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A Completeness Theorem for Dissipative Conformable Fractional Sturm-Liouville Operator in Singular Case

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Abstract. This paper is concerned with the singular dissipative conformable fractional Sturm-Liouville operators. A completeness theorem for these operators is proved.

1. Introduction

In a boundary value problem if the differential expression and the coefficients are finite at each point in the defined range, the problem is regular; the problem is named as singular problem if the range is unlimited and at least one of the coefficients increases at least at one point of the range. In regular differential expressions, the boundary conditions are given directly by the values at that point, where as in the singular state these boundary conditions can not be given easily ([1, 9, 13]). Therefore, it is difficult to solve singular problems. Another difficulty that arises in the analysis of singular problems is in which space element will be the solution of the problem. This was overcome by H. Weyl's analysis. Weyl showed that a solution of a differential equation is necessarily quadratic integrable. In case all solutions are quadratic integrable, in case of limit-circle to differential expression; otherwise it is called a limit-point case. In the literature, there are many sufficient conditions for a differential equation to be in the limit-circle or limit-point case ([1, 7-13, 30-32]).

A method used in the analysis of boundary value problems is the operator method. An operator that is compatible with the problem is established and the problem is analyzed. The established operator can be self-adjoint or non-self-adjoint. Dissipative operators constitute an important class of non-self-adjoint operators. All eigenvalues of a dissipative operator are in the closed upper half plane; but this analysis is not sufficient. There are some methods to complete the analysis. Some of these methods are Krein's method, Lidskii's method and Livsic's method. There are analyzes in the literature with these methods ([15]-[29]).

There are more and more applications of spectral problems that do not merge spontaneously. For example, interesting non-classical wavelets can be derived from eigenfunctions and related functions for non-self-adjoint spectral problems. Therefore, such problems are becoming more and more noticeable, in

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particular with regard to the separation of the spectrum and the completeness of eigenfunctions. One of the most important problems for Sturm–Liouville operators is the completeness of the root function system of these operators. Furthermore, creating a basis for the system of root functions is another important issue for these operators. It can be said that there are few studies on the second issue, because it relates to the asymptotic behavior of the eigenvalues of the individual operators and it is difficult to observe this asymptotic behavior in general. Interest in fractional differential equations has been increasing in recent years, and the recently introduced definition of the appropriate fractional derivative includes a limit rather than an integral. Khalil and et al. have redefined the definition of conformable fractional derivative and conformable fractional integral using the classical derivative definition. In their work, Khalil and et. presented linearity condition, product rule, division rule, fractional Rolle theorem and fractional mean value theorem for conformable fractional derivative [3]. Later in [4], Abdeljawad gave the definition of left and right conformable fractional derivatives, the definition of higher order fractional integral, fractional Grönwall inequality, chain rule and partial integration formulas for conformable fractional derivatives, fractional force series expansion and Laplace transformation. Conformable fractional derivative, aims to broaden the definition of classical derivative carrying the natural features of the classical derivative. In addition, with the help of the conformable differential equations obtained by the definition of derivative aims at a new look for differential equation theory [5]. In [6], the researchers have addressed the conformable fractional Sturm–Liouville problem and formulated the self-associated conformable fractional Sturm–Liouville problem for this problem. Then the eigenfunction of the conformable fractional Sturm–Liouville problem was examined by examining the Green function. In [14], Gulsen et al. studied the conformable fractional Sturm–Liouville equation with separated boundary conditions on an arbitrary time scale T and extended some main spectral qualities of the standard Sturm–Liouville equation to the conformable fractional case. In [13], the authors studied the singular conformable sequential equation with distributional potentials. They established Weyl's theory in the frame of conformable derivatives. In the present paper, using Livsic's theorem, we shall show that the system of eigenfunctions and associated functions of a conformable fractional Sturm–Liouville problem with one singular endpoint are complete in $L_\alpha^2(I)$.

2. Preliminaries

In this section, we provide some preliminaries for proving the main results. Let T denote the linear non-selfadjoint operator in the Hilbert space with domain $D(T)$. The element $x \in D(T)$, $x \neq 0$ is called a root vector of T corresponding to the eigenvalue λ if $(T - \lambda I)^m x = 0$ for some $m \in \mathbb{N} := \{1, 2, ...\}$. The root vectors for λ span a linear subspace of $D(T)$, called the root lineal for λ. The algebraic multiplicity of λ is the dimension of its root lineal. If a root vector is not an eigenvector, it is called an associated vector. The completeness of the system of all eigenvectors and associated vectors of T is equivalent to the completeness of the system of all root vectors of this operator. Let T be an arbitrary compact operator acting in the Hilbert space H. Let $\left\{\xi_j(T)\right\}_{j\in\mathbb{N}}$ be a sequence of all nonzero eigenvalues of T arranged by considering algebraic multiplicity and with decreasing modulus, and $\nu(T)$ $(\leq \infty)$ is a sum of algebraic multiplicities of all nonzero eigenvalues of T. If T is a nuclear operator, then $\sum_{j=1}^{\nu(T)} \left|\xi_j(T)\right| < +\infty$ and if T is a Hilbert–Schmidt operator, then $\sum_{j=1}^{\nu(T)} \left|\xi_j(T)\right|^2 < +\infty$. We will denote the class of all nuclear and Hilbert–Schmidt operators in H by σ_1 and σ_2, respectively. If $T \in \sigma_1$, then $\sum_{j=1}^{\nu(T)} \xi_j(T)$ is called the trace of T and is denoted by TrT.

