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Stochastic Helmholtz Problem and Convergence in Distribution

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Abstract. In the present paper, the solvability of the stochastic Helmholtz problem is investigated in the class of stochastic differential equations equivalent in distribution. Earlier, by additional variables method the Helmholtz problem was investigated in the class of stochastic differential equations equivalent almost surely (a.s.). The study of the stochastic Helmholtz problem in the class of equations equivalent in distribution allows us to significantly expand the region of its solvability. This is due to the possibility of using well-known methods of the theory of stochastic processes, such as the method of the phase space transformation, the method of absolutely continuous change of measure, and the method of random change of time. In that paper stochastic equations of the Lagrangian structure equivalent in distribution are constructed by the given second order Ito stochastic equations using the methods of phase space transformation, absolutely continuous measure transformation and random time substitution. The obtained results are illustrated by specific examples.

1. Introduction

The theory of dynamics inverse problems has been quite fully developed for ordinary differential equations (ODEs) [1-9]. This theory goes back to the work by Yerugin [10], in which a set of ODEs is constructed given an integral curve. In [1,2], Galiullin proposed a classification of the main types of dynamics inverse problems for ODEs and developed general methods for their solving. In recent decades, growing interest in the Helmholtz problem [11] has led to a new phase in studying inverse problems for differential systems. The dynamics inverse problems related to the Helmholtz problem are described in [12]. The classical Helmholtz problem consists in constructing equivalent differential equations in the Lagrange form by the given second-order ODEs. Mayer [13] and Suslov [14] showed independently that the classical Helmholtz conditions are not only necessary but also sufficient for the transition from the Newtonian equations to equivalent Lagrangian ones. The solvability of dynamics inverse problems for Ito stochastic differential equations is studied in [15-19]. Solving the Helmholtz problem in a wider class of differential equations allows one to extend the well-developed mathematical methods of classical mechanics to this

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class. Moreover, since the Helmholtz problem is also an inverse problem of calculus of variations, it is often useful to replace the problem of finding the solution of a differential equation with an equivalent problem of finding the extremals of a functional constructed from this equation. Of special note in this regard is Santilli's two-volume monograph [20,21] that provides a variety of aspects in the study of the Helmholtz problem. The monograph is devoted to the problem of representing a second-order ODE in the form of the Lagrange, Hamilton and Birkhoff equations. In [22-24], methods for solving the Helmholtz problem are developed for partial differential equations (PDEs). In [20,21,25], the authors present their studies on the Helmholtz problem, mainly for ODEs and PDEs, as well as a historical overview of the development and generalization of the problem. In contrast to this paper, the stochastic Helmholtz problem is solved in the class of stochastic differential equations equivalent a.s. [26].

Let us consider the following equations:

$$d\dot{y} = Y_1(y, \dot{y}, t)dt + Y_2(y, \dot{y}, t)d\xi, \tag{a}$$

$$d\dot{z} = Z_1(z, \dot{z}, t)dt + Z_2(z, \dot{z}, t)d\xi \tag{b}$$

Definition 1.1. [27] *We say that equations (a) and (b) are equivalent a.s. if , for all* $t \geq t_0$, $y(t_0) = z(t_0)$, $\dot{y}(t_0) = \dot{z}(t_0)$ *a.s. implies* $y(t, t_0, y_0, \dot{y}_0) = z(t, t_0, z_0, \dot{z}_0)$, $\dot{y}(t, t_0, y_0, \dot{y}_0) = \dot{z}(t, t_0, z_0, \dot{z}_0)$ *a.s.*

The equivalence of equations (a) and (b) in distribution is understood in the following sense.

Definition 1.2. [27] *Equations (a) and (b) are d-equivalent (or equivalent in distribution) if for* $(y(t_0)^T, \dot{y}(t_0)^T)^T$ *and* $(z(t_0)^T, \dot{z}(t_0)^T)^T$ *, with the same initial distributions in* R^{2n}*, the distribution laws of the processes* $(y(t)^T, \dot{y}(t)^T)^T$ *and* $(z(t)^T, \dot{z}(t)^T)^T$ *coincide in the space* $W^{2n} = C([0, \infty) \to R^{2n})$.

Following R.M. Santilli [20], we present some concepts, necessary in the sequel , that take into account the action of random perturbing forces.

