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$Z^\circ-$Ideals in $MV-$Algebras of Continuous Functions

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Abstract. In this paper, we study $MV-$algebra of continuous functions $C(X)$ and maximal ideals of $C(X)$. Furthermore, $Z-$ideal and $Z^\circ-$ideal of $C(X)$ are introduced and proved that every $Z^\circ-$ideal in $C(X)$ is a $Z-$ideal but the converse is not true and every finitely generated $Z-$ideal is a basic $Z^\circ-$ideal. Also, we investigate meet and join of two $Z-$ideals ($Z^\circ-$ideal) of $C(X)$. Complemented elements of $C(X)$ are examined and their properties have been studied. In particular, the relationship between generated ideal by them and $Z-$ideals ($Z^\circ-$ideals) is proved. Finally, we investigate some property of $Z^\circ-$ideals in basically disconnected space and extremally disconnected space.

1. Introduction

C.C. Chang introduced MV-algebras as algebraic models for Łukasiewicz logic to give its algebraic analysis and proved completeness of Łukasiewicz logic with respect to the variety of all MV-algebras. ([7]). Chang's completeness theorem states that any MV-algebra equation holding in the standard MV-algebra over the interval [0,1] will hold in every MV-algebra. These algebras relate to the above mentioned system of logic in the same manner as Boolean algebras relate to two classical valued logic.

The first studies of MV-algebras ([3, 5, 7]) were strictly confined to applications to the Łukasiewicz propositional and predicate logics. From this period to the second half of the eighties there were a few scattered results dealing with MV-algebras presented. Since the second half of the eighties there has been a renewal of interest in MV-algebras and their influence has now been extended to other areas of mathematics. In particular, MV-algebras apply to fuzzy set theory ([4, 6]), and most notably, by the work of D. Mundici ([12]), to AF C*-algebras and lattice ordered abelian groups ([13]). By the work of Mundici ([12]) we know that whenever there is a lattice ordered abelian group with a strong order unit, there is a corresponding MV-algebra.

Considering any topological space (shortly in the sequel, space) and [0, 1] endowed with the natural topology, the family C(X) of all [0, 1]-valued continuous functions defined on X has a structure of MV-algebra, induced pointwise by the MV -operations on [0, 1]. The same operations induce on $[0, 1]^X$, if X is a nonempty set, the MV-algebra of all the fuzzy sets of X, called usually Bold algebra of fuzzy sets of X ([2]).

However in this work we study MV-algebra of continuous functions and ideals. We establish a relation between MV-algebras and topological space X. We do not claim profundity but it is always a matter of

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interest when two seemingly disparate parts of mathematics touch hands. One would like to know just how accidental such a link may be. In the work at hand the link is probably not a fluke. The types of regular rings herein are studied widely. One suspects there is also a direct link between these structures and the Łukasiewicz infinite valued logic.

In this paper, $C(X)$ is the $MV-$algebra of all continuous function on completely regular space X to standard $MV-$algebra $([0,1],\oplus,*,0)$. For each $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$, is the zero set of f. For $M \subseteq X$, by $intM$ and $\overline{M}$ we mean the interior and the closure of M, respectively. We study maximal ideals of $C(X)$ and show that if X is a compact space, then subset of $C(X)$ such that every element of that equal to zero for a unique $x \in X$ is a maximal ideal. Subsequent, according to the definition $Z-$ideal and $Z^\circ-$ideal in $MV-$algebra A and they are connection with maximal ideals and minimal prime ideals of $MV-$algebra A search for equivalent definitions of them in $MV-$algebra $C(X)$. By establishing between intersection of the minimal prime ideals containing a and annihilator of a for all a in $MV-$algebra A it has been proved that every $Z^\circ-$ideal in $C(X)$ is a $Z-$ideal. By providing an example, it turned out that the converse is not necessary true but it has been shown that every finitely generated $Z-$ideal is a basic $Z^\circ-$ideal and equivalent by generated ideal with a complemented elements of $C(X)$. It is clear that meet of two $Z-$ideal ($Z^\circ-$ideal) of $C(X)$ is a $Z-$ideal ($Z^\circ-$ideal). Also, join of two $Z-$ideal is proved that is a $Z-$ideal but showed not necessarily join of two $Z^\circ-$ideal is not a $Z^\circ-$ideal unless X is a basically disconnected. We prove that an element $f \in C(X)$ is not a zero divisor if and only if interior zero set of f is non empty and if $Z(f)$ is a clopen subset of X, then generated ideal by f is equivalent by generated ideal with a complemented element of $C(X)$. It is proved that if every ideal in $C(X)$ consisting of zero divisors is a $Z^\circ-$ideal, then every $f \in C(X)$ where $\emptyset \neq Z(f) \subsetneq X$ is a zero divisor. Finally, we made a connection between basically disconnected space and extremally disconnected space by basic $Z^\circ-$ideals of $C(X)$.

2. Preliminaries

We recollect some definitions and results which will be used in the sequel:

Definition 2.1. ([7]) An $MV-$algebra is a structure $(A, \oplus, {}^*, 0)$ where $\oplus$ is a binary operation, $*$, is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $x, y \in A$:
$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(x^*)^* = x$,
$(MV3)$ $0^* \oplus x = 0^*$,
$(MV4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Note that we have $1 = 0^*$ and the auxiliary operation $\odot$ which are as follows:

$$x \odot y = (x^* \oplus y^*)^*.$$

We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Also for any two elements $x, y \in A$, $x \leq y$ if and only if $x^* \oplus y = 1$ if and only if $x \odot y^* = 0$

Lemma 2.2. ([8]) *In each $MV-$algebra A, the following relations hold for all $x, y, z \in A$:*
(1) If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$, $x \wedge z \leqslant y \wedge z$,
(2) $x, y \leq x \oplus y$ and $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$ and $x^n = x \odot x \odot \cdots \odot x \leq x$,
(3) If $x \leq y$ and $z \leq t$, then $x \oplus z \leq y \oplus t$,
(4) $x \wedge (y \oplus z) \leqslant (x \wedge y) \oplus (x \wedge z)$, $x \wedge (x_1 \oplus ... \oplus x_n) \leqslant (x \wedge x_1) \oplus ... \oplus (x \wedge x_n)$, for all $x_1, ..., x_n \in A$; in particular $(mx) \wedge (ny) \leqslant mn(x \wedge y)$, for every $m, n \geq 0$.

For any $MV-$algebra A we shall denote by $B(A)$ the set of all complemented elements of $L(A)$ such that $L(A)$ is distributive lattice with 0 and 1.
In the paper A is an $MV-$algebra.

Theorem 2.3. ([15]) *For every element e in A, the following conditions are equivalent:*
(1) $e \in B(A)$,
(2) $e \vee e^* = 1$,
(3) $e \wedge e^* = 0$,
(4) $e \oplus e = e$,
(5) $e \odot e = e$.

Definition 2.4. ([8]) An ideal of A is a nonempty subset I of A satisfying the following conditions:
($I1$) If $x \in I,\ y \in A$ and $y \leq x$, then $y \in I$,
($I2$) If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(A)$ the set of all ideals of A.