The determinant

$$\det(I - \xi T) = \prod_{j=1}^{\nu(T)} \left[1 - \xi\xi_j(T)\right] \text{ where } T \in \sigma_1$$

where $T \in \sigma_1$ is called the characteristic determinant of T and is denoted by $D_T(\xi)$. $D_T(\xi)$ is an entire function of ξ.

For any $T \in \sigma_2$, the product

$$\widetilde{D}_T(\xi) = \prod_{j=1}^{\nu(T)} \left[1 - \xi\xi_j(T)\right] e^{\xi\xi_j(T)} \tag{1}$$

is also an entire function of ξ, called the regularized characteristic determinant of T.

If the operator $I - \xi T$ has a bounded inverse defined on the whole space H, then the complex number ξ is called an F -regular point (regular in the sense of Fredholm) for T.

Let T_1 and T_2 be linear bounded operators in H and $T_1 - T_2 \in \sigma_1$. If the point ξ is an F-regular point of T_2, then

$$(I - \xi T_1)(I - \xi T_2)^{-1} = I - \xi(T_1 - B)(I - \xi T_2)^{-1},$$

where $\xi(T_1 - T_2)(I - \xi T_2)^{-1} \in \sigma_1$. Consequently, the determinant

$$D_{\frac{T_1}{T_2}}(\xi) = \det\left[(I - \xi T_1)(I - \xi T_2)^{-1}\right]$$

makes sense and is called the determinant of perturbation of the operator T_2 by the operator $K = T_1 - T_2$.

Theorem 2.1 ([2], p.172). *If $T_1,\ T_2 \in \sigma_2,\ T_1 - T_2 \in \sigma_1$ and μ is an F-regular point of T_2, then*

$$D_{\frac{T_1}{T_2}}(\xi) = \frac{\widetilde{D}_{T_1}(\xi)}{\widetilde{D}_{T_2}(\xi)} e^{\xi Tr(T_2 - T_1)}.$$

Definition 2.2. *An operator T is called dissipative if* $\operatorname{Im}(Tx, x) \geq 0$, *for all* $x \in D(T)$.

Theorem 2.3 ([2], p.177). *If T_1 and T_2 are bounded dissipative operators and $T_1 - T_2 \in \sigma_1$, then for any $\beta_0 \in \left(0, \frac{\pi}{2}\right)$, the relation*

$$\lim_{\rho\to\infty} \frac{1}{\rho} \ln\left|D_{\frac{T_1}{T_2}}\left(\rho e^{i\beta}\right)\right| = 0$$

holds uniformly with respect to β in the sector

$$\left\{\lambda : \lambda = \rho e^{i\beta},\ 0 < \rho < \infty,\ \left|\frac{\pi}{2} - \beta\right| < \beta_0\right\}.$$

.

Definition 2.4 (([2])). *Let x be an entire function. If for each $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$, such that*

$$|x(\lambda)| \leq C_\varepsilon e^{\varepsilon|\lambda|},\ \lambda \in \mathbb{C}, \tag{2}$$

then f is called an entire function of order ≤ 1 of growth and minimal type.

From (2), it is clear that

$$\lim_{|\lambda|\to\infty} \sup \frac{\ln|x(\lambda)|}{|\lambda|} \leq 0. \tag{3}$$

It is known that each function x, having properties (2) and $x(0) = -1$, has the representation

$$x(\lambda) = -\lim_{r\to\infty} \prod_{|\lambda_j|\leq r} \left(\frac{\lambda_j - \lambda}{\lambda_j}\right), \tag{4}$$

and also the limit $\lim_{r\to\infty} \prod_{|\lambda_j|\leq r} \frac{1}{\lambda_j}$ exists and is finite ([7]).

Theorem 2.5 (Livšic [2], p. 226). *Let T be compact dissipative operator on H and let $T_{\mathrm{Im}} \in \sigma_1$, where $2iT_{\mathrm{Im}} = T - T^*$. The system of all root vectors of T is complete in H, if and only if*

$$\sum_{j=1}^{\nu(T)} \mathrm{Im}\, \xi_j(T) = spT_{\mathrm{Im}}.$$

Definition 2.6 ([4]). *Assume α be a positive number with $0 < \alpha < 1$. A function $x : (0, \infty) \longrightarrow \mathbb{R} := (-\infty, \infty)$ the conformable fractional derivative of order α of x at $t > 0$ was defined by*

$$T_\alpha x(t) = \lim_{\varepsilon \to 0} \frac{x\left(t + \varepsilon t^{1-\alpha}\right) - x(t)}{\varepsilon}, \tag{5}$$