Definition 1.3. *The equation*

$$d\dot{x}_\nu = F_\nu(x, \dot{x}, t)dt + \sigma_{\nu j}(x, \dot{x}, t)d\xi^j, \ \left(\nu = \overline{1, n}, \quad j = \overline{1, r}\right) \tag{1}$$

is called the kinematic form of Newton's equation in the presence of random perturbations.

The equation

$$A_{\nu i}(x, \dot{x}, t)d\dot{x}_i + B_\nu(x, \dot{x}, t)dt = \sigma_{\nu j}(x, \dot{x}, t)d\xi^j, \ \left(\nu = \overline{1, n}, \quad j = \overline{1, r}\right) \tag{2}$$

is called the basic form of Newton's equation in the presence of random perturbations.

The equation

$$d\left(\frac{\partial L}{\partial \dot{x}_k}\right) - \frac{\partial L}{\partial x_k}dt = \sigma'_{kj}(x, \dot{x}, t)d\xi^j, \ \left(k = \overline{1, n}, \quad j = \overline{1, r}\right) \tag{3}$$

is called the stochastic equation of the Lagrangian structure.

In what follows, we suppose that the functions included in the above equations are necessarily smooth and satisfy the existence and uniqueness theorem for the Cauchy problem in the class of Ito stochastic differential equations [27].

In particular, following [27], we assume that in (1) the vector function $F(z, t)$ and the matrix $\sigma(z, t)$:

(i) are continuous in t and satisfy the Lipschitz condition with respect to the variable z

$$\|\sigma(z', t) - \sigma(z'', t)\|^2 + \|F(z', t) - F(z'', t)\|^2 \leq L\left(1 + \|z' - z''\|^2\right) \ \text{for all } z', z'' \in R^{2n},$$

(ii) the conditions for linear growth

$$\|\sigma(z,t)\|^2 + \|F(z,t)\|^2 \leq L\left(1 + \|z\|^2\right)$$

are satisfied for all $z \in R^{2n}$, where $z = \left(x^T, \dot{x}^T\right)^T$.

Let (Ω, U, P) be a probability space with a flow $\{U_t\}$. Here $\{\xi_1(t,\omega), \ldots, \xi_r(t,\omega)\}$ is a system of random processes with independent increments that can be represented as a sum of processes [28]: $\xi = \xi_0 + \int c(y)P^0(t,dy)$, where $\xi = (\xi_1(t,\omega), \ldots, \xi_r(t,\omega))^T$, ξ_0 is a vector Wiener process, P^0 is a Poisson process, $P^0(t,dy)$ is the number of jumps of the process P^0 in the interval $[0,t]$ that fall onto the set dy; $c(y)$ is the vector function that maps the space R^{2n} to the value space R^k of the process $\xi(t)$ for all t.

Equations (1), (2), (3) are the second-order Ito stochastic equations. By $d\xi^j$ we mean a differential in the Ito sense [29].

This problem was considered in [20] in the absence of random perturbations ($\sigma_{\nu j} \equiv \sigma'_{\nu j} \equiv 0$). Here and below, we assume summation over repeated indices of the factors. The indices i, k and ν range from 1 to n and the index j ranges from 1 to r. We exclude from consideration the case of degenerate equations and assume that $\det(A_{\nu i}) \neq 0$ in equation (2).

Equations (1) or (2) describe the models of mechanical systems that account for the effect of external random forces and are of great practical importance. Here are some examples: the motion of an artificial Earth satellite under the action of gravitational and aerodynamic forces [30], the fluctuation drift of a heavy gyroscope in a gimbal [31], etc.

2. Problem statement

Given the second-order system of Ito stochastic equations of the form (1), reduce the system to equivalent equations of the Lagrangian structure.

The study of the stochastic Helmholtz problem in the class of equations equivalent in distribution allows us to significantly expand the region of its solvability. This is due to the possibility of using well-known methods of the theory of stochastic processes, such as the method of the phase space transformation, the method of absolutely continuous change of measure, and the method of random change of time.