Definition 2.5. ([8]) Let I be an ideal of A. If $I \neq A$, then I is a proper ideal of A.
•([8]) A proper ideal I of A is called prime ideal if for all $x, y \in A, x \wedge y \in I$, then $x \in I$ or $y \in I$.
We denote by $Spec(A)$ the set of all prime ideals of an $MV-$algebra A.
•([8]) An ideal I of A is called a minimal prime ideal of A:
1)$I \in Spec(A)$;
2) *If there exists* $Q \in Spec(A)$ *such that* $Q \subseteq I$, *then* $I = Q$.
We denote by $Min(A)$ the set of all minimal prime ideals of A.
•([15]) An ideal I of A is called maximal if and only if for each ideal $J \neq I$, if $I \subseteq J$, then $J = A$.
We denote by $Max(A)$ the set of all maximal ideals of A.

Definition 2.6. ([15]) Let X be a nonempty subset of A. Then $Ann(X)$ is the annihilator of X defined by:

$$Ann(X) = \{a \in A : a \wedge x = 0, \forall x \in X\}.$$

Remark 2.7. ([15]) Let $X \subseteq A$. The ideal of A generated by X will be denoted by $(X]$. We have
(1)$(X] = \{a \in A \mid a \leqslant x_1 \oplus x_2 \oplus ... \oplus x_n,\ for\ some\ n \in \mathbb{N}\ and\ x_1, ..., x_n \in X\}$. In particular, $(a] = \{x \in A \mid x \leqslant na,\ for\ some\ n \in \mathbb{N}\}$.
We denote by $(a_1, a_2, ..., a_n]$, *the ideal of A generated by* $X = \{a_1, a_2, ..., a_n\}$.
(2) For $I_1, I_2 \in Id(A)$,

$$I_1 \wedge I_2 = I_1 \cap I_2\ ,\ I_1 \vee I_2 = (I_1 \cup I_2] = \{a \in A : a \leq x \oplus y\ ;\ x \in I_1, y \in I_2\}.$$

Definition 2.8. ([10]) Let X be a nonempty subset of A. The set of all zero-divisors of X is denoted by $Z_X(A)$ and is defined as follows:

$$Z_X(A) = \{a \in A : \exists 0 \neq x \in X\ such\ that\ x \wedge a = 0\}.$$

Zero element of an $MV-$algebra is a zero divisor, which is called trivial zero divisor. We denote by Z_A the set of all zero divisors of A.

One can easily show that $Ann(X) \subseteq Z_X(A)$.

Notation: let $a \in A$. Define

$$M(a) = \{M \in Max(A) : a \in M\} \qquad P(a) = \{P \in Min(A) : a \in P\}.$$

$$M_a = \bigcap\{M : M \in Max(A), a \in M\} \quad P_a = \bigcap\{P : P \in Min(A), a \in P\}.$$

If I is an ideal of A, define

$$M_I = \bigcap\{M : M \in Max(A), I \subseteq M\} \quad P_I = \bigcap\{P : P \in Min(A), I \subseteq P\}.$$

Theorem 2.9. ([9]) *Let* $P \in Min(A)$ *and* I *be finitely generated ideal. Then* $I \subseteq P$ *if and only if* $Ann(I) \nsubseteq P$.

Lemma 2.10. ([9]) *If* $0 \neq x \in A$, *then there exists* $P \in Min(A)$ *such that* $x \notin P$.

Definition 2.11. ([1]) (1) A proper ideal I of A is called a $Z^{\circ}-$ideal if $P_a \subseteq I$, for each $a \in I$.
(2) A proper ideal I of A is called a $Z-$ideal if $M_a \subseteq I$, for each $a \in I$.

Remark 2.12. ([1]) If a is a zero divisor of $MV-$algebra of A, then P_a is a $Z^{\circ}-$ideal which is called a basic $Z^{\circ}-$ideal. Also, every intersection of $Z^{\circ}-$ideals ($Z-$ideals) is a $Z^{\circ}-$ideal ($Z-$ideal).

Proposition 2.13. ([1]) *If* $a \in A$ *and* X *is a subset of* A, *then*
(1) $P_a = \{b \in A | Ann(a) \subseteq Ann(b)\}$,
(2) $P_a = Ann(Ann(a))$
(3) $Ann(Ann(Ann(X)) = Ann(X)$.

Theorem 2.14. ([1]) *Every* $Z-$*ideal of* A *is the intersection of the minimal prime ideals containing it.*

We have $[0, 1]$ and $[-\infty, +\infty]$ are homeomorphic, so we can replace the definitions that depend on $[-\infty, +\infty]$ with $[0, 1]$. Such as the following definition:

Definition 2.15. ([11]) A space X is said to be completely regular provided that it is a Husdorff space such that, whenever F is a closed set and x is a point in its complement, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f(F) = \{0\}$.

Remark 2.16. ([11]) Let $f \in C(X)$. The set $Z(f) = \{x \in X : f(x) = 0\}$ is called zero set and $X \setminus Z(f)$ is called cozero-set.

Definition 2.17. ([11]) A space X is said to be extremally disconnected if every open set has an open closure; X is basically disconnected if every cozero-set has an open closure.

Lemma 2.18. ([14]) *Let* $X = A \cup B$ *such that* A *and* B *be closed subsets of* X. *Also, let* $f : A \to Y$ *and* $g : B \to Y$ *be continuous functions. If* $f(x) = g(x)$ *for all* $x \in A \cap B$, *then there exists continuous function* $h : X \to Y$ *such that* $h(x) = f(x)$ *for all* $x \in A$, *and* $h(x) = g(x)$ *for all* $x \in B$.

Theorem 2.19. ([14]) *Let* X *be a topological space. If* ζ *is a collection of compact subsets of* X *such that every finite intersection of elements* ζ *be nonempty, then intersection of all the elements of* ζ *is nonempty.*

Theorem 2.20. ([14]) *Let* X *be a compact space and* $f \in C(X)$. *Then there exist* $c, d \in X$ *such that* $f(c) \leq f(x) \leq f(d)$, *for all* $x \in X$.

3. Ideal theory of $C(X)$

Let X be a completely regular space. In this paper, we denote by $C(X)$ the $MV-$algebra of all continuous functions on topological space X to standard $MV-$algebra $([0, 1], \oplus, *, 0)$. For every $f, g \in C(X)$ we define $(f \oplus g)(x) = f(x) \oplus g(x)$, $f^*(x) = (f(x))^*$ and $0(x) = 0$, for all $x \in X$. Obviously, $(C(X), \oplus, *, 0)$ is an $MV-$algebra. Let $f \in C(X)$ and I be an ideal of $C(X)$. Define

$$Z(f) = \{x \in X : f(x) = 0\} \qquad Z(X) = \{Z(f) : f \in C(X)\}$$

$$Z(I) = \{Z(f) : \forall f \in I\} \qquad Z^{-1}(Z(I)) = \{f \in C(X) : Z(f) \in Z(I)\}.$$

Lemma 3.1. Let $f_1, f_2 \in C(X)$. Then
(1) $Z(f_1 \oplus f_2) = Z(f_1) \cap Z(f_2)$,
(2) $intZ(f_1 \oplus f_2) = intZ(f_1) \cap intZ(f_2)$.

Proof. (1) It is clear.