and the fractional derivative at 0 is defined

$$(T_\alpha x)(0) = \lim_{\varepsilon \to 0^+} T_\alpha x(\varepsilon).$$

Definition 2.7 ([4]). *The conformable fractional integral starting from 0 of a function x of order $0 < \alpha \le 1$ is defined by*

$$(I_\alpha x)(t) = \int_0^t s^{\alpha-1} x(s) ds = \int_0^t x(s) d_\alpha s.$$

Lemma 2.8 ([4]). *Assume that x is a continuous function on $(0, b)$ and $0 < \alpha < 1$. Then, we have*

$$T_\alpha I_\alpha x(t) = x(t),$$

for all $t > 0$.

Theorem 2.9 ([4]). *Let $x, y : [0, b] \to \mathbb{R}$ be two functions such that x and y are conformable fractional differentiable. Then, we have*

$$\int_0^b y(t) T_\alpha(x)(t) d_\alpha t + \int_0^b x(t) T_\alpha(y)(t) d_\alpha t = x(b) y(b) - x(0) y(0).$$

Let $L^2_\alpha(I)$ be the space of all complex-valued functions defined on $I = [0, b)$ such that

$$\|y\| := \sqrt{\int_0^b |y(t)|^2 d_\alpha t} < \infty,$$

where $0 < b \le \infty$. The space $L^2_\alpha(I)$ is a Hilbert space with the inner product

$$\langle x, y \rangle := \int_0^b x(t) \overline{y(t)} d_\alpha t,$$

where $x, y \in L^2_\alpha(I)$.

Theorem 2.10 ([4]). *Let x be a continuous, nonnegative function on an interval I and δ and k be nonnegative constants such that*

$$x(t) \le \delta + k \int_0^t x(s) d_\alpha s, \text{ where } t \in I.$$

Then for all $t \in I$, we have

$$x(t) \le \delta e^{k \frac{t^\alpha}{\alpha}}. \tag{6}$$

3. Main Results

We consider the following conformable fractional Sturm–Liouville problem

$$l[y] = -T_\alpha(p(t)T_\alpha)y(t) + q(t)y(t) = \lambda y, \text{ on } I, \tag{7}$$

where $I = [0, b)$. The coefficients $p(.)$ and $q(.)$ are real-valued functions on I and satisfy the conditions $\frac{1}{p(.)}, q(.) \in L^1_{\alpha,loc}(I)$. Throughout this paper, we assume that the endpoint b is singular.

The maximal operator corresponding to (7)is defined as follows:

$$L_{\max} y := l[y],$$

with

$$D_{\max} := \left\{ y \in L^2_\alpha(I) : y, T_\alpha y \in AC_{loc}(I),\ l[y] \in L^2_\alpha(I) \right\},$$

where $AC_{loc}(I)$ denotes the class of complex-valued functions which are absolutely continuous on all compact sub-intervals of I.

Let

$$D_{\min} := \{ y \in D_{\max} : y(0) = p(0) T_\alpha y(0) = 0,\ [y, \chi](b) = 0 \}.$$

for arbitrary $\chi \in D_{\max}$, where

$$[y, \chi](t) = p(t) \left\{ y(t)\overline{T_\alpha \chi(t)} - T_\alpha y(t)\overline{\chi(t)} \right\},\ t \in I.$$

The operator $L_{\min}$, that is the restriction of the operator $L_{\max}$ to $D_{\min}$ is called the *minimal operator* and the equalities $L_{\max} = L^*_{\min}$ holds. Further, $L_{\min}$ is closed symmetric operator with deficiency indices (d, d), where $d = 1$ or $d = 2$ ([1, 12, 13, 30]).

Theorem 3.1 ([6]). *For $y_1, y_2 \in D_{\max}$, we have the following Green's formula*

$$\int_0^b l[y_1](t)\overline{y_2(t)} d_\alpha t - \int_0^b y_1(t)\overline{l[y_2](t)} d_\alpha t = [y_1, y_2](b) - [y_1, y_2](0). \tag{8}$$

In this study, we will assume that the Weyl's limit-circle case holds for the expressions $l[.]$, i.e., $L_{\min}$ has the deficiency indices $(2, 2)$. In this context, we first give a sufficient condition for the expression $l[.]$ to be limit-circle case. We employed Everitt's method in the following theorem (see [11]).

Theorem 3.2. *Let the coefficients p and q satisfy the following conditions:*

$$\begin{array}{ll} (i) & q \in C(I), T_\alpha p,\ T_\alpha q \in AC_{loc}(I) \text{ and } T^2_\alpha p, T^2_\alpha q \in L^2_{\alpha,loc}(I), \\ (ii) & q(t) < 0 \text{ and } p(t) > 0 \text{ for all } t \in I, \\ (iii) & (-pq)^{-\frac{1}{2}} \in L^2_\alpha(I), \\ (iv) & T_\alpha\{pT_\alpha(pq)(-pq)^{-\frac{1}{2}}\} \in L^2_\alpha(I), \end{array} \tag{9}$$

then $l[.]$ is in the limit-circle case at b.

