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Asymptotic Behaviour of Negative Eigenvalues of an Operator Differential Equation

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Abstract. In this work, we find the asymptotic formulas for the sum of the negative eigenvalues smaller than $-\varepsilon$ ($\varepsilon > 0$) of a self-adjoint operator L defined by the following differential expression $\ell(y) = -(p(x)y^{'}(x))^{'} - Q(x)y(x)$ with the boundary condition $y(0) = 0$ in the space $L_2(0, \infty; H)$.

1. Introduction and History

One of the main problems of spectral theory of the differential equations is to investigate asymptotic behaviour of eigenvalues and is studied by many mathematicians. There exists important applications in mathematics and mathematical physics.In particular, unbounded self-adjoint operators are of paramount importance for quantum mechanics. Accordingly, this research area has been receiving growing attention since 1960s.

Skachek [1] in 1963 obtained eigenvalue asymptotics for scalar differential operator. Kostyuchenko and Levitan [2] examined the spectrum of Sturm-Liouville Operator. Afterthat, Gorbacuk, M.L [3], Gorbacuk, V. and Gorbacuk, M.L [4-5], Otelbayev, M. [6], Solomyak, M.Z [7] investigated the spectrum of the differential operator with operator coefficient. Maksudov, F.G et all [8] studied asymptotics of the spectrum of high order differential operator. Adiguzelov, E. et all [9] obtained asymptotic formulas for eigenvalues of Sturm-Liouville Operator with singularity. Furthermore, Hashimoglu [10], Baksi and Ismayılov [11], Sezer[12] derived eigenvalue asymptotics for various differential operators. Only a few works in literature concentrate on differential operators with operator coefficient. The aim of our work is to establish asymptotic formulas for negative eigenvalues of Sturm-Liouville problem (1). This paper divided into three parts. The first part outlines some historical background, related researches in the theory and describes the problem. The second part presents formulation of regularized trace and gives main results. The last part contains the proof of theorems.

Return to our problem:

Let H be an infinite dimensional separable Hilbert space. Let us consider the operator L in the Hilbert space $L_2(0, \infty; H)$ defined by the differantial expression

$$\ell(y) = -(p(x)y^{'}(x))^{'} - Q(x)y(x) \tag{1}$$

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and with the boundary condition $y(0) = 0$.
Assume that the scalar function $p(x)$ and the operator function $Q(x)$ satisfy the following conditions:
p1) For every $x \in [0, \infty)$, there are positive constants c_1,c_2 such that $c_1 \leq p(x) \leq c_2$,
p2) The function $p(x)$ has continuous and bounded derivative,
p3) The function $p(x)$ is not decreasing in the interval $[0, \infty)$,
Q1) For every $x \in [0, \infty)$ the operator $Q(x) : H \to H$ is self-adjoint, compact and positive,
Q2) The operator $Q(x)$ is monotone decreasing,
Q3) $Q(x)$ is a continuous operator function with respect to the norm in $B(H)$ and $\lim_{x\to\infty} \|Q(x)\| = 0$.
$D(L)$ denotes the set of all functions $y(x) \in L_2(0, \infty; H)$ satisfying the following conditions:
y1) $y(x)$ and $y^{'}(x)$ are absolute continuous with respect to the norm in the space H in every infinite interval $[0, a]$,
y2) $\ell(y) = -(p(x)y^{'}(x))^{'} - Q(x)y(x) \in L_2(0, \infty; H)$,
y3) $y(0) = 0$ and, $(Ly)(x) = -(p(x)y^{'}(x))^{'} - Q(x)y(x)$.
It is proved that the operator $D(L) \to L_2(0, \infty; H)$ is self-adjoint, semi bounded-below and the negative part of the spectrum of the operator L is discreate [13]. Let $-\lambda_1 \leq -\lambda_2 \leq ... \leq -\lambda_n \leq ...$ be negative eigenvalues of the operator L. In this work, we find asymptotic formulas for the sum

$$\sum_{-\lambda_i<-\varepsilon} \lambda_i \ (\varepsilon > 0),$$

as $\varepsilon \to +0$.