The method of the phase space transformation. We consider the transformation of the phase space in terms of velocities

$$\dot{\tilde{x}}_i = \varphi_i(x, \dot{x}, t) \tag{4}$$

under the assumption that $\varphi_i \in C^{1\,2\,1}_{x\,\dot{x}\,t}$ and there is an inverse transformation $\psi_\nu = \psi_\nu(x, \dot{\tilde{x}}, t) \in C^{1\,2\,1}_{x\,\dot{x}\,t}$ such that

$$\dot{x}_\nu = \psi_\nu(x, \dot{\tilde{x}}, t). \tag{5}$$

Using the Ito stochastic differentiation rule [28], we calculate the differential

$$d\dot{\tilde{x}}_i = \frac{\partial \varphi_i}{\partial t}dt + \frac{\partial \varphi_i}{\partial x_k}dx_k + \frac{\partial \varphi_i}{\partial \dot{x}_k}d\dot{x}_k + \frac{1}{2}\sigma_{lj}\sigma_{kj}\frac{\partial^2 \varphi_i}{\partial \dot{x}_l \partial \dot{x}_k}dt + \frac{\partial \varphi_i}{\partial \dot{x}_k}\sigma_{kj}d\xi.$$

Hence, we obtain the relation

$$d\dot{\tilde{x}}_i = \tilde{F}_i(x, \dot{\tilde{x}}, t)dt + \tilde{\sigma}_{ij}(x, \dot{\tilde{x}}, t)d\xi^j, \tag{6}$$

where $\tilde{F}_i = \left[\frac{\partial \varphi_i}{\partial t} + \frac{\partial \varphi_i}{\partial x_k}\dot{x}_k + \frac{\partial \varphi_i}{\partial \dot{x}_k}F_k + \frac{1}{2}\sigma_{lj}\sigma_{kj}\frac{\partial^2 \varphi_i}{\partial \dot{x}_l \partial \dot{x}_k}\right]\Bigg|_{\dot{x}_\nu = \psi_\nu(x,\dot{\tilde{x}},t)}$, $\tilde{\sigma}_{ij} = \frac{\partial \varphi_i}{\partial \dot{x}_\nu}\sigma_{\nu j}\Big|_{\dot{x}_\nu = \psi_\nu(x,\dot{\tilde{x}},t)}$.

Further, we assume that the functions φ_i for the given F_i and σ_{ij} are such that in the space $(x, \dot{\tilde{x}})$, the following relations (Helmholtz conditions [20]) hold with respect to variables $\tilde{F}_i$ and $\tilde{\sigma}_{ij}$:

$$\frac{\partial \tilde{F}_\nu}{\partial \dot{\tilde{x}}_\mu} + \frac{\partial \tilde{F}_\mu}{\partial \dot{\tilde{x}}_\nu} = 0, \tag{7}$$

$$\frac{\partial^2 \tilde{F}_\nu}{\partial \dot{\tilde{x}}_\mu \partial \dot{\tilde{x}}_k} - \frac{\partial^2 \tilde{F}_\mu}{\partial \dot{\tilde{x}}_\nu \partial \dot{\tilde{x}}_k} = 0, \tag{8}$$

$$\frac{\partial \tilde{F}_\nu}{\partial \dot{\tilde{x}}_\mu} - \frac{\partial \tilde{F}_\mu}{\partial \dot{\tilde{x}}_\nu} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{\tilde{x}}_k \frac{\partial}{\partial x_k}\right\}\left(\frac{\partial \tilde{F}_\nu}{\partial \dot{\tilde{x}}_\mu} - \frac{\partial \tilde{F}_\mu}{\partial \dot{\tilde{x}}_\nu}\right), \tag{9}$$

Conditions (7)-(9) ensure the existence of some $\tilde{L} = \tilde{L}(x, \dot{\tilde{x}}, t)$ and the transition from equations (6) to equations of the Lagrangian structure

$$d\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{x}}_i}\right) - \frac{\partial \tilde{L}}{\partial x_i} dt = \tilde{\sigma}_{ij}(x, \dot{\tilde{x}}, t) d\xi^j. \tag{10}$$

Theorem 2.1. *Let there exist a transformation* (4) $\varphi_i = \varphi_i(x, \dot{x}, t) \in C^{1\,2\,1}_{x\,\dot{x}\,t}$ *that has the inverse* (5) $\psi_\nu = \psi_\nu(x, \dot{x}, t) \in C^{1\,2\,1}_{x\,\dot{x}\,t}$ *and satisfies conditions* (7)-(9). *Then equation* (1) *defined in the space* $(x, \dot{x})$ *is equivalent in distribution to the equation of the Lagrangian structure* (10) *in the space* $(x, \dot{\tilde{x}})$.

Remark 2.2. *Theorem* (2.1) *is convenient to use when the functions* F_i *in the problem of direct representation do not satisfy the Helmholtz conditions [20] in the space* $(x, \dot{x})$. *In the case when the classical Helmholtz conditions are satisfied, it is sufficient to set* $\varphi_i \equiv \dot{x}_i$ *so that conditions* (7) - (9) *coincide with the Helmholtz conditions in the space*$(x, \dot{x})$.