Proof. This proof is based on the ideas in ([10, 11]). It follows from Green's formula that $\lim_{t\to b^-}[y, z](t)$ exists and is finite for all $y, z \in D_{\max}$. Furthermore, it is known that $l[.]$ is limit-point at b if and only if $\lim_{t\to b^-}[y, z](t) = 0$ for all $y, z \in D_{\max}$ (see ([10])). Thus to establish that $l[.]$ is limit-circle at b it is sufficient to produce one pair y, z of elements of $D_{\max}$ such that

$$\lim_{t\to b^-}[y, z](t) \neq 0. \tag{10}$$

We take $y = z$ and determine y by

$$y(t) = \{-p(t)q(t)\}^{-\frac{1}{2}} \exp\left[i \int_0^t \left\{ -\frac{q(s)}{p(s)} \right\}^{\frac{1}{2}} d_\alpha s \right], \tag{11}$$

where $t \in I$. A calculation shows that

$$T_\alpha y = \left[\frac{i(-q)^{\frac{1}{2}}}{p^{\frac{1}{2}}} + \frac{1}{4} \frac{T_\alpha(pq)}{(-pq)^{\frac{1}{2}}} \right] \exp[...]$$

and

$$T_\alpha^2 y = \left[\begin{array}{c} -\frac{(-q)^{\frac{1}{2}}}{p^{\frac{1}{2}}} + \frac{i}{4} \frac{T_\alpha(pq)}{p^{\frac{1}{2}}(-q)^{\frac{1}{2}}} - \frac{i}{4} \frac{T_\alpha p}{(-pq)^{\frac{1}{2}}} - \frac{3i}{4} \frac{(-q)^{\frac{1}{2}} T_\alpha p}{p^{\frac{1}{2}}} \\ +\frac{1}{4} \frac{T_\alpha^2(qp)}{(-pq)^{\frac{1}{2}}} + \frac{5}{16} \frac{\{T_\alpha(pq)\}^2}{(-pq)^{\frac{1}{2}}} \end{array} \right] \exp[...].$$

From these results we obtain

$$[y, y](t) = -2i, \tag{12}$$

where $t \in I$ and, with details of the calculation omitted,

$$l[y] = -pT_\alpha^2 y - T_\alpha p y + qy = -\frac{1}{4} T_\alpha \{pT_\alpha(pq)(-pq)^{-\frac{1}{2}}\} \exp[...]. \tag{13}$$

From (11) and conditions (ii) and (iii) of the Theorem we see that $y \in L_\alpha^2(I)$; from (11) and condition (i) we have $T_\alpha f \in AC_{\alpha,loc}(I)$; from (13) and condition (iv) that $l[y] \in L_\alpha^2(I)$; thus $y \in D_{\max}$. From (10) and (12) it now follows that the differential expression $l[.]$ is in the limit-circle case at b. $\square$

Let $\phi(t, \lambda)$ and $\psi(t, \lambda)$ two linearly independent solutions in $L_\alpha^2(I)$ of the equation. (7) and satisfy the initial conditions

$$\begin{aligned} &\phi(0, \lambda) = \cos\beta,\ p(0)T_\alpha\phi(0, \lambda) = \sin\beta, \\ &\psi(0, \lambda) = -\sin\beta,\ p(0)T_\alpha\psi(0, \lambda) = \cos\beta, \end{aligned} \tag{14}$$

where $\beta \in \mathbb{R}$. They are entire functions of λ ([6]). Further, they are real functions for real values of λ. Since the operator $L_{\min}$ has the deficiency indices $(2, 2)$, the solutions $\phi(t, \lambda)$ and $\psi(t, \lambda)$ belong to $L_\alpha^2(I)$.

Let $r(t) = \phi(t, 0)$ and $v(t) = \psi(t, 0)$. So $r(t)$ and $v(t)$ are solutions of the equation $l[y] = 0$, satisfying the initial conditions

$$\begin{aligned} &r(0) = \cos\beta,\ p(0)T_\alpha r(0) = \sin\beta, \\ &v(0) = -\sin\beta,\ p(0)T_\alpha v(0) = \cos\beta. \end{aligned}$$

Then, we have $r, v \in L_\alpha^2(I)$; moreover $r, v \in D_{\max}$. Consequently for each $v \in D_{\max}$ the values $[y, r](b)$ and $[y, v](b)$ exist and are finite. Let $D(L)$ denote the set of all functions $y \in D_{\max}$ satisfying the boundary conditions

$$y(0)\cos\beta + p(0)T_\alpha y(0)\sin\beta = 0,\ [y, r](b) - h[y, v](b) = 0, \tag{15}$$

where $h \in \mathbb{C}$ and $\operatorname{Im} h > 0$. In $L_\alpha^2(I)$ we define the operator L with the domain $D(L)$ and $Ly = l[y]$ for all $y \in D(L)$.