(2) If $x \in intZ(f_1) \cap intZ(f_2)$, then there exist open subsets U_1 and U_2 of X such that $x \in U_1 \subseteq intZ(f_1)$ and $x \in U_2 \subseteq intZ(f_2)$. Put $U = U_1 \cap U_2$. Obviously, $U \subseteq intZ(f_1) \cap intZ(f_2)$. Hence $U \subseteq int(Z(f_1) \cap Z(f_2))$, so $U \subseteq Z(f_1) \cap Z(f_2)$. Thus $x \in Z(f_1) \cap Z(f_2)$ then $f_1(x) = f_2(x) = 0$, hence $(f_1 \oplus f_2)(x) = 0$ so $x \in Z(f_1 \oplus f_2)$. On the other hand U is an open subset of X such that $x \in U$ so $x \in U \subseteq Z(f_1 \oplus f_2)$. Then $x \in intZ(f_1 \oplus f_2)$, implies that $intZ(f_1) \cap intZ(f_2) \subseteq intZ(f_1 \oplus f_2)$. Now, if $y \in intZ(f_1 \oplus f_2)$, then $y \in Z(f_1 \oplus f_2)$. So $(f_1 \oplus f_2)(y) = 0$, thus $f_1(y) = f_2(y) = 0$ which implies that $Z(f_1 \oplus f_2) \subseteq Z(f_1)$ and $Z(f_1 \oplus f_2) \subseteq Z(f_2)$. Hence $intZ(f_1 \oplus f_2) \subseteq intZ(f_1)$ and $intZ(f_1 \oplus f_2) \subseteq intZ(f_2)$. So $intZ(f_1 \oplus f_2) \subseteq intZ(f_1) \cap intZ(f_2)$. Therefore $intZ(f_1 \oplus f_2) = intZ(f_1) \cap intZ(f_2)$. □

Theorem 3.2. *Let $\tau = \{intZ(f) : f \in C(X)\}$. Then τ is a topological basis for X.*

Proof. By Lemma 3.1(2), it is sufficient to show that for an open set U and $x \in U$, there exists $f \in C(X)$ such that $x \in intZ(f) \subseteq U$. If U is an open subset of X and $x \in U$, then there exists $g \in C(X)$ such that $g(X \setminus U) = \{0\}$ and $g(x) = 1$. Put $f = |(g - (1/4)) \wedge 0|$. Obviously,

$$x \in intZ(f) \subseteq Z(f) = \{x \in X : g(x) \geq 1/4\} = g^{-1}([1/4, 1]) \subseteq U.$$

Therefore τ is a basis for X. □

Example 3.3. Let $X = \mathbb{R}$ and (a, b) be an open interval in $\mathbb{R}$. Put

$$f(x) = \begin{cases} 1 & x \in (-\infty, a-1] \\ -x + a & x \in (a-1, a) \\ 0 & x \in [a, b] \\ x - b & x \in (b, b+1) \\ 1 & x \in [b, \infty) \end{cases}$$

Obviously, $(a, b) = intZ(f)$. Then $\tau = \{intZ(f) : f \in C(\mathbb{R})\}$ is a basis for standard topology on $\mathbb{R}$.

Lemma 3.4. Let I be an ideal of $C(X)$. Then $Z(I)$ is closed under finite intersections and supersets.

Proof. Let $Z_1, Z_2 \in Z(I)$. Then there exist $f_1, f_2 \in I$ such that $Z_1 = Z(f_1), Z_2 = Z(f_2)$. Hence $f_1 \oplus f_2 \in I$, so $Z(f_1 \oplus f_2) \in Z(I)$. By Lemma 3.1(1), $Z(f_1) \cap Z(f_2) \in Z(I)$. Let $Z_1 \in Z(I), Z' \in Z(X)$ and $Z_1 \subseteq Z'$. Then there exist $f_1 \in I$ and $f \in C(X)$ such that $Z_1 = Z(f_1)$ and $Z' = Z(f)$. Hence $f_1 \wedge f \in I$, so $Z(f_1 \wedge f) \in Z(I)$. Obviously, $Z(f) = Z(f_1 \wedge f)$ thus $Z' \in Z(I)$. □

Proposition 3.5. *If I is an ideal of $C(X)$, then $Z^{-1}(Z(I))$ is an ideal of $C(X)$. Also $I \subseteq Z^{-1}(Z(I))$.*

Proof. Obviously, $Z^{-1}(Z(I))$ is a nonempty subset of $C(X)$. Let $f, g \in Z^{-1}(Z(I))$. Then $Z(f), Z(g) \in Z(I)$. By Lemma 3.4, we get that $Z(f) \cap Z(g) \in Z(I)$ thus $Z(f \oplus g) \in Z(I)$, then $f \oplus g \in Z^{-1}(Z(I))$. Let $f \in Z^{-1}(Z(I)), g \in C(X)$ and $g \leq f$. Then $Z(f) \in Z(I)$ and $Z(f) \subseteq Z(g)$. It follows from Lemma 3.4, that $Z(g) \in Z(I)$ thus $g \in Z^{-1}(Z(I))$. Obviously, $I \subseteq Z^{-1}(Z(I))$. □

Lemma 3.6. *Let $f, g \in C(X)$. Then the following statements are equivalent:*
(1) $P_g \subseteq P_f$,
(2) $P(f) \subseteq P(g)$,
(3) $intZ(f) \subseteq intZ(g)$,
(4) $Ann(f) \subseteq Ann(g)$.

Proof. ($1 \Rightarrow 2$) Let $P \in P(f)$. Then $P_f \subseteq P$, hence $P_g \subseteq P$. So $g \in P$, thus $P \in P(g)$. Then $P(f) \subseteq P(g)$.

($2 \Rightarrow 1$) It is clear.

($3 \Rightarrow 4$) Let $intZ(f) \subseteq intZ(g)$ and $h \in Ann(f)$. Then $(h \wedge f)(x) = 0$, for all $x \in X$ which implies that $h(x) = 0$ or $f(x) = 0$, so $X \setminus Z(h) \subseteq Z(f)$. Since $Z(h)$ is closed subset of X we get $int(X \setminus Z(h)) = Z(h)$. Hence

$$X \setminus Z(h) \subseteq intZ(f) \subseteq intZ(g) \subseteq Z(g).$$

Then $(g \wedge h)(x) = 0$, for all $x \in X$. Therefore $h \in Ann(g)$.

($4 \Rightarrow 3$) Let $Ann(f) \subseteq Ann(g)$. To prove that $intZ(f) \subseteq intZ(g)$, it suffices to show that $intZ(f) \subseteq Z(g)$. Suppose $x \in intZ(f)$ and $x \notin Z(g)$. Since $x \notin X \setminus intZ(f)$, then there exists $0 \neq h \in C(X)$ such that $h(X \setminus intZ(f)) = \{0\}$ and $h(x) = 1$. Clearly, $(h \wedge f)(x) = 0$ and $(h \wedge g)(x) \neq 0$, which is impossible.

