group invertible with $ab=ba$. Then $a+b$ is central group invertible if and only if $1+a^{©}b$ is central group invertible. In this case,*

$$\begin{aligned}(a+b)^{©}&=(1+a^{©}b)^{©}a^{©}+b^{©}a^{\pi}\\&=a^{©}(1+a^{©}b)^{©}bb^{©}+a^{©}b^{\pi}+b^{©}a^{\pi}\end{aligned}$$

and $(1+a^{©}b)^{©}=a^{\pi}+a(a+b)^{©}$, where $a^{\pi}=1-aa^{©}$, $b^{\pi}=1-bb^{©}$.

Proof. ($\Leftarrow$): From Lemma 2.1 and Lemma 4.2, it is clear that $a+b$ is group invertible, and $(a+b)^{\#}=(1+a^{©}b)^{©}a^{©}+b^{©}a^{\pi}$

Let $\varphi=1+a^{©}b$ and $x=\varphi^{©}a^{©}+b^{©}a^{\pi}$. Next we show $x(a+b)\in C(R)$. In fact,

$$\begin{aligned}x(a+b)&=\varphi^{©}a^{©}a+\varphi^{©}a^{©}b+b^{©}a^{\pi}b\\&=\varphi^{©}a^{©}a+\varphi^{©}\varphi-\varphi^{©}+a^{\pi}b^{©}b\\&=\varphi\varphi^{©}-\varphi^{©}a^{\pi}+a^{\pi}b^{©}b.\end{aligned}$$

For arbitrary $x\in R$, since $a^{\pi}\in C(R)$, we have $\varphi xa^{\pi}=xa^{\pi}\varphi$, then $\varphi^{©}xa^{\pi}=xa^{\pi}\varphi^{©}$. Thus $\varphi^{©}a^{\pi}x=x\varphi^{©}a^{\pi}$. Now we get $\varphi^{©}a^{\pi}\in C(R)$, that is $x(a+b)\in C(R)$. Therefore $a+b$ is central group invertible, and $(a+b)^{©}=(1+a^{©}b)^{©}a^{©}+b^{©}a^{\pi}$.

($\Rightarrow$): Similarly from Lemma 2.1 and Lemma 4.2, we have $1+a^{©}b$ is group invertible, and $(1+a^{©}b)^{\#}=a^{\pi}+a(a+b)^{©}$. Then

$$\begin{aligned}(1+a^{©}b)^{\#}(1+a^{©}b)&=a^{\pi}+a(a+b)^{©}+a(a+b)^{©}a^{©}b\\&=a^{\pi}+aa^{©}a(a+b)^{©}+aa^{©}b(a+b)^{©}\\&=a^{\pi}+aa^{©}(a+b)(a+b)^{©}\in C(R).\end{aligned}$$

Thus $1+a^{©}b$ is central group invertible, and $(1+a^{©}b)^{©}=a^{\pi}+a(a+b)^{©}$. □

Remark 4.4. *Let a,b be central group invertible. Then either condition of $a^2b=aba$ or $b^2a=bab$ implies $ab=ba$. In fact, according to the proof of Lemma 3.2, we have $a^{©}b=ba^{©}$, thus $ab=ba$.*

Given any Dedekind-finite ring R with $2 \in R^{-1}$, Zhou et al.[20] proved that if $a, b \in R$ are group invertible with $abb^{\#} = baa^{\#}$ or $bb^{\#}a = aa^{\#}b$, then $a + b$ and $a - b$ are also group invertible. Next, we investigate central group invertibility of $a+b$ and $a-b$ under the condition of $abb^{©} = baa^{©}$, and give their explicit representations. Moreover, it is worth noting that we can drop the condition that the ring R is Dedekind-finite ring.

Theorem 4.5. *Let $a, b \in R$ be central group invertible with $abb^{©} = baa^{©}$ and $2 \in R^{-1}$. Then*

(i) $(a + b)^{©} = a^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©}$.

(ii) $(a - b)^{©} = a^{©} - b^{©}$.

Proof. (i) Since $abb^{©} = baa^{©}$, we have

$$\begin{aligned}
a^{©}b &= a^{©}aa^{©}b = aa^{©}bb^{©},\\
b^{©}a &= b^{©}bb^{©}a = bb^{©}aa^{©},\\
ab^{©} &= ab^{©}bb^{©} = baa^{©}b^{©} = aa^{©}bb^{©},\\
ba^{©} &= ba^{©}aa^{©} = abb^{©}a^{©} = bb^{©}aa^{©}.
\end{aligned}$$

Thus $a^{©}b = b^{©}a = ab^{©} = ba^{©}$. Also we have

$$\begin{aligned}
&a^{©}ab^{©} = a^{©}ba^{©},\\
&a^{©}ab^{©} = a^{©}abb^{©}b^{©} = a^{©}aa^{©}bb^{©} = a^{©}bb^{©},\\
&a^{©}ba^{©} = a^{©}ab^{©} = a^{©}bb^{©},\\
&a^{©}a^{©}b = a^{©}ba^{©} = a^{©}bb^{©}.
\end{aligned}$$