Lemma 3.3. $\ker L = \{0\}$.

Proof. Let $y \in D(L)$ and $Ly = 0$. Then, we get

$$-T_\alpha(p(t)T_\alpha)y(t) + q(t)y(t) = 0 \tag{16}$$

and the function y satisfies the boundary condition (15). Therefore, there exists the constants a_1 and a_2 , such that $y(t) = a_1 r(t) + a_2 v(t)$. Substituting this in the boundary conditions (15), we find $a_1 = a_2 = 0$; consequently $y = 0$. □

From Lemma 3.3., we get that there exists the inverse operator L^{-1}. Let us consider the functions $v(t)$ and $u(t) = r(t) - hv(t)$. These functions belong to $L^2_\alpha(I)$. The first satisfies the boundary condition at 0 in (15) and the second at b. The functions $v(t)$ and $u(t)$ form a fundamental system of solutions of (16).

Let Y denote the integral operator defined by the formula

$$Yf = \int_0^\infty G(t,x)f(x)d_\alpha x, \tag{17}$$

where $f \in L^2_\alpha(I)$ and

$$G(t,x) = \begin{cases} v(t)u(x), & 0 \le t \le x \\ v(x)u(t), & x \le t < b. \end{cases} \tag{18}$$

Since

$$\int_0^b \int_0^b |G(t,x)|^2 \, d_\alpha t d_\alpha x < \infty$$

we get that $Y \in \sigma_2$ ([6]). It is easy to verify that $Y = L^{-1}$([6]). Consequently, the root lineals of the operators L and Y coincide and, therefore, the completeness in $L^2_\alpha(I)$. of the system of all eigenvectors and associated vectors of L is equivalent to the completeness of those for Y. Since the algebraic multiplicity of nonzero eigenvalues of a compact operator is finite, each eigenvector of L may have only a finite number of linear independent associated vectors.

Theorem 3.4. *Every nontrivial solution y of (7) in $[0,c]$, $c < b$, and its conformable fractional derivative $T_\alpha y$ are entire functions of λ of order at most $\frac{1}{2}$.*

Proof. Let $v = pT_\alpha y$ then $T_\alpha y = (q-\lambda)y$. Fix λ and let prime $T_{\alpha,t}$ denote conformable fractional differentiation with respect to t. Then

$$T_{\alpha,t}[|\lambda||y|^2 + |v|^2] = T_{\alpha,t}[|\lambda|\overline{y}y + \overline{v}v]$$

$$= |\lambda|(\frac{1}{p}y\overline{v} + \frac{1}{p}v\overline{y}) + \overline{v}(q-\lambda)y + v(\overline{q} - \overline{\lambda})\overline{y}.$$

From this and the elementary inequality

$$2|ab| \le \frac{|\lambda||a|^2 + |b|^2}{\sqrt{|\lambda|}}, \ |\lambda| \ne 0,$$

we get

$$T_{\alpha,t}[|\lambda||y|^2 + |v|^2] \le \frac{|\lambda||y|^2 + |v|^2}{\sqrt{|\lambda|}} \left(|\lambda| \frac{1}{|p|} + |q| + |\lambda| \right)$$

and hence

$$T_{\alpha,t}\left[\log(|\lambda||y|^2 + |v|^2\right] \le \sqrt{|\lambda|}\frac{1}{|p|} + \frac{1}{\sqrt{|\lambda|}}|q| + \sqrt{|\lambda|}.$$

An integration yields

$$\begin{aligned}|\lambda||y(t,\lambda)|^2+|v(t,\lambda)|^2 &\leq Ce^{\frac{1}{\sqrt{|\lambda|}}\int_0^t |q|d_\alpha s}e^{\sqrt{|\lambda|}\int_0^t(\frac{1}{|p|}+1)d_\alpha s}\\ &\leq Be^{M\sqrt{|\lambda|}},\end{aligned}$$

where M and B are positive constants such that

$$0<M=\int_0^t \frac{1}{|p|}+|w|)d_\alpha s<\infty$$

and

$$e^{(\frac{1}{\sqrt{|\lambda|}}\int_0^t |q|)d_\alpha s}<B<\infty.$$

Then, we have

$$\begin{aligned}\left|y(t,\lambda)\right| &\leq Be^{M\sqrt{|\lambda|}},\ 0\leq t\leq c,\ |\lambda|\geq\delta>0,\\ |(pT_\alpha y)(t,\lambda)| &\leq Be^{M\sqrt{|\lambda|}},\ 0\leq t\leq c,\ |\lambda|\geq\delta>0.\end{aligned}$$

This completes the proof. □

Let

$$c_1(\lambda)=[\psi(t,\lambda),r(t)](b),\ c_2(\lambda)=[\psi(t,\lambda),v(t)](b),$$

where $\psi(t,\lambda)$ the solution of (7). It is clear that

$$\sigma_d(L)=\{\lambda\in\mathbb{C}:\ c(\lambda)=0\},$$

where $\sigma_d(L)$ denotes the set of all eigenvalues of L and

$$c(\lambda)=c_1(\lambda)-hc_2(\lambda). \tag{19}$$

Theorem 3.5. *The functions $c_1(\lambda)$ and $c_2(\lambda)$ are entire functions of order ≤ 1 of growth and minimal type.*

Proof. We set

$$c_{b_k,1}(\lambda)=[\psi(t,\lambda),r(t)](b_k),\ c_{b_k,2}(\lambda)=[\psi(t,\lambda),v(t)](b_k),$$

where $b_k\in I$.