($4 \Leftrightarrow 1$) By Proposition 2.13 (2,3), we get that $P_g \subseteq P_f$ if and only if $Ann(Ann(g)) \subseteq Ann(Ann(f))$ if and only if $Ann(Ann(Ann(f))) \subseteq Ann(Ann(Ann(g)))$ if and only if $Ann(f) \subseteq Ann(g)$. □

Corollary 3.7. (1) *Let $f, g \in C(X)$. Then $intZ(f) = intZ(g)$ if and only if $Ann(f) = Ann(g)$.*
(2) *Let $f \in C(X)$. Then $P_f = \{g \in C(X) | intZ(f) \subseteq intZ(g)\}$.*

Proposition 3.8. *Let I be an ideal of $C(X)$. Then the following statements are equivalent:*
(1) *I is a $Z^\circ-$ideal,*
(2) *$intZ(f) \subseteq intZ(g)$ and $f \in I$ imply that $g \in I$.*

Proof. ($1 \Rightarrow 2$) Let $intZ(f) \subseteq intZ(g)$ and $f \in I$. Since I is a $Z^\circ-$ideal, hence $P_f \subseteq I$. It follows from Lemma 3.6, that $P_g \subseteq P_f$, so $g \in I$.

($2 \Rightarrow 1$) For every $f \in I$, We must show that $P_f \subseteq I$. Let $g \in P_f$. Obviously, $P_g \subseteq P_f$ by Lemma 3.6, we get that $intZ(f) \subseteq intZ(g)$, then $g \in I$. □

Corollary 3.9. Let J be a $Z^\circ-$ideal of $C(X)$ and $f \in J$. Then $intZ(f) \neq \emptyset$.

Proof. Let $intZ(f) = \emptyset$. Then $intZ(f) = intZ(i)$ such that $i(x) = 1$, for all $x \in X$. Hence $i \in J$, so $J = C(X)$ that is impossible. □

The following is an example that the join of two $Z^\circ-$ideal in $C(X)$ is a proper ideal that is not a $Z^\circ-$ideal. In addition, It has been shown that every ideal contains a $Z^\circ-$ideal in $C(X)$ is not necessary a $Z^\circ-$ideal.

Example 3.10. (1) Let $X = \mathbb{R}, I = \{f \in C(X) : [0, \infty) \subseteq Z(f)\}$ and $J = \{f \in C(X) : (-\infty, 0] \subseteq Z(f)\}$. Obviously, I and J are $Z^\circ-$ideals of $C(X)$. Define $k_1(x) = 1 \wedge |x|$, for all $x \in (-\infty, 0)$ and $k_1(x) = 0$, for all $x \in [0, \infty)$, $k_2(x) = 1 \wedge x$, for all $x \in (0, \infty)$ and $k_2(x) = 0$, for all $x \in (-\infty, 0]$, $k(x) = |x|$, for all $x \in [-1, 1]$ and $k(x) = 1$, for all $x \in \mathbb{R} \setminus [-1, 1]$. Obviously, $k_1 \in I, k_2 \in J$ and $k = k_1 \oplus k_2$. Since $k \in I \vee J$ and $intZ(k) = \emptyset$, then $I \vee J$ is not a $Z^\circ-$ideal.

(2) Let $I = \{f \in C(X) : (-\infty, 1] \subseteq Z(f)\}$ and $J = \{f \in C(X) : (-\infty, 0] \cup \{\frac{1}{2}\} \subseteq Z(f)\}$.

It is claimed that I is a $Z^\circ-$ideal. Let $f, g \in C(X)$ such that $(-\infty, 1] \subseteq Z(f)$ and $(-\infty, 1) = intZ(f) = intZ(g)$. On the other hand $\overline{(-\infty, 1)} = (-\infty, 1] \subseteq Z(g)$, hence $g \in I$. We deduce that I is a $Z^\circ-$ideal. Now, J is not a $Z^\circ-$ideal, since for each $f, g \in C(X)$ such that $Z(f) = (-\infty, 0] \cup \{\frac{1}{2}\}$ and $Z(g) = (-\infty, 0] \cup \{\frac{1}{3}\}$, we have $intZ(f) = intZ(g), f \in I$ and $g \notin I$. Obviously, I is a subset of J. Hence every ideal contains a $Z^\circ-$ideal in $C(X)$ is not necessary a $Z^\circ-$ideal.

Now, we are going to investigate topological spaces X, such that the join of two $Z^\circ-$ideals in $C(X)$ is either a $Z^\circ-$ideal or $C(X)$.

Theorem 3.11. *If I and J are $Z^\circ-$ideals of $C(X)$ and X is a basically disconnected space, then $I \vee J$ is either a $Z^\circ-$ideal or $C(X)$.*

Proof. Let I and J be two $Z^\circ-$ideal in $C(X)$ and suppose that $I \vee J \neq C(X)$. Let $f \in I \vee J$, and $intZ(f) \subseteq intZ(g)$, for some $g \in C(X)$. It is claimed that $g \in I \vee J$. Since $f \in I \vee J$, then $f \leq h \oplus k$ where $h \in I$ and $k \in J$. We consider two cases:

Case 1. if $h(x) = k(x) = 0$, for all $x \in X$, then $f = 0$. So $Z(f) = X$ hence $intZ(f) = X$, thus $intZ(g) = X$, then $Z(g) = X$. So $g(x) = 0$, for all $x \in X$. We obtain $g \in I \vee J$.

Case 2. if $h \neq 0$ and $k \neq 0$, we show that $g \in I \vee J$. Now, since X is a basically disconnected space, $intZ(k)$ and $intZ(h)$ are closed subsets of X. By Corollary 3.9, we have $intZ(h) \neq \emptyset$ and $intZ(k) \neq \emptyset$. Put $A = X \setminus intZ(k)$, then A and $intZ(k)$ are disjoint clopen subsets of X. Thus there exists $k' \in C(X)$ such that

$k'(A) = \{1\}$ and $k'(intZ(k)) = \{0\}$. So $Z(k') = intZ(k)$. Then $intZ(k') = intZ(k)$ thus $k' \in J$. Similarly, there exists $h' \in C(X)$ such that $Z(h') = intZ(h)$ and $h' \in I$. Obviously, $Z(h) \cap Z(k) \subseteq Z(f)$ then $intZ(h) \cap intZ(k) \subseteq intZ(f)$, so $Z(h') \cap Z(k') \subseteq intZ(f)$. Thus $Z(h' \oplus k') \subseteq intZ(f)$, hence $Z(h' \oplus k') \subseteq intZ(g)$, then $Z(h' \oplus k') \subseteq Z(g)$. We consider two cases:

Case 1. if $x \in Z(g)$, then $g(x) = 0$ So $g(x) \leq h'(x) \oplus k'(x)$.

Case 2. if $x \notin Z(g)$, then $x \notin Z(h' \oplus k')$. So $x \notin Z(h')$ or $x \notin Z(k')$, hence $h'(x) = 1$ or $k'(x) = 1$, thus $(h' \oplus k')(x) = 1$. So $g(x) \leq (h' \oplus k')(x)$.