Let $x = a^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©}$. Using the above equations, we can get

$$\begin{aligned}
x(a + b) &= (a^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©})(a + b)\\
&= a^{©}a + a^{©}b + b^{©}a + b^{©}b - \frac{3}{2}a^{©}bb^{©}a - \frac{3}{2}a^{©}bb^{©}b\\
&= a^{©}a + a^{©}b + b^{©}a + b^{©}b - \frac{3}{2}a^{©}b - \frac{3}{2}a^{©}b\\
&= aa^{©} + bb^{©} - a^{©}b\\
&= aa^{©} + bb^{©} - aa^{©}bb^{©} \in C(R).
\end{aligned}$$

Similarly $(a + b)x = aa^{©} + bb^{©} - a^{©}b$. In addition, we have

$$\begin{aligned}
&x(a + b)x\\
&= (a^{©}a + b^{©}b - a^{©}b)(a^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©})\\
&= a^{©} + a^{©}ab^{©} - \frac{3}{2}a^{©}bb^{©} + b^{©}ba^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©} - a^{©}ba^{©} - a^{©}bb^{©} + \frac{3}{2}a^{©}ba^{©}bb^{©}\\
&= a^{©} + a^{©}ab^{©} - \frac{3}{2}a^{©}bb^{©} + a^{©}bb^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©} - a^{©}ba^{©} - a^{©}bb^{©} + \frac{3}{2}a^{©}a^{©}b\\
&= a^{©} + a^{©}bb^{©} - \frac{3}{2}a^{©}bb^{©} + a^{©}bb^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©} - a^{©}bb^{©} - a^{©}bb^{©} + \frac{3}{2}a^{©}b^{©}b\\
&= a^{©} + b^{©} - \frac{3}{2}a^{©}bb^{©}\\
&= x
\end{aligned}$$

and

$$\begin{aligned}
&(a+b)-(a+b)^2x \\
&=(a+b)-(a+b)(aa^{©}+bb^{©}-a^{©}b) \\
&=a+b-aaa^{©}-abb^{©}+aa^{©}b-baa^{©}-bbb^{©}+ba^{©}b \\
&=a+b-a-abb^{©}+abb^{©}-abb^{©}-b+abb^{©} \\
&=0.
\end{aligned}$$

Hence, x is the central group inverse of $a+b$, and $(a+b)^{©}=a^{©}+b^{©}-\frac{3}{2}a^{©}bb^{©}$.

(2) Let $y=a^{©}-b^{©}$. Then we have

$$\begin{aligned}
y(a-b)&=(a^{©}-b^{©})(a-b) \\
&=a^{©}a-a^{©}b-b^{©}a+b^{©}b \\
&=a^{©}a+b^{©}b-2aa^{©}bb^{©}\in C(R).
\end{aligned}$$

Similarly $(a-b)y=aa^{©}+bb^{©}-2a^{©}b$. Also we have

$$\begin{aligned}
y(a-b)y&=(a^{©}a+b^{©}b-2a^{©}b)(a^{©}-b^{©}) \\
&=a^{©}aa^{©}-a^{©}ab^{©}+b^{©}ba^{©}-b^{©}bb^{©}-2a^{©}ba^{©}+2a^{©}bb^{©} \\
&=a^{©}-a^{©}ab^{©}+a^{©}bb^{©}-b^{©}-2a^{©}ab^{©}+2a^{©}bb^{©} \\
&=a^{©}-b^{©} \\
&=y
\end{aligned}$$

and

$$\begin{aligned}
(a-b)-(a-b)^2y&=(a-b)-(a-b)(aa^{©}+bb^{©}-2a^{©}b) \\
&=a-b-aaa^{©}-abb^{©}+2aa^{©}b+baa^{©}+bbb^{©}-2ba^{©}b \\
&=a-b-a-abb^{©}+2abb^{©}+abb^{©}+b-2abb^{©} \\
&=0.
\end{aligned}$$

Hence y is the central group inverse of $a-b$, and $(a-b)^{©}=a^{©}-b^{©}$. □

Remark 4.6. *There exist $a,b\in R$ such that a,b are central group invertible with $abb^{©}=baa^{©}$. For example, let $R=\mathbb{Z}_2S_3$. We take $a=(1)+(123)+(132)$, $b=(1)+(123)$, then we can obtain $a^{©}=(1)+(123)+(132)$ and $b^{©}=(1)+(132)$. After calculation, we have $abb^{©}=baa^{©}=0$.*

Remark 4.7. *From the proof of Theorem 4.5, it is easy to obtain that $abb^{©}=baa^{©}$ implies $ab=ba$. However the reverse is not true in general. For example, let $R=\mathbb{Z}_2S_3$. We take $a=(12)+(13)$, $b=(123)+(132)$, then $a^{©}=(12)+(13)$, $b^{©}=(123)+(132)$. By calculation, we have $ab=ba$, but $abb^{©}\neq baa^{©}$.*

Remark 4.8. *If the condition $2\in R^{-1}$ in Theorem 4.5 is replaced with $2=0$, then it is easy to check that $(a+b)^{©}=a^{©}+b^{©}$.*

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