It follows from Theorem 3.4. that the functions $\psi(b_k,\lambda)$ and $T_\alpha\psi(b_k,\lambda)$ are entire functions of of order $\frac{1}{2}$ for arbitrary fixed b_k. Hence, the functions a $c_{b_k,1}(\lambda)$ and $c_{b_k,2}(\lambda)$ are entire functions of of order $\frac{1}{2}$. Now we shall prove that the entire function $c_{b_k,,j}(\lambda)$ converges to $c_j(\lambda)$ as $b_k\to b$, uniformly in λ in each compact set of the complex plane $\mathbb{C}$.

Let $y=y(t,\lambda)$ be a solution of (7); then

$$y=[y,v](t)\,r-[y,r](t)\,v. \tag{20}$$

If we define

$$f_1(t,\lambda)=[y,r](t),\ f_2(t,\lambda)=[y,v](t),$$

then following ([9]), we get that $f_1(t,\lambda)$ and $f_2(t,\lambda)$ satisfy a system of first order conformable fractional equations

$$T_{\alpha,t} f_1(t,\lambda) = \lambda y(t,\lambda) r(t),\ T_{\alpha,t} f_2(t,\lambda) = \lambda y(t,\lambda) v(t),\ t \in I.$$

Using (20), we obtain

$$T_{\alpha,t} f(t,\lambda) = \lambda \Omega(t) f(t,\lambda),\ t \in I. \tag{21}$$

$$T_{\alpha,t} f(t,\lambda) = T_{\alpha,t} \begin{bmatrix} f_1(t,\lambda) \\ f_2(t,\lambda) \end{bmatrix} = \begin{bmatrix} \lambda y(t,\lambda) r(t) \\ \lambda y(t,\lambda) v(t) \end{bmatrix}$$

$$= \lambda \begin{bmatrix} [y,v] r^2 - [y,r] vr \\ [y,v] vr - [y,r] v^2 \end{bmatrix} = \lambda \begin{bmatrix} f_2 r^2 - \ f_1 vr \\ f_2 vr - \ f_1 v^2 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} -vr & r^2 \\ -v^2 & rv \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where

$$f(t,\lambda) = \begin{bmatrix} f_1(t,\lambda) \\ f_2(t,\lambda) \end{bmatrix},\ \Omega(t) = \begin{bmatrix} -r(t)v(t) & r^2(t) \\ -v^2(t) & r(t)v(t) \end{bmatrix},$$

and the elements $\Omega(t)$ are in $L^1_\alpha(I)$. For

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

we put $\|w\| = |w_1| + |w_2|$ and the norm of a square 2×2 matrix will be denoted by $\|.\|$. The inclusion $\|\Omega(.)\| \in L^1_\alpha(I)$ holds.

If $y(t,\lambda) = \psi(t,\lambda)$, then the system (21) is equivalent to the integral equation

$$f(t,\lambda) = f(b_k,\lambda) + \lambda \int_{b_k}^{t} \Omega(s) f(s,\lambda) d_\alpha s,\ \ t \in I, \tag{22}$$

where

$$f(b_k,\lambda) = \begin{bmatrix} f_1(b_k,\lambda) \\ f_2(b_k,\lambda) \end{bmatrix} = \begin{bmatrix} [y,r](b_k) \\ [y,v](b_k) \end{bmatrix} = \begin{bmatrix} c_{b_k,1}(\lambda) \\ c_{b_k,2}(\lambda) \end{bmatrix},$$
$$f(0,\lambda) = \begin{bmatrix} f_1(0,\lambda) \\ f_2(0,\lambda) \end{bmatrix} = \begin{bmatrix} [y,r](0) \\ [y,v](0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$
$$f(b,\lambda) = \begin{pmatrix} c_1(\lambda) \\ c_2(\lambda) \end{pmatrix}.$$

From (22) and (6), we find to

$$\left\| f(t,\lambda) \right\| \le \left\| f(b_k,\lambda) \right\| \exp\left(|\lambda| \int_{b_k}^{t} \|\Omega(s)\| \, d_\alpha s \right);$$

hence

$$\left\| f(b,\lambda) - f(b_k,\lambda) \right\| \ \ \le \ \ |\lambda| \exp\left\{ |\lambda| \int_0^b \|\Omega(s)\| \, ds \right\} \int_{b_k}^{b} \|\Omega(s)\| \, d_\alpha s \tag{23}$$

$$\left\| f(b,\lambda) \right\| \ \ \le \ \ \exp\left(|\lambda| \int_{b_k}^{b} \|\Omega(s)\| \, d_\alpha s \right) \left\| f(b_k,\lambda) \right\|. \tag{24}$$

It follows from (23) that $c_{b_k,j}(\lambda)$ converges to $c_j(\lambda)$ as $b_k \to b$ ($k \to \infty$), uniformly in λ in a compact set. Consequently $c_{,j}(\lambda)$, $j = 1, 2$, are entire functions.