Therefore $g(x) \leq (h' \oplus k')(x)$, for all $x \in X$. Hence $g \in I \vee J$. □

Theorem 3.12. *Let $x_0 \in X$. Then $I = \{f \in C(X) : f(x_0) = 0\}$ is a maximal ideal of $C(X)$.*

Proof. Obviously, $I \in Id(C(X))$. Let $Q \in Id(C(X))$ be such that $I \subsetneq Q \subseteq C(X)$. Then there exists $g_1 \in Q$ such that $g_1 \notin I$. So $g_1(x_0) \neq 0$. Put $U = \{x \in X : (1/2)g_1(x_0) < g_1(x)\} = \{x \in X : x \in g_1^{-1}((1/2)g_1(x_0), 1]\}$, hence U is an open subset of X and $C = X \setminus U$ is a closed subset of X. Thus there exists $f \in C(X)$ such that $f(C) = 1$ and $f(x_0) = 0$, imply that $f \in I$. Put $g = f \oplus g_1$. Obviously, $g \in Q$. Now, we consider two cases:

Case 1: if $x \in C$, then $f(x) = 1$ hence $g(x) = 1$.

Case 2: if $x \in U$, then $(1/2)g_1(x_0) < g_1(x)$.

Therefore $0 < (1/2)g_1(x_0) \leq g(x)$, for all $x \in X$. So, by Archimedean property there exists $m \in \mathbb{N}$ such that $mg(x) = 1$, for all $x \in X$. Hence $Q = C(X)$. Therefore $I \in Max(C(X))$. □

Now, converse of Theorem 3.12, is proved with an extra condition.

Theorem 3.13. *If X is a compact space, then every maximal ideal M of $C(X)$ has the form M^x for a unique $x \in X$ where*

$$M^x = \{f \in C(X) : f(x) = 0\}.$$

Proof. First we show that $\bigcap_{f \in M} Z(f) \neq \emptyset$. By Theorem 2.19, it suffices to show that $Z(f_1) \cap Z(f_2) \cap ... \cap Z(f_n) \neq \emptyset$, where $f_i \in M$, for all $1 \leq i \leq n$. Let $Z(f_1) \cap Z(f_2) \cap ... \cap Z(f_n) = \emptyset$. Hence $(\oplus_{i=1}^n f_i)(x) \neq 0$, for all $x \in X$. So by Theorem 2.20, there exist $x' \in X$ and $p \in (0, 1]$ such that $min(\oplus_{i=1}^n f_i)(x') = p$. Then there exists $t \in \mathbb{N}$ such that $t(\oplus_{i=1}^n f_i)(x) = 1$, for all $x \in X$ this implies that $M = C(X)$, which is a contradiction. Hence there exists $x \in \bigcap_{f \in M} Z(f)$ such that $M = \{f \in C(X) : f(x) = 0\}$. If there exists $y \in \bigcap_{f \in M} Z(f)$ such that $x \neq y$, then $M \subsetneq \{f \in C(X) : f(x) = f(y) = 0\}$, which is a contradiction. □

Example 3.14. If $X = [0, 1]$, then $I = \{f \in C(X) : f(1/2) = 0\}$ and $J = \{f \in C(X) : f(1/3) = 0\}$ are maximal ideals of $C(X)$. Hence $C(X)$ is not a local $MV-$algebra.

Lemma 3.15. *Let $f, g \in C(X)$. Then the following statements are equivalent:*
(1) $M_g \subseteq M_f$,
(2) $M(f) \subseteq M(g)$.

Proof. ($1 \Rightarrow 2$) Let $M \in M(f)$. Then $M \in Max(C(X))$ and $f \in M$. So $M_f \subseteq M$ thus $M_g \subseteq M$. We obtain $g \in M$, hence $M \in M(g)$.

($2 \Rightarrow 1$) Let $t \in M_g$ but $t \notin M_f$. Then there exists $M \in Max(C(X))$ such that $f \in M$ and $t \notin M$. Since $M(f) \subseteq M(g)$, thus $g \in M$. Hence $t \notin M_g$, which is impossible. □

Lemma 3.16. *If $f, g \in C(X)$ and $M(f) \subseteq M(g)$, then $Z(f) \subseteq Z(g)$.*

Proof. Let $x \in Z(f)$. Then $f \in M$ such that $M \in Max(C(X))$. Hence $M \in M(f)$, so $M \in M(g)$. Thus $g \in M$ implies that $x \in Z(g)$. We deduce that $Z(f) \subseteq Z(g)$. □

The following example shows that the converse Lemma 3.16, is not necessary correct.

Example 3.17. Let $f, g \in C((0,1))$, such that $f(x) = sin(\frac{\pi x}{2})$ and $g(x) = |\frac{1}{2} - sin(\frac{\pi x}{2})|$, for all $x \in (0,1)$. Then $Z(f) = \emptyset$ and $Z(g) = \{\frac{1}{2}\}$, we deduce that $Z(f) \subseteq Z(g)$. It is claimed that $M(f) \nsubseteq M(g)$. If $M(f) \subseteq M(g)$, then there exists $M \in M(g)$ such that $f \in M$. On the other hand,

$$\frac{1}{2} = |sin(\frac{\pi x}{2}) + \frac{1}{2} - sin(\frac{\pi x}{2})| \leqslant |sin(\frac{\pi x}{2})| + |\frac{1}{2} - sin(\frac{\pi x}{2})| = sin(\frac{\pi x}{2}) + |\frac{1}{2} - sin(\frac{\pi x}{2})|, \forall x \in (0,1)$$

Put $K := f \oplus g$. Hence $k \in M$, such that $k(x) \geq \frac{1}{2}$, for all $x \in (0,1)$. We obtain $i \in M$, such that $i(x) = 1$, for all $x \in (0,1)$ that is impossible. Therefore $M(f) \nsubseteq M(g)$.

Proposition 3.18. *Let I be an ideal of $C(X)$. Then the following statements are equivalent:*
(1) *I is a $Z-$ideal,*
(2) *$Z(f) \subseteq Z(g)$ and $f \in I$ imply that $g \in I$.*

Proof. (1 ⇒ 2) Let $Z(f) \subseteq Z(g)$ and $f \in I$. Then $intZ(f) \subseteq intZ(g)$. It follows from Lemma 3.6, that $Ann(f) \subseteq Ann(g)$. Since $f \in I$ by Theorem 2.14, we have $f \in P_I$, i.e, $f \in P$, for each $P \in Min(C(X))$ such that $I \subseteq P$. It follows from Theorem 2.9, that $Ann(f) \nsubseteq P$, thus $Ann(g) \nsubseteq P$. By Theorem 2.9, so $g \in P$. Hence $g \in P_I$. So $g \in I$.

(2 ⇒ 1) Let $f \in I$. We must show that $M_f \subseteq I$. Let $g \in M_f$, obviously $M_g \subseteq M_f$. By Lemma 3.15 and Lemma 3.16, we get that $Z(f) \subseteq Z(g)$, then $g \in I$. □

Corollary 3.19. (1) *If I is a $Z^{\circ}-$ideal of $C(X)$, then I is a $Z-$ideal.*
(2) *Let J be a $Z-$ideal of $C(X)$ and $f \in J$. Then $Z(f) \neq \emptyset$.*

Now, we give examples that are shown every $Z-$ideal is not necessary a $Z^{\circ}-$ideal and there exists an ideal of $C(X)$ such that is not a $Z-$ideal nor $Z^{\circ}-$ideal.