The method of absolutely continuous change of measure.

Following the Girsanov method [27], we consider the transformation of the drift coefficient.

Assume that (Ω, U_t, P) is an initial probability space, U_t is some flow of σ-algebras, and $\xi^j(s)$ is the Wiener process with respect to U_t. If $\phi_t(\omega)$ is some non-negative functional, measurable with respect to U_t, such that $M\phi_t(\omega) = 1$, then we can consider a new probability measure $\hat{P}(A) = \int_A \phi_t(\omega) P(d\omega)$ on (Ω, U_t).

Further, if $\phi_t(x, \dot{x}, t) = exp\left\{\int_0^t b_j(x, \dot{x}, t) d\xi^j(s) - \frac{1}{2}\int_0^t |b_j(x, \dot{x}, s)|^2 ds\right\}$, then, by the Girsanov theorem [27], the process

$$\hat{\xi}^j(t) = \xi^j(t) - \int_0^t b_j(x, \dot{x}, s) ds, \tag{11}$$

is the Wiener process with respect to the flow of σ-algebras$\{U_t\}$ on the probability space $(\Omega, U_t, \hat{P})$.

Let us apply the Girsanov transformation (11) to equation (1). Differentiating (11), we have $d\xi^j(t) = d\hat{\xi}^j(t) + b_j(x, \dot{x}, t)dt$. Then, upon transition from (Ω, U_t, P) to $(\Omega, U_t, \hat{P})$, equation (1) takes the form

$$d\dot{x}_i = \hat{F}_i(x, \dot{x}, t)dt + \sigma_{kj}(x, \dot{x}, t) d\hat{\xi}^j, \tag{12}$$

where $\hat{F}_i = F_i + \sigma_{ij}(x, \dot{x}, t) b_j(x, \dot{x}, t)$.

Let us now suppose that for given F_i and σ_{ij} the functions $b_j(x, \dot{x}, t)$ are such that the Helmholtz conditions

$$\frac{\partial \hat{F}_\nu}{\partial \dot{x}_\mu} + \frac{\partial \hat{F}_\mu}{\partial \dot{x}_\nu} = 0, \tag{13}$$

$$\frac{\partial^2 \hat{F}_\nu}{\partial \dot{x}_\mu \partial \dot{x}_k} - \frac{\partial^2 \hat{F}_\mu}{\partial \dot{x}_\nu \partial \dot{x}_k} = 0, \tag{14}$$

$$\frac{\partial \hat{F}_\nu}{\partial \dot{x}_\mu} - \frac{\partial \hat{F}_\mu}{\partial \dot{x}_\nu} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{x}_k \frac{\partial}{\partial x_k}\right\}\left(\frac{\partial \hat{F}_\nu}{\partial \dot{x}_\mu} - \frac{\partial \hat{F}_\mu}{\partial \dot{x}_\nu}\right) \tag{15}$$

hold with respect to the variables $\hat{F}_i$. These conditions ensure the existence of a certain Lagrangian $\hat{L} = \hat{L}(x, \dot{x}, t)$ and the transition from (12) to the equivalent equations of the Lagrangian structure

$$d\left(\frac{\partial \hat{L}}{\partial \dot{x}_i}\right) - \frac{\partial \hat{L}}{\partial x_i} dt = \sigma_{ij}(x, \dot{x}, t) d\hat{\xi}^j. \tag{16}$$

Theorem 2.3. *Let there exist the Girsanov transformation* (11) *such that for given F_i and σ_{ij} conditions* (13) - (15) *are satisfied. Then the given equation* (1) *is equivalent in distribution to the equation of the Lagrangian structure* (16).

Remark 2.4. *In Theorem* (2.3), *conditions* (13) - (15) *are the Helmholtz conditions for the functions $\hat{F}_i$ in equation* (12). *Suppose that the Helmholtz conditions for these functions are not satisfied, i.e.the following relations hold:*

$$\frac{\partial \hat{F}_\nu}{\partial \dot{x}_\mu} + \frac{\partial \hat{F}_\mu}{\partial \dot{x}_\nu} = \alpha_{\nu\mu} \neq 0, \tag{17}$$

$$\frac{\partial^2 \hat{F}_\nu}{\partial \dot{x}_\mu \partial \dot{x}_k} - \frac{\partial^2 \hat{F}_\mu}{\partial \dot{x}_\nu \partial \dot{x}_k} = \beta_{\nu\mu k} \neq 0, \tag{18}$$

$$\frac{\partial \hat{F}_\nu}{\partial \dot{x}_\mu} - \frac{\partial \hat{F}_\mu}{\partial \dot{x}_\nu} - \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{x}_k \frac{\partial}{\partial x_k}\right\}\left(\frac{\partial \hat{F}_\nu}{\partial \dot{x}_\mu} - \frac{\partial \hat{F}_\mu}{\partial \dot{x}_\nu}\right) = \gamma_{\nu\mu k} \neq 0, \tag{19}$$