For $b_k = 0$, from (24) we get

$$\left\| f(b,\lambda) \right\| \leq \exp\left(|\lambda| \int_0^b \|\Omega(x)\| \, d_\alpha x \right);$$

hence $c_{,j}(\lambda)$ are of not higher than the first order. Since, for arbitrary fixed b, the functions $c_{b,j}(\lambda)$, $j = 1, 2$, are entire functions of λ of order $\frac{1}{2}$, from (24) we obtain that the entire functions $c_{,j}(\lambda)$, $j = 1, 2$, are of minimal type. □

Using Green's formula (8), we have

$$c_1(\lambda) = [\psi(t,\lambda), r(t)](b) = -1 + \lambda \int_0^b \psi(t,\lambda) r(t) d_\alpha t \tag{25}$$

$$c_2(\lambda) = [\psi(t,\lambda), v(t)](b) = \lambda \int_0^b \psi(t,\lambda) v(t) d_\alpha t. \tag{26}$$

From (19), (25) and (26) we find that $c(0) = -1$.

We will use the well-known formula

$$[y_1, y_2](t) = [y_1, v](t)[r, y_2](t) - [y_1, r](t)[v, y_2](t), \; t \in I, \tag{27}$$

where $y_1, y_2 \in D_{\max}$.

Theorem 3.6. *The operator L is dissipative.*

Proof. If $y \in D(L)$, then by the formula (8) we get

$$\langle Ly, y \rangle - \langle y, Ly \rangle = [y, y](b) - [y, y](0). \tag{28}$$

From the boundary condition (15) we have $[y, r](b) = h[y, v](b)$ and $[y, y](0) = 0$. From (27) we obtain

$$\begin{aligned}[y, y](b) &= [y, v](b)[r, y](b) - [y, r](b)[v, y](b) \\ &= -\overline{h}|[y, v](b)|^2 + h|[y, v](b)|^2 = 2i(\operatorname{Im} h)|[y, v](b)|^2\end{aligned}$$

and this proves

$$\operatorname{Im}\langle Ly, y \rangle = (\operatorname{Im} h)|[y, v](b)|^2 \geq 0.$$

□

Since $u(t) = r(t) - hv(t)$, setting $h = h_1 + ih_2$ we get from (17) in view of (18) that $Y = Y_1 + iY_2$, where

$$Y_1 f = \int_0^b G_1(t,x) f(t) d_\alpha t, \; Y_2 f = \int_0^b G_2(t,x) f(t) d_\alpha t$$

and

$$G_1(t,x) = \begin{cases} v(t)[r(x) - hv(x)], & 0 \leq t \leq x \\ v(x)[r(t) - hv(t)], & x \leq t \leq b, \end{cases}$$

$$G_2(t,x) = -h_2 v(t) v(x), \; h_2 = \operatorname{Im} h > 0.$$

Then we get

$$\langle Y_2 f, f\rangle = \langle \int_0^b G_2(t,x) f(t) d_\alpha t, f\rangle$$

$$= \int_0^b \left\{ \left(\int_0^b G_2(t,x) f(t) d_\alpha t \right) \overline{f(x)} \right\} d_\alpha x$$

$$= -h_2 \int_0^b \left\{ \left(\int_0^b v(t) v(x) f(t) \right) \overline{f(x)} \right\} d_\alpha t$$

$$= -h_2 \int_0^b v(x) f(x) d_\alpha x \int_0^b v(t) f(t) d_\alpha t$$

$$= -h_2 \left| \int_0^b v(t) f(t) d_\alpha t \right|^2 \leq 0$$

The operator Y_1 is the self-adjoint Hilbert –Schmidt operator in $L^2_\alpha(I)$ and Y_2 is the self-adjoint one-dimensional operator in $L^2_\alpha(I)$, and $(Y_2 f, f) \leq 0$ for all $f \in L^2_\alpha(I)$.

Let L_1denote the operator generated in $L^2_\alpha(I)$ by the expression $l[y]$ and the boundary conditions

$$\begin{aligned} y(0)\cos\beta + p(0) T_\alpha y(0) \sin\beta &= 0, \\ [y, r](b) - h_1 [y, v](b) &= 0. \end{aligned}$$

It is easy to verity that Y_1 is the inverse of $L_1 : Y_1 = L_1^{-1}$. Let $Z = -Y$ and $Z = Z_1 + iZ_2$, where $Z_1 = -Y_1,\ Z_2 = -Y_2$.