Example 3.20. (1) Let $X = \mathbb{R}$. Then $I = \{f \in C(X) : [0,1] \cup \{2\} \subseteq Z(f)\}$ is a $Z-$ideal, but I is not a $Z^{\circ}-$ideal, since for each $f, g \in C(X)$ such that $Z(f) = [0,1] \cup \{2\}$ and $Z(g) = [0,1] \cup \{3\}$, we have $intZ(f) = intZ(g)$, $f \in I$ and $g \notin I$.

(2) Let $X = [0,1], I = (f]$ such that $f(x) = x$, for all $x \in X$ and suppose that $g(x) = \sqrt{x}$, for all $x \in X$. Obviously, $Z(f) = Z(g) = 0$, $intZ(f) = intZ(g) = \emptyset$. It is claimed that $g \notin I$. If $g \in I$, then there exists $n \in \mathbb{N}$ such that $\sqrt{x} \leq nx$, for each $x \in [0,1]$. Hence $1 \leq n\sqrt{x}$, for each $x \in [0,1]$ which is impossible. Therefore I is not a $Z-$ideal nor $Z^{\circ}-$ideal.

Theorem 3.21. *If I and J are $Z-$ideals of $C(X)$, then $I \vee J$ is a $Z-$ideal of $C(X)$.*

Proof. Let $Z(g) \subseteq Z(f)$ and $g \in I \vee J$. Then $g \leq g_1 \oplus g_2$ such that $g_1 \in I$ and $g_2 \in J$. Obviously, $Z(g_1) \cap Z(g_2) \subseteq Z(g)$, so $Z(g_1) \cap Z(g_2) \subseteq Z(f)$. Define

$$h(x) = \begin{cases} 0 & x \in Z(g_1) \cap Z(g_2) \\ f(x)(\frac{g_1(x)}{g_1(x) + g_2(x)}) & x \notin Z(g_1) \cap Z(g_2) \end{cases}$$

and

$$k(x) = \begin{cases} 0 & x \in Z(g_1) \cap Z(g_2) \\ f(x)(\frac{g_2(x)}{g_1(x) + g_2(x)}) & x \notin Z(g_1) \cap Z(g_2) \end{cases}$$

Now, we show that h and k are continuous functions. Let $\varepsilon > 0$ and $x_0 \in Z(g_1) \cap Z(g_2)$. Since $f \in C(X)$ then there exists open subset V of X such that $f(V) \subseteq (-\varepsilon, \varepsilon)$. On the other hand $h(x) \leq f(x) < \varepsilon$, for all $x \in V$. So $h(V) \subseteq (-\varepsilon, \varepsilon)$, thus h is continuous at x_0. Therefore $h \in C(X)$. Similarly, it is proved that $k \in C(X)$. Obviously, $f = h \oplus k, Z(g_1) \subseteq Z(h)$ and $Z(g_2) \subseteq Z(k)$. So $h \in I$ and $k \in J$, we deduce that $f \in I \vee J$. □

It was shown in Example 3.20(1), that every $Z-$ideal is not a $Z^\circ-$ideal. Now, conditions are provided that $Z-$ideals connection by $Z^\circ-$ideals.

Lemma 3.22. *Let $e \in A$. Then $e \in B(A)$ if and only if $Ann(e) = (e^*]$.*

Proof. If $e \in B(A)$, then $e \wedge e^* = 0$. We obtain $e^* \in Ann(e)$, hence $(e^*] \subseteq Ann(e)$. Let $x \in Ann(e)$. Then $x \wedge e = 0$, thus $x \odot a = 0$. So $x \leq e^*$, then $x \in (e^*]$. Hence $Ann(e) \subseteq (e^*]$. Therefore $Ann(e) = (e^*]$. Converse is clear. □

Corollary 3.23. *Let $e \in B(A)$. Then $P_e = (e]$.*

Proof. Let $e \in B(A)$. Then $Ann(e) = (e^*]$, so $Ann(Ann(e)) = Ann((e^*])$. By Lemma 3.22 and Proposition 2.13 (2), we get that $P_e = (e]$. □

Lemma 3.24. (1) *If $f \in C(X)$, then $Ann(f) = \{0\}$ if and only if $intZ(f) = \emptyset$.*
(2) *If $e \in B(C(X))$, then $Z(e)$ is an open subset of X.*
(3) *Let $f \in C(X)$ be such that $inf(f(X\backslash Z(f)) \neq 0$ and $Z(f)$ be an open subset of X. Then there exists $e \in B(C(X))$ such that $(e] = (f]$.*

Proof. (1) Let $i \in C(X)$ be such that $i(x) = 1$, for all $x \in X$. Obviously, $Ann(i) = \{0\}, Z(i) = \emptyset$ and $intZ(i) = \emptyset$. Now, if $Ann(f) = \{0\}$, then $Ann(f) = Ann(i)$. It follows from Corollary 3.7(1), that $intZ(f) = intZ(i)$. Hence $intZ(f) = \emptyset$.

Conversely, if $intZ(f) = \emptyset$, then $intZ(f) = intZ(i)$. It follows from Corollary 3.7(1), that $Ann(f) = Ann(i)$ implies that $Ann(f) = \{0\}$.

(2) By hypothesis $e \in B(C(X))$, so $(e \oplus e)(x) = e(x)$, for all $x \in X$. We deduce that $e(x) \oplus e(x) = min\{2e(x), 1\} = e(x)$, for all $x \in X$. Hence $e(x) = 0$ or $e(x) = 1$, for all $x \in X$. Put $K = \{x : e(x) = 1\}$. Obviously, $Z(e) \cap K = \emptyset$ and $Z(e) \cup K = X$, then K and $Z(e)$ are clopen subsets of X. Therefore $intZ(e) = Z(e)$.

(3) Define $e : X \to [0, 1]$ by $e(x) = 0$, for all $x \in Z(f)$ and $e(x) = 1$, for all $x \notin Z(f)$. By Lemma 2.18, we get that $e \in C(X)$. Obviously, $e \in B(C(X))$ and $Z(f) = intZ(f) = Z(e)$. It is claimed that $(f] = (e]$. we consider two cases:

Case 1. if $x \in Z(f)$, then $e(x) = f(x) = 0$. So $e(x) \leq f(x)$.

Case 2. if $x \notin Z(f)$, then $f(x) \neq 0$. By hypothesis $inff(x) \neq 0$, we imply that there exists $n \in \mathbb{N}$ such that $nf(x) = 1$. So $e(x) \leq nf(x)$.