2. Main Results

The main purpose of this section is to obtain some formulas for the negative eigenvalues of the operator L. Let $\alpha_1(x) \geq \alpha_2(x) \geq \ldots \geq \alpha_j(x) \geq \ldots$ be the eigenvalues of the operator $Q(x) : H \to H$. Since the operator function $Q(x)$ is monotone decreasing, the functions $\alpha_1(x), \alpha_2(x), \ldots, \alpha_j(x), \ldots$ are also monotone decreasing [14]. Moreover, since

$$\alpha_1(x) = \sup_{\|f\|=1}(Q(x)f, f),$$

[15] and

$$\|Q(x)\| = \sup_{\|f\|=1} |(Q(x)f, f)| = \sup_{\|f\|=1}(Q(x)f, f),$$

[16], $\alpha_1(x) = \|Q(x)\|$. Let

$$\psi_j(\varepsilon) = \sup\{x \in [0, \infty) : \ \alpha_j(x) \geq \varepsilon\} \ (j = 1, 2, \ldots) \tag{2}$$

and ψ_1 denotes the inverse function of α_1. On the other hand, since
$\lim_{x\to\infty} \alpha_1(x) = 0$, the function α_1 has a continuous inverse function defined in the interval $(0, \alpha_1(0)]$. We consider the following operators:
1) L^0 and $L^{'}$ be operators in the space $L_2(0, \psi_1(\varepsilon); H)$, which are formed by expression (1) and with the boundary conditions $y(0) = y(\psi_1(\varepsilon)) = 0$, $y^{'}(0) = y^{'}(\psi_1(\varepsilon)) = 0$, respectively. Here, $\varepsilon \in (0, \alpha_1(0)]$.
2)L_i and $L_i^{'}$ be operators in the space $L_2(x_{i-1}, x_i; H)$ which are formed by expression (1) and with the boundary conditions $y(x_{i-1}) = y(x_i) = 0$, $y^{'}(x_{i-1}) = y^{'}(x_i) = 0$, respectively.
3)$L_{i(1)}$ be operator in the space $L_2(x_{i-1}, x_i; H)$ which is formed by the differential expression $-p(x_i)y^{''}(x) - Q(x_i)y(x)$ and with the boundary conditions $y(x_{i-1}) = y(x_i) = 0$.
4) $L_{i(1)}^{'}$ be operator in the space $L_2(x_{i-1}, x_i; H)$ which is formed by the differential expression $-p(x_{i-1})y^{''}(x) - Q(x_{i-1})y(x)$ and with the boundary conditions $y^{'}(x_{i-1}) = y^{'}(x_i) = 0$.
Divide the interval $[0, \psi_1(\varepsilon)]$ by the intervals at the length

$$\delta = \frac{\psi_1(\varepsilon)}{[\psi_1^k] + 1} \tag{3}$$

Here, $k \in (0,1)$ is constant number and ε is any positive number satisfying the inequality $\psi_1^k(\varepsilon) \geq 2$. [.] shows the greatest integer function whose value at any number x is the greatest integer less than or equal to x. Let the partition points of the interval $[0, \psi_1(\varepsilon)]$ be $0 = x_0 < x_1 < \ldots < x_M = \psi_1(\varepsilon)$. Let $N(\lambda), N^0(\lambda), N'(\lambda), n_i(\varepsilon)$ and $n_{i(1)}(\lambda)$ be numbers of eigenvalues smaller than $-\lambda$ ($\lambda > 0$) of the operators L, L^0, L', L_i, and $L_{i(1)}$, respectively. Let us write $n_i, n_{i(1)}$ instead of $n_i(\varepsilon), n_{i(1)}(\varepsilon)$, respectively. If $Q(x)$ and $p(x)$ satisfy the conditions $Q1) - Q3)$ and $p1) - p3)$ respectively,then the inequalities

$$N^0(\varepsilon) \leq N(\varepsilon) \leq N'(\varepsilon) \tag{4}$$

are satisfied, [13]. We can similarly show that the inequality (5)

$$N^0(\lambda) \leq N(\lambda) \leq N'(\lambda) \quad (\forall \lambda \in [\varepsilon, \infty)) \tag{5}$$