Then, taking into account that $\hat{F}_i = F_i + \sigma_{ij} b_j$, the conditions (17) - (19) *take the form*

$$\alpha_{\nu\mu} + \frac{\partial \sigma_{\nu j} b_j}{\partial \dot{x}_\mu} + \frac{\partial \sigma_{\mu j} b_j}{\partial \dot{x}_\nu} = 0, \tag{20}$$

$$\beta_{\nu\mu k} + \frac{\partial^2 \sigma_{\nu j} b_j}{\partial \dot{x}_\mu \partial \dot{x}_k} - \frac{\partial^2 \sigma_{\mu j} b_j}{\partial \dot{x}_\nu \partial \dot{x}_k} = 0, \tag{21}$$

$$\gamma_{\nu\mu k} + \frac{\partial \sigma_{\nu j} b_j}{\partial \dot{x}_\mu} - \frac{\partial \sigma_{\mu j} b_j}{\partial \dot{x}_\nu} - \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{x}_k \frac{\partial}{\partial x_k}\right\}\left(\frac{\partial \sigma_{\nu j} b_j}{\partial \dot{x}_\mu} - \frac{\partial \sigma_{\mu j} b_j}{\partial \dot{x}_\nu}\right) = 0. \tag{22}$$

Thus, Theorem (2.3) implies the following statement.

Corollary 2.5. *Let the diffusion matrix* $(\sigma_{\nu j})$ *and the Girsanov transformation functions* b_j *satisfy relations* (20) - (22), *in which the functions* $\alpha_{\nu\mu}, \beta_{\nu\mu k}$ *and* $\gamma_{\nu\mu k}$ *are determined from* (17) - (19). *Then the given equation* (1) *is equivalent in distribution to the equation of the Lagrangian structure* (16).

Remark 2.6. *It follows from Corollary* (2.5) *that the possibility of analytically constructing the Lagrangian and, therefore, the equivalent equation of the Lagrangian structure in the class of stochastic equations, in addition to the drift coefficient* F_i, *substantially depends on the diffusion matrix* $(\sigma_{\nu j})$.

The method of random change of time. Let us consider the time conversion

$$t = \int_0^\tau v(y, \dot{y}, s) ds \tag{23}$$

or $\frac{d\tau}{dt} = \frac{1}{v(y,\dot{y},\tau)}$. Then, by [27], equation (1) is equivalent to the equation

$$d\dot{x} = \overline{F}_i(x, \dot{x}, \tau) d\tau + \overline{\sigma}_{ij}(x, \dot{x}, \tau) d\xi^j, \tag{24}$$

here $\overline{F}_i = \frac{F_i(x,\dot{x},\tau)}{v(x,\dot{x},\tau)}, \quad \overline{\sigma}_{ij} = \frac{\sigma_{ij}(x,\dot{x},\tau)}{\sqrt{v(x,\dot{x},\tau)}}$.

We assume that for given F_i and σ_{ij} the function v satisfies the Helmholtz conditions with respect to the variables $\overline{F}_i$:

$$\frac{\partial \overline{F}_\nu}{\partial \dot{x}_\mu} + \frac{\partial \overline{F}_\mu}{\partial \dot{x}_\nu} = 0, \tag{25}$$

$$\frac{\partial^2 \overline{F}_\nu}{\partial \dot{x}_\mu \partial \dot{x}_k} - \frac{\partial^2 \overline{F}_\mu}{\partial \dot{x}_\nu \partial \dot{x}_k} = 0, \tag{26}$$

$$\frac{\partial \overline{F}_\nu}{\partial \dot{x}_\mu} - \frac{\partial \overline{F}_\mu}{\partial \dot{x}_\nu} = \frac{1}{2}\left\{\frac{\partial}{\partial \tau} + \dot{x}_k \frac{\partial}{\partial x_k}\right\}\left(\frac{\partial \overline{F}_\nu}{\partial \dot{x}_\mu} - \frac{\partial \overline{F}_\mu}{\partial \dot{x}_\nu}\right). \tag{27}$$