We will denote by λ_j and δ_k the eigenvalues of the operators L and L_1, respectively. Then the eigenvalues of Z are $\left(\frac{-1}{\lambda_j}\right)$ and eigenvalues at Z_1 are $\left(\frac{-1}{\delta_k}\right)$. Since L_1 is a self-adjoint operator, therefore $\operatorname{Im}\delta_k = 0$ for all k.

Theorem 3.7. $spZ_2 = \sum_j \operatorname{Im}\left(-\frac{1}{\lambda_j}\right)$.

Proof. Using Theorem 2.1 for $A = T_1$ and $B = T$ we obtain

$$D_{\frac{Z_1}{Z}}(\xi) = \frac{\tilde{D}_{Z_1}(\xi)}{\tilde{D}_Z(\xi)} e^{i\xi\ TrZ_1}, \tag{29}$$

and by (1) we get

$$\tilde{D}_Z(\xi) = \prod_j \frac{\left(\frac{\lambda_j+\xi}{\lambda_j}\right)}{e^{\frac{\xi}{\lambda_j}}},\ \tilde{D}_{Z_1}(\xi) = \prod_k \frac{\left(\frac{\delta_k+\xi}{\delta_k}\right)}{e^{\frac{\xi}{\delta_k}}}.$$

We set

$$c(\xi) = c_1(\xi) - hc_2(\xi),\ d(\xi) = c_1(\xi) - h_1 c_2(\xi),$$

where the functions $c_1(\xi)$ and $c_2(\xi)$ are given by (25) and (26). The eigenvalues of Y and Y_1 coincide with the root of the functions $c(\xi)$ and $d(\xi)$, respectively. By Theorem 3.5, the functions $c(\xi)$ and $d(\xi)$ are entire functions of order ≤ 1 of growth and minimal type and $c(0) = d(0) = -1$; thus, by (4), we have

$$c(\xi) = -\prod_j \left(\frac{\lambda_j + \xi}{\lambda_j}\right), \; d(\xi) = -\prod_k \left(\frac{\delta_k - \xi}{\delta_k}\right).$$

Then we get

$$\tilde{D}_Z(\xi) = -c(-\xi)e^{-\xi \sum_j (\frac{1}{\lambda_j})}, \; \tilde{D}_{Z_1}(\xi) = -d(-\xi)e^{-\xi \sum_k (\frac{1}{\delta_k})}.$$

It follows from (29) that

$$D_{\frac{Z_1}{Z}}(\xi) = \frac{d(-\xi)}{a(-\xi)} e^{\left(\xi \sum_j \frac{1}{\lambda_j} - \xi \sum_k \frac{1}{\delta_k} + i\xi TrZ_2\right)}.$$

If we take $\xi = it$ (where $0 < t < \infty$), then we get

$$D_{\frac{Z_1}{Z}}(\xi)(it) = \frac{d(-it)}{a(-it)} \exp\left(\xi \sum_j \frac{1}{\lambda_j} - \xi \sum_k \frac{1}{\delta_k} + i\xi TrZ_2\right)$$

and

$$\begin{aligned} &\frac{1}{t}\log\left|D_{\frac{Z_1}{Z}}(\xi)(it)\right| \\ &= \frac{1}{t}\log|d(-it)| - \frac{1}{t}\log|c(-it)| - \sum_j \operatorname{Im}\frac{1}{\lambda_j} - spZ_2. \end{aligned} \tag{30}$$

From Theorem 2.3 and (3) we get

$$\lim_{t\to\infty} \frac{1}{t}\log\left|D_{\frac{Z_1}{Z}}(it)\right| = 0, \tag{31}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log|c(-it)| \leq 0, \; \limsup_{t\to\infty} \frac{1}{t}\log|d(-it)| \leq 0. \tag{32}$$

On the other hand, for $t > 0$,

$$\left|\frac{\lambda_j + it}{\lambda_j}\right|^2 \geq 1, \; \left|\frac{\delta_k + it}{\delta_k}\right|^2 \geq 1,$$

and we have $|d(-it)| \geq 1$, $|c(-it)| \geq 1$ for all $t > 0$. Consequently, we have

$$\frac{1}{t}\log|c(-it)| \geq 0, \; \frac{1}{t}\log|d(-it)| \geq 0.$$

It follows from (32) that

$$\lim_{t\to\infty} \frac{1}{t}\log|a(-it)| = \lim_{t\to\infty} \frac{1}{t}\log|d(-it)| = 0. \tag{32}$$

By (30), (31) and (32), we deduce that

$$\sum_j \operatorname{Im}\left(-\frac{1}{\lambda_j}\right) = spZ_2.$$

□

Thus the operator Z carries out all the conditions of Theorem 2.5. Hence we have

Theorem 3.8. *The system of all root vectors of the dissipative operator Z (also of Y) is complete in $L^2_\alpha(I)$.*

Since the completeness in $L^2_\alpha(I)$ of the system of all root vectors of the dissipative operator Z is equivalent to the completeness of L, we have

Theorem 3.9. *The system of all eigenvectors and associated vectors (or root vectors) of the dissipative operator L is complete in $L^2_\alpha(I)$.*

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