is satisfied.
Let $-\mu_{i(1)1} \leq -\mu_{i(1)2} \leq -\mu_{i(1)3} \leq \ldots$ be eigenvalues of the operator $L_{i(1)}$ and

$$a_j(x,t) = \alpha_j(x) - p(x)(\frac{\pi t}{\delta})^2 \quad (j = 1, 2, \ldots) \tag{6}$$

$$b_j(\varepsilon, x) = \frac{\delta}{\pi}\sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} \quad (j = 1, 2, \ldots) \tag{7}$$

$$\beta_j(\varepsilon, x) = \int_0^{b_j(\varepsilon,x)} a_j(x,t)dt \quad (j = 1, 2, \ldots) \tag{8}$$

$$\varphi_{i,j}(\varepsilon) = min\{x_{i+1}, \psi_j(\varepsilon)\} \qquad (i = 1, 2, \ldots, M-1) \tag{9}$$

Lemma 1 *If the operator function Q(x) and the scalar function p(x) satisfy the conditions Q1)-Q3) and p1)-p3), then we have*

$$\sum_{m=1}^{n_{i(1)}} \mu_{i(1)m} > \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_i) > \varepsilon}} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \beta_j(\varepsilon, x)dx - 3 \sum_{\substack{j \\ \alpha_j(0) > \varepsilon}} \alpha_j(0)$$

for small positive values of ε.

Theorem 2 *If the operator function Q(x) and the scalar function* $p(x)$ *satisfy the conditions* $Q1) - Q3), p1) - p3)$, *then we have*

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i > \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx - const. \sum_{j=1}^{l_\varepsilon} \int_0^{\delta} \alpha_j^{\frac{3}{2}}(x)dx - const.\psi_1^k(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

for small positive values of ε. *Here,* $l_\varepsilon = \sum_{\alpha_j(0) \geq \varepsilon} 1$.

Lemma 3 *If the operator function* $Q(x)$ *and the scalar function* $p(x)$ *satisfy the conditions* $Q1) - Q3)$, $p1) - p3)$, *then the inequality*

$$\sum_{m=1}^{n'_{i(1)}} \mu'_{i(1)m} \leq \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_{i-1}) > \varepsilon}} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx + \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \ (i = 2, 3, \ldots)$$

is satisfied for the small positive values of ε.

Theorem 4 *If the operator functions $Q(x)$ and the scalar function $p(x)$ satisfy the condition $Q1)-Q3)$, and $p1)-p3)$, then we have*

$$\sum_{i=1}^{N(\varepsilon)}\lambda_i < \sum_{m=1}^{n_1'}\mu_m' + \frac{1}{\delta}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx + \frac{\psi_1(\varepsilon)}{\delta}\sum_{j=1}^{l_\varepsilon}\alpha_j(0)$$

for the small positive values of ε.

Theorem 5 *If the operator function $Q(x)$ and the scalar function $p(x)$ satisfy the conditions $Q1)-Q3)$, and $p1)-p3)$, then we have*

$$\sum_{i=1}^{N(\varepsilon)}\lambda_i < \frac{1}{\delta}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx + const.\sum_{j=1}^{l_\varepsilon}\int_0^{\delta}\alpha_j^{\frac{3}{2}}(x)dx + const.\psi_1^k(\varepsilon)\sum_{j=1}^{l_\varepsilon}\alpha_j(0)$$

for small positive values of ε.

Let us denote the function of the form $ln_0x = x$, $ln_nx = ln(ln_{n-1}x)$ by ln_nx $\quad(n = 0,1,2,\ldots)$ and we suppose that the function $\alpha_1(x) = \|Q(x)\|$ satisfies the following conditions:
$\alpha1)$ There are a number $\xi > 0$ and a natural number $n \geq 1$ such that the function $\alpha_1(x) - (ln_nx)^{-\xi}$ is neither negative nor monotone increasing in the interval $[b,\infty)$ $\quad(b > 0)$.
$\alpha2)$ For every $\eta > 0$, $\lim_{x\to\infty}\alpha_1(x)x^{k_0-\eta} = \lim_{x\to\infty}[\alpha_1(x)x^{k_0+\eta}]^{-1} = 0$. Here, k_0 is a constant in the interval $(0,\frac{2}{3})$.
We are at the position to give to the main results.