These conditions guarantee the existence of some $\overline{L} = \overline{L}(x, \dot{x}, \tau)$ and the transition from equations (24) to the equations of the Lagrangian structure

$$d\left(\frac{\partial \overline{L}}{\partial \dot{x}_i}\right) - \frac{\partial \overline{L}}{\partial x_i} d\tau = \overline{\sigma}_{ij}(x, \dot{x}, \tau) d\xi^j. \tag{28}$$

Theorem 2.7. *Let the random time transformation* (23) *be such that conditions* (25) - (27) *are satisfied. Then the given equation* (1) *is equivalent in distribution to the equation of the Lagrangian structure* (28).

3. Examples

Let us solve the Helmholtz problem for specific equations using the proven statements.

Example 1. Let us consider the plane motion of a symmetric satellite in a circular orbit. We assume that a change in pitch occurs under the action of gravitational and aerodynamic forces [30]

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \sigma(\theta, \dot{\theta})\dot{\xi}, \tag{29}$$

where θ is a pitch angle, and the functions f and σ are of the form

$$f = Ql\sin 2\theta - Q[g(\theta) + \eta\dot{\theta}], \;\; \sigma = Q\delta[g(\theta) + \eta\dot{\theta}]. \tag{30}$$

1) Let the phase space transformation (4) have the form $\dot{\tilde{\theta}} = \varphi(\theta, \dot{\theta}) = \dot{\theta} - a\theta$, here a is an unknown coefficient. Using the Ito's rule of stochastic differentiation and expressions (30), we find the coefficients $\tilde{f}$ and $\tilde{\sigma}$ of the transformed equation $\ddot{\tilde{\theta}} = \tilde{f}(\theta, \dot{\theta}) + \tilde{\sigma}(\theta, \dot{\theta})\dot{\xi}$. Then, from the requirement $\partial\tilde{f}/\partial\dot{\tilde{\theta}} = 0$ (the Helmholtz condition in the one-dimensional case [20]), we determine $a = -Q\eta$. As a result, equation (29) is transformed to the form

$$\ddot{\tilde{\theta}} = Q\,l\,sin2\theta - Qg(\theta) + Q\delta[g(\theta) + \eta(\dot{\tilde{\theta}} - Q\eta\theta)]\dot{\xi}. \tag{31}$$

We will seek the Lagrange function in the form $L = \frac{1}{2}\dot{\tilde{\theta}}^2 + W(\theta) = \frac{1}{2}(\dot{\theta} + Q\eta\theta)^2 + W(\theta)$ and determine $W(\theta)$ so that (31) is equivalent in distribution to the equation of Lagrangian structure.

A simple computation shows that the stochastic equation of the Lagrangian structure with given L is equivalent to the equation $\ddot{\tilde{\theta}} = Q^2\eta^2\theta + W_\theta + \sigma'\dot{\xi}$. Hence, comparing the obtained equation and equation (31), we determine $W(\theta)$ as $W = -\frac{1}{2}Q\,l\,cos2\theta - QG(\theta) - Q\eta\theta$, where $G = \int g(\theta)d\theta$. Under the assumption $\sigma' = \sigma = Q\delta[g(\theta) + \eta\dot{\theta}]$, it follows from Theorem (2.1) that the transformation φ with $a = -Q\eta$ ensures the existence of the Lagrangian $L = \frac{1}{2}\dot{\tilde{\theta}}^2 - Q(\frac{1}{2}l\,cos2\theta + G(\theta) + \eta\theta + \frac{1}{2}Q\eta^2\theta^2)$ and the construction of the equation of Lagrangian structure $\frac{d}{dt}\frac{\partial L}{\partial\dot{\tilde{\theta}}} - \frac{\partial L}{\partial\theta} = \tilde{\sigma}(\theta, \dot{\tilde{\theta}})\dot{\xi}$, which is equivalent in distribution to equation (29).

2) In the one-dimensional case, the Girsanov transformation (11) takes the form $\hat{\xi} = \xi - \int_0^t b(\theta, \dot{\theta})ds$. By this transformation, equation (29) is converted into the equation

$$\ddot{\theta} = \hat{f}(\theta, \dot{\theta}) + \hat{\sigma}(\theta, \dot{\theta})\dot{\hat{\xi}}, \tag{32}$$

where$\hat{f} = f + \sigma b$ and $\hat{\sigma} = \sigma$.