Hence $e \in (f]$, thus $(e] \subseteq (f]$. Now, it is clear that $(f] \subseteq (e]$. Then $(e] = (f]$. □

Proposition 3.25. *Let K be finitely generated $Z-$ideal in $C(X)$. Then K is a basic $Z^\circ-$ideal and there exists $e \in B(C(X))$ such that $K = (e]$.*

Proof. Let $K = (f_1, f_2, ..., f_n]$ and $f := f_1 \oplus f_2 \oplus ... \oplus f_n$. Obviously, $f \in K$ hence $(f] \subseteq K$. On the other hand, $f_i \leq f$, for all $1 \leq i \leq n$, hence $f_i \in (f]$, for all $1 \leq i \leq n$. Thus $\{f_1, f_2, ..., f_n\} \subseteq (f]$, so $(f_1, f_2, ..., f_n] \subseteq (f]$, implies that $K \subseteq (f]$. Hence $K = (f]$. Obviously, $Z(f) = Z(\sqrt{f})$ and $f \in K$, by hypothesis and Proposition 3.18, hence $\sqrt{f} \in K$. So there exists $n \in \mathbb{N}$ such that $\sqrt{f(x)} \leq nf(x)$, for all $x \in X$. Now, $f(x) \neq 0$, for all $x \notin Z(f)$, hence $\sqrt{f(x)} \leq nf(x)$. So $1 \leq n\sqrt{f(x)}$, thus $(1/n) \leq \sqrt{f(x)}$ implies that $(1/n^2) \leq f(x)$. So $\beta = \{x \in X : (1/n^2) \leq f(x)\} = \{x \in X : f(x) \in [1/n^2, 1]\}$ is a closed subset of X and $X = Z(f) \cup \beta$, imply that $Z(f)$ is a clopen subset of X. It follows from Lemma 3.24 (3), that there exists $e \in B(C(X)$ such that $(e] = (f]$. By Corollary 3.23, we get that $K = (e] = P_e$. □

Now, we give an example for previous proposition.

Example 3.26. Let $X = (0, 1) \cup (1, 2)$ and $I = (f_1, f_2]$ such that

$$f_1(x) = \begin{cases} 0 & x \in (0,1) \\ 1/2x & x \in (1,2) \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & x \in (0,1) \\ x/4 & x \in (1,2) \end{cases}$$

We define $f = f_1 \oplus f_2$ and $e \in C(X)$ such that $e(x) = 0$, for all $x \in (0,1)$ and $e(x) = 1$, for all $x \in (1,2)$. Obviously, $I = (f]$ is a $Z-$ideal and $I = (e] = P_e$.

Theorem 3.27. (1) *Every basic $Z^\circ-$ideal in $C(X)$ is principal if and only if X is basically disconnected.*
(2) *Any arbitrary intersections of basic $Z^\circ-$ideals in $C(X)$ is principal if and only if X is extremally disconnected.*

Proof. (1) Suppose that every basic $Z^\circ-$ideal in $C(X)$ is principal. To prove that $\overline{X - Z(f)}$ is open subset of X, it suffices to show that $intZ(f)$ is closed, for all $f \in C(X)$. We consider two cases:

Case 1. if f is not a zero divisor, then $Ann(f) = \{0\}$. It follows from Lemma 3.24(1), that $intZ(f) = \emptyset$.

Case 2. if f is a zero divisor, then $Ann(f) \neq \{0\}$. By hypothesis, there exists $g \in C(X)$ such that $P_f = (g]$. By Corollary 3.19(1) and Proposition 3.25, there exists $e \in B(C(X))$ such that $P_f = (e]$. Now, it is claimed that $intZ(f) = intZ(e)$. Obviously, $f \in P_f = (e]$, so $f \in (e]$. Hence there exists $n \in \mathbb{N}$ such that $f(x) \leq ne(x)$, for all $x \in X$. Thus $Z(e) \subseteq Z(f)$ implies that $intZ(e) \subseteq intZ(f)$. Let $e \in P_f$. Obviously, by Corollary 3.7(2), we have $intZ(f) \subseteq intZ(e)$. Now, by Lemma 3.24(2), we get that $intZ(f) = Z(e)$.

Conversely, let X be a basically disconnected space and $f \in C(X)$ with $Ann(f) \neq \{0\}$. By Lemma 3.24 (1), we have $F = intZ(f) \neq \emptyset$ is a closed subset of X. Define $e_1 : F \to [0,1]$ where $e_1(F) = 0$ and $e_2 : X \setminus F \to [0,1]$ where $e_2(X \setminus F) = 1$. It follows from Lemma 2.18, that there exists $e \in C(X)$ such that

$$e(x) = \begin{cases} e_1(x) & x \in F \\ e_2(x) & x \in X \setminus F \end{cases}$$

Obviously, $e \in B(C(X))$ and it is claimed that $P_f = (e]$. Let $g \in P_f$. We consider two cases:

Case 1. if $x \in X \setminus F$, then $e(x) = 1$. Hence $g(x) \leq e(x)$, for all $x \in X \setminus F$.

Case 2. if $x \in F = intZ(f)$, hence by hypothesis $g \in P_f$ and Lemma 3.24 (2), we get that $intZ(f) \subseteq intZ(g)$. Then $x \in intZ(g)$ so $x \in Z(g)$, hence $g(x) = 0$. We obtain $g(x) \leq e(x)$, for all $x \in F$.

Then $g \in (e]$ hence $P_f \subseteq (e]$. Let $k \in (e]$. Then there exists $n \in \mathbb{N}$ such that $k(x) \leq (ne)(x)$, for all $x \in X$. So $Z(e) \subseteq Z(k)$, hence $intZ(f) \subseteq Z(k)$ then $intZ(f) \subseteq intZ(k)$. It follows from Corollary 3.7(2), that $k \in P_f$ thus $(e] \subseteq P_f$. Therefore $(e] = P_f$.

(2) Suppose that every intersection of basic $Z^\circ-$ideals is principal and G is an open subset of X. By Theorem 3.2, there exists $S \subseteq C(X)$ such that $G = \bigcup_{f \in S} intZ(f)$ and $intZ(f) \neq \emptyset$. By Lemma 3.24, we have $Ann(f) \neq \{0\}$ so f is a zero divisor of $C(X)$. By hypothesis, there exists $g \in C(X)$ such that $\bigcap_{f \in S} P_f = (g]$. Then $(g]$ is a $Z^\circ-$ideal so by Corollary 3.19, $(g]$ is a $Z-$ideal. It follows from Proposition 3.25, that there exists $e \in B(C(X))$ such that $(g] = (e]$. Now, by Lemma 3.24(2), this shows that $Z(g) = Z(e)$ is an open subset of X. It is claimed that $\overline{G} = Z(g)$. Let $x \in G = \bigcup_{f \in S} intZ(f)$. Then there exists $f \in S$ such that $x \in intZ(f)$. Because $(g] = \bigcap_{f \in S} P_f$, for every $f \in S$ thus $g \in P_f$ by Corollary 3.7(2), we get that $intZ(f) \subseteq intZ(g)$. Hence $intZ(f) \subseteq Z(g)$, so $G \subseteq Z(g)$ implies that $\overline{G} \subseteq Z(g)$. Let $x \in Z(g)$ and $x \notin \overline{G}$. Then there exists $h \in C(X)$ such that $h(x) = 1$ and $h(\overline{G}) = \{0\}$. Let $\beta \in intZ(f)$, for every $f \in S$. Then $\beta \in G$, hence $\beta \in \overline{G}$ so $h(\beta) = 0$. We deduce that $\beta \in Z(h)$, so $intZ(f) \subseteq Z(h)$. Hence $intZ(f) \subseteq intZ(h)$ thus $h \in P_f$ implies that $h \in \bigcap_{f \in S} P_f = (g]$.Therefore there exists $n \in \mathbb{N}$ such that $h(\alpha) \leq (ng)(\alpha)$, for all $\alpha \in X$. On the other hand, $h(x) = 1$ and $g(x) = 0$, which is a contradiction. Hence $Z(g) \subseteq \overline{G}$, therefore $Z(g) = \overline{G}$.