Theorem 6 *If the operator function $Q(x)$, the scaler function $p(x)$, and $\alpha(x)$ satisfy the conditions $Q1)-Q3)$, $p1)-p3)$ and $\alpha1)$, respectively. In addition, the series $\sum_{j=1}^{\infty}[\alpha_j(0)]^m$ is convergent for a constant $m \in (0,\infty)$, then the asymptotic formula*

$$\sum_{-\lambda_i<-\varepsilon}\lambda_i = \frac{1}{3\pi}\left[1 + O(e^{-\varepsilon^{-\beta}})\right]\sum_j\int_{\alpha_j(x)\geq\varepsilon}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}(2\alpha_j(x)+\varepsilon)dx$$

is satisfied as $\varepsilon \to +0$. Here, β is a positive constant.

Theorem 7 *If the operator function $Q(x)$, the scalar function $p(x)$ and $\alpha(x)$ satisfy the conditions $Q1)-Q3)$, $p1)-p3)$ and $\alpha1)-\alpha2)$ respectively. In addition, the series $\sum_{j=1}^{\infty}[\alpha_j(0)]^m$ is convergent for a constant m satisfying the condition*

$$0 < m < \frac{(2-3k_0)^2}{2k_0(4-3k_0)} \tag{10}$$

then the asymptotic formula

$$\sum_{-\lambda_i<-\varepsilon}\lambda_i = \frac{1}{3\pi}[1 + O(\varepsilon^{t_0})]\sum_j\int_{\alpha_j(x)\geq\varepsilon}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}(2\alpha_j(x)+\varepsilon)dx$$

is satisfied as $\varepsilon \to 0$, where t_0 is a positive constant.

3. Proofs

Proof. [Proof of Lemma 1] Let us consider the operator $L_{i(1)}$ which is formed by differential expression $-p(x_i)y''(x) - Q(x_i)y(x)$ with the boundary conditions $y(x_{i-1}) = y(x_i) = 0$. The eigenvalues of the operator $L_{i(1)}$ are $p(x_i)\Big(\frac{m\pi}{x_i - x_{i-1}}\Big)^2 - \alpha_j(x_i)$ $(m = 1, 2, \ldots; j = 1, 2, \ldots)$, therefore $n_{i(1)}$ is the number of pairs (m, j) $(m, j \geq 1)$ satisfying the inequality

$$p(x_i)(\frac{m\pi}{\delta})^2 - \alpha_j(x_i) < -\varepsilon \quad (\delta = x_i - x_{i-1}) \tag{11}$$

By using (5),(6) and (11), we obtain

$$\sum_{m=1}^{n_{i(1)}} \mu_{i(1)m} = \sum_{\substack{j \\ \alpha_j(x_i) > \varepsilon}} \sum_{\substack{m \\ a_j(x_i, m) > \varepsilon}} a_j(x_i, m)$$

$$\geq \sum_{\substack{j \\ \alpha_j(x_i) > \varepsilon}} \sum_{m=1}^{[b_j(\varepsilon, x_i)]-1} a_j(x_i, m) \tag{12}$$

For the sum $\sum\limits_{m=1}^{[b_j(\varepsilon, x_i)]-1} a_j(x_i, m)$ in (12)

$$\sum_{m=1}^{[b_j(\varepsilon, x_i)]-1} a_j(x_i, m) \geq \int_1^{b_j(\varepsilon, x_i)-2} a_j(x_i, t)dt = \int_0^{b_j(\varepsilon, x_i)} a_j(x_i, t)dt - \int_0^1 a_j(x_i, t)dt$$

$$- \int_{b_j(\varepsilon, x_i)-2}^{b_j(\varepsilon, x_i)} a_j(x_i, t)dt > \int_0^{b_j(\varepsilon, x_i)} \Big(a_j(x_i, t)dt - 3\alpha_j(x_j)\Big)dt$$