The Helmholtz condition $\partial\hat{f}/\partial\dot{\theta} = 0$ takes the following form:

$$\frac{\partial f}{\partial\dot{\theta}} + \frac{\partial\sigma}{\partial\dot{\theta}}b + \frac{\partial b}{\partial\dot{\theta}}\sigma = 0. \tag{33}$$

Denoting $b(\theta, \dot{\theta}) = l_0(\theta) + \mu_0\dot{\theta}$ and substituting (30) into (32), we obtain

$$-Q\eta + Q\delta\eta l_0(\theta) + 2Q\delta\eta\dot{\theta}\mu_0 + Q\delta g(\theta)\mu_0 = 0.$$

We set $\mu_0 = 0$, then $l_0(\theta) = 1/\delta = const$ and, therefore, $b = 1/\delta = const$. Hence, equation (32) can be written in the form

$$\ddot{\theta} = Q\,l\,sin2\theta + [Q\delta g(\theta) + Q\delta\eta\dot{\theta}]\dot{\hat{\xi}}. \tag{34}$$

We suppose that the Lagrange function has the form of $L = \frac{1}{2}\dot{\theta}^2 + \beta(\theta)$ and determine $\beta(\theta)$ so that the stochastic equation of Lagrangian structure is equivalent to the transformed equation (34). Since the given Lagrange function L leads to the equation $\ddot{\theta} = \beta_\theta + \sigma'\dot{\xi}$, we determine, taking into account

(30), $\beta_\theta = Ql\,sin2\theta$ and then $\beta(\theta) = -\frac{1}{2}Ql\,cos2\theta$. Under the assumption that $\sigma' = \sigma$ we conclude that the Lagrangian, enabling the representation of equation (34) in the form of the stochastic equation of Lagrangian structure $\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = \sigma(\theta,\dot\theta)\dot{\xi}$, is of the form $L = \frac{1}{2}\dot\theta^2 - \frac{1}{2}Ql\,cos2\theta$.

3) Applying the transformation of random change of time (23) for equation (29), we get the equation

$$\ddot\theta = \overline{f}(\theta,\dot\theta) + \overline{\sigma}(\theta,\dot\theta)\dot\xi, \tag{35}$$

where $\overline{f} = f/v(\theta,\dot\theta), \overline{\sigma} = \sigma(\theta,\dot\theta)/\sqrt{v(\theta,\dot\theta)}$.

Further, the Helmholtz condition $\partial\overline{f}/\partial\dot\theta = 0$ implies

$$\frac{vf_{\dot\theta} - fv_{\dot\theta}}{v^2},\ v^2 \neq 0,\ vf_{\dot\theta} - fv_{\dot\theta} = 0. \tag{36}$$

We set $v(\theta,\dot\theta) = l_1(\theta) + \mu_1(\theta)$ and substitute (30) into (36), so we obtain the relation $-l_1(\theta)Q\eta - Ql\,sin2\theta\mu_1 + Qg(\theta)\mu_1 = 0$.

Taking $\mu_1 = 1$ we get $l_1(\theta) = \frac{-l\,sin2\theta+g(\theta)}{\eta}$. Hence, the function $v(\theta,\dot\theta)$ that ensures the fulfillment of the Helmholtz conditions takes the form $v(\theta,\dot\theta) = \frac{-l\,sin2\theta+g(\theta)}{\eta}$, and equation (35) is transformed to the form

$$\ddot\theta = -Q\eta + \frac{Q\delta[g(\theta)+\eta\dot\theta]}{\sqrt{\frac{-l\,sin2\theta+g(\theta)}{\eta}}}\dot\xi. \tag{37}$$

Suppose that the Lagrange function is of the form $L = \frac{1}{2}\dot\theta^2 + \beta(\theta)$, where the function $\beta(\theta)$ is yet unknown. The Lagrange function L leads to the stochastic equation $\ddot\theta = \beta_\theta + \sigma'\dot\xi$. Taking into account (30), we have $\beta_\theta = -Q\eta$. This implies $\beta(\theta) = -Q\eta\theta$. Further, under the assumption that $\sigma' = \overline{\sigma} = \frac{Q\delta[g(\theta)+\eta\dot\theta]}{\sqrt{\frac{-l\,sin2\theta+g(\theta)}{\eta}}}$, we conclude that the Lagrangian $L = \frac{1}{2}\dot\theta^2 - Q\eta\theta$ provides the representation of equation (36) in the form of the stochastic equation of Lagrangian structure $\frac{d}{d\tau}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = \overline{\sigma}(\theta,\dot\theta)\dot\xi$.