Conversely, let X be an extremally disconnected space and $I = \bigcap_{f \in S} P_f$ be such that $S \subseteq C(X)$. Obviously, $G = \overline{\bigcup_{f \in S} intZ(f)}$ is an open and closed subset of X. Define $e_1 : G \to [0,1]$ where $e_1(G) = 0$ and $e_2 : X \setminus G \to [0,1]$ where $e_2(X \setminus G) = 1$. It follows from Lemma 2.18, that there exists $e \in C(X)$ such that

$$e(x) = \begin{cases} e_1(x) & x \in G \\ e_2(x) & x \in X \setminus G \end{cases}$$

Obviously, $e \in B(C(X))$ and by Lemma 3.24(2), we get that $Z(e)$ is an open subset of X hence $intZ(f) \subseteq intZ(e)$, for all $f \in S$. It follows from Corollary 3.7(2), that $e \in P_f$, for all $f \in S$. So $e \in \bigcap_{f \in S} P_f = I$, then $(e] \subseteq I$. Let $g \in I = \bigcap_{f \in S} P_f$. Then $g \in P_f$, for all $f \in S$. It follows from Corollary 3.7(2), that $intZ(f) \subseteq intZ(g)$, for all $f \in S$ thus $intZ(f) \subseteq Z(g)$, for all $f \in S$ therefore $G = \overline{\bigcup_{f \in S} intZ(f)} \subseteq Z(g)$. Since $X = G \cup (X \setminus G)$ we consider two cases:

Case 1. if $x \in G$, then $x \in Z(g)$. So $g(x) \leq e(x)$, for all $x \in G$.

Case 2. if $x \in X \setminus G$, then $e(x) = 1$. Hence $g(x) \leq e(x)$, for all $x \in X \setminus G$.

Thus $g \in (e]$, we obtain $I \subseteq (e]$. Therefore $I = (e]$. □

Theorem 3.28. *Let every ideal of $C(X)$ containing zero divisors be a $Z^\circ-$ideal. Then every $f \in C(X)$ with $\emptyset \neq Z(f) \subsetneq X$ is a zero divisor and P_f is a principal ideal of $C(X)$.*

Proof. We consider two cases:

Case 1. if $f(X)$ is a finite subset of $[0, 1]$ and $\{a_0 = 0, a_1, ..., a_n\} \subseteq [0, 1]$ be such that $f^{-1}(0) = A_0 = Z(f), f^{-1}(a_1) = A_1, ..., f^{-1}(a_n) = A_n$ and $X = A_0 \cup A_1 \cup ... \cup A_n$. We define $g : X \to [0, 1]$ such that $g(x) = 1$, for all $x \in Z(f)$ and $g(x) = 0$, for all $x \notin Z(f)$. Obviously, by Lemma 2.18, we have $g \in C(X)$. Hence $(f \wedge g)(x) = min\{f(x), g(x)\} = 0$, for all $x \in X$, then f is a zero divisor.

Case 2. if $f(X)$ is an infinite subset of $[0, 1]$ and $a, b \in f(X)$ such that $0 < a < b < 1$. Define

$$C = \{y \in X : a \geq f(y)\} = \{y \in X : f(y) \in [0, a]\} = f^{-1}([0, a])$$
$$B = \{y \in X : a \leq f(y)\} = \{y \in X : f(y) \in [a, 1]\} = f^{-1}([a, 1]).$$

Obviously, since $0 < a < b < 1$, neither B contains C nor C contains B. It is clear that B and C are closed subset of X such that $X = B \cup C$ and $B \cap C = \{x \in X : f(x) = a\}$. Define $\phi_1 : C \to [0, 1]$ where $\phi_1(x) = f(x)$, for all $x \in C$ and $\phi_2 : B \to [0, 1]$ where $\phi_2(B) = a$. Obviously $\phi_1(B \cap C) = \phi_2(B \cap C)$. It follows from Lemma 2.18, that

$$\phi(x) = \begin{cases} \phi_1(x) & x \in C \\ \phi_2(x) & x \in B \end{cases}$$

is a continuous function. Put $h(x) = |\phi(x) - f(x)|$, for all $x \in X$. It is clear that $h \neq 0$ and $Z(h) = C$. Similarly, there exists $g \in C(X)$ such that $Z(g) = B$ and $g \neq 0$. Then $(f \wedge g \wedge h)(x) = 0$, for all $x \in X$ so $h \in Ann(f \wedge g)$ hence $Ann(f \wedge g) \neq \{0\}$ thus $f \wedge g$ is a zero divisor of $C(X)$. Since $f \wedge g \in (f \wedge g]$, so by hypothesis, $(f \wedge g]$ is a $Z^\circ-$ideal and by Corollary 3.19, we get that $(f \wedge g]$ is a $Z-$ideal. Now, by Proposition 3.25, there exists $e \in B(C(X))$ such that $(f \wedge g] = (e]$. It follows from Lemma 3.24(2), that $Z((f \wedge g]) = Z(e) = intZ(e)$. Obviously, $Z(f \wedge g) = Z(f) \cup Z(g) = Z(e)$ and $Z(f) \cap Z(g) = \emptyset$, then $Z(f) = Z(f \wedge g) \setminus Z(g) = Z(f \wedge g) \cap (Z(g))^c$. Hence $Z(f)$ is an open subset of X and $Z(f) = intZ(f)$. By hypothesis $intZ(f) \neq \emptyset$ and by Lemma 3.24 (2), we have $Ann(f) \neq \{0\}$. Hence f is a zero divisor of $C(X)$.

On the other hand, f is a zero divisor, then $(f]$ is a $Z^\circ-$ideal so $P_f \subseteq (f]$. Obviously, $(f] \subseteq P_f$, hence $(f] = P_f$. □

4. Conclusion

We investigated ideals of continuous of functions $C(X)$ and concluded that if X is compact space, then every maximal ideal M of $C(X)$ has the form M^x for a unique $x \in X$ where $M^x = \{f \in C(X) : f(x) = 0\}$. For every element f of $Z-$ideal ($Z^\circ-$ideal) has $Z(f) \neq \emptyset$ ($intZ(f) \neq \emptyset$) and every $Z^\circ-$ideal is a $Z-$ideal but converse is not true. Also join of two $Z-$ideals is a $Z-$ideal but if X is a basically disconnected, then join of two $Z^\circ-$ideals is a $Z^\circ-$ideal. If g is a complemented element of $C(X)$, then $Z(g)$ is an open subset of X and if g is an element of $C(X)$ such that $inf(g(X)) \neq 0$ then $Z(g)$ is an open subset of X. Furthermore, every $f \in C(X)$ is not a zero divisor if and only if $intZ(f) \neq \emptyset$. It is proved that for each complemented element e of $C(X), Ann(e) = (e^*]$ and $Z(e)$ is an open subset of X. Also, we conclude that there exist connections between elements X and $C(X)$, for example for every $f, g \in C(X)$, if $M(f) \subseteq M(g)$, then $Z(f) \subseteq Z(g)$ and $intZ(f) \subseteq intZ(g)$ if and only if $Ann(f) \subseteq Ann(g)$.

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