$$= \beta_j(\varepsilon, x_i) - 3\alpha_j(x_i) \tag{13}$$

is obtained. If we consider that the functions $\beta_j(\varepsilon, x)$ $(j = 1, 2, \ldots)$ are decreasing, from (9),(12) and (13)

$$\sum_{m=1}^{n_{i(1)}} \mu_{i(1)m} > \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_i) > \varepsilon}} \int_{x_i}^{x_{i+1}} \beta_j(\varepsilon, x_i)dx - 3 \sum_{\substack{j \\ \alpha_j(0) > \varepsilon}} \alpha_j(0)$$

$$\geq \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_i) > \varepsilon}} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \beta_j(\varepsilon, x_i)dx - 3 \sum_{\substack{j \\ \alpha_j(0) > \varepsilon}} \alpha_j(0)$$

is obtained. □

Proof. [Proof of Theorem 2] We can easily show that $L_i < L_{i(1)}$. In this case, it is known that

$$n_i(\lambda) \geq n_{i(1)}(\lambda) \tag{14}$$

[17]. On the other hand, from variation principles of R. Courant [18], we have

$$N^0(\lambda) \geq \sum_{i=1}^{M} n_i(\lambda). \tag{15}$$

From (14) and (15)

$$N^0(\lambda) \geq \sum_{i=1}^{M} n_{i(1)}(\lambda) \quad (\lambda \geq \varepsilon) \tag{16}$$

is obtained. From (5) and (16)

$$N(\lambda) \geq \sum_{i=1}^{M} n_{i(1)}(\lambda) \quad (\forall \lambda \geq \varepsilon) \tag{17}$$

is found. By using (17), we can show that the inequality

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \geq \sum_{i=1}^{M} \sum_{m=1}^{n_{i(1)}} \mu_{i(1)m} \tag{18}$$

is satisfied. By the Lemma 2 and (18)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \geq \sum_{i=1}^{M-1} \Big\{ \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x)>\varepsilon}} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \beta_j(\varepsilon, x)dx - 3 \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \Big\}$$

$$= \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_i)>\varepsilon}} \sum_{i} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \beta_j(\varepsilon, x)dx - 3(M-1) \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{19}$$

is obtained. Since the functions $\alpha_j(x)$ $(j = 1, 2, \ldots)$ are decreasing, we have

$$\sum_{\substack{j \\ \alpha_j(x_i)>\varepsilon}} \sum_{i} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \beta_j(\varepsilon, x)dx = \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i \\ \alpha_j(x_i)>\varepsilon}} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \beta_j(\varepsilon, x)dx. \tag{20}$$

From (19) and (20)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \geq \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i \\ \alpha_j(x_i)>\varepsilon}} \int_{x_i}^{\varphi_{i,j}(\varepsilon)} \beta_j(\varepsilon, x)dx - 3M \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{21}$$

is obtained. Putting (9) into the right-hand side of inequality (21)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \geq \frac{1}{\delta} \sum_{\alpha_j(x_1)>\varepsilon} \Big[\int_{x_1}^{x_2} \beta_j(\varepsilon, x)dx + \int_{x_2}^{x_3} \beta_j(\varepsilon, x)dx + \ldots + \int_{x_{i_0}}^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx \Big]$$

$$-3M \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{22}$$

is found. Here, i_0 is a natural number satisfying the following condition:

$$x_{i_0} < \psi_j(\varepsilon) \leq x_{i_0+1}.$$

By using (9) and (22)

$$\sum_{i=1}^{N(\varepsilon)}\lambda_i \geq \frac{1}{\delta}\sum_{\psi_j(\varepsilon)>x_1}\int_{x_1}^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx - 3M\sum_{j=1}^{l_\varepsilon}\alpha_j(0) = \frac{1}{\delta}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx$$

$$-\frac{1}{\delta}\sum_{\psi_j(\varepsilon)<x_1}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx - \frac{1}{\delta}\sum_{\psi_j(\varepsilon)\geq x_1}\int_0^{x_1}\beta_j(\varepsilon,x)dx - 3M\sum_{j=1}^{l_\varepsilon}\