Example 2. Consider the second-order nonlinear differential equation describing the motion of the inner ring of a gyroscope in a gimbal [31]

$$\ddot\beta + 2\nu\dot\beta + f(\beta) = \dot\xi, \tag{38}$$

where β is the rotation angle of the inner ring . Here the coefficient of white noise is $\sigma = 1$.

1) As in Example 1, we define the phase space transformation as$\dot{\tilde\beta} = \phi_i(\beta,\dot\beta) = \dot\beta - a\beta$. The Helmholtz conditions are fulfilled for $a = -2\nu$ and equation (38) takes the form

$$\dot{\tilde\beta} = -g(\beta) + \dot\xi. \tag{39}$$

We take the Lagrange function to be equal to $L = \frac{1}{2}\dot\beta^2 + \gamma(\beta)$ and, similarly to the item 1) of Example 1, we determine γ and σ' as $\gamma(\beta) = -2\nu - G(\beta) - 4\nu^2\beta^2/2,\quad \sigma' = \sigma = 1,\quad$ where $\frac{d}{d\beta}G(\beta) = g(\beta)$.

We conclude that the Lagrangian $L = \frac{1}{2}\dot{\tilde\beta}^2 - (2\nu + G(\beta) + 4\nu^2\beta^2/2)$ provides the representation of equation (38), defined in the space $(\beta,\dot\beta)$, in the form of the stochastic equation of Lagrangian structure $\frac{d}{dt}(\frac{\partial L}{\partial\dot{\tilde\beta}}) - \frac{\partial L}{\partial\beta} = \dot\xi$.

2) Let us define the function $b(\beta,\dot\beta)$ of the Girsanov transformation (11) for equation (38) in the form

$$b = l_1(\beta) + \mu_1\dot\beta. \tag{40}$$

It follows from the Helmholtz condition for the one-dimensional case [20] that $l_1(\beta) = 0$ and $b = 2\nu\dot{\beta}$. Thus, equation (38) is transformed to the form $\ddot{\beta} =- g(\beta) + \dot{\xi}$.

Let $L = \frac{1}{2}\dot{\beta}^2 + \gamma(\beta)$. Then the equation of Lagrangian structure is equivalent to the equation $\ddot{\beta} = \gamma_\beta + \sigma'\dot{\xi}$. We thus determine $\gamma(\beta) = -G(\beta)$, where $\frac{d}{d\beta}G(\beta) = g(\beta)$, $\sigma' = \sigma = 1$.

Hence, the Girsanov transformation with the function b of the form (40) ensures the existence of the Lagrange function $L = \frac{1}{2}\dot{\beta}^2 - G(\beta)$ and the representation of equation (38) in the form of the stochastic equation of the Lagrangian structure $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\beta}}) - \frac{\partial L}{\partial \beta} = \dot{\xi}$.

3) We define the random change of time (23) by $v = l_0(\beta) + \dot{\beta}$. From the Helmholtz condition we have $l_0(\beta) = g(\beta)/2\nu$. Consequently, equation (38) is transformed to the form

$$\ddot{\beta} = -2\nu + \sqrt{\frac{2\nu}{g(\beta) + 2\nu\dot{\beta}}}\dot{\xi}. \tag{41}$$

We take the Lagrange function $L = \frac{1}{2}\dot{\beta}^2 + \gamma(\beta)$ and, comparing equation (41) with $E(L) = \sigma'\dot{\xi}$, we determine $\gamma(\beta) = -2\nu\beta$ and $\sigma' = \frac{\sigma}{\sqrt{v}} = \sqrt{\frac{2\nu}{g(\beta)+2\nu\dot{\beta}}}$.

Consequently, the Lagrangian, enabling the representation of equation (38) in the form of the stochastic equation of Lagrangian structure $\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\beta}}) - \frac{\partial L}{\partial \beta} = \sqrt{\frac{2\nu}{g(\beta)+2\nu\dot{\beta}}}\dot{\xi}$ is of the form $L = \frac{1}{2}\dot{\beta}^2 - 2\nu\beta$.

In [26], the method of additional variables was applied to the problems in Examples 1 and 2 to construct Lagrangians in the class of stochastic differential equations equivalent a.s. by the given equations (29) and (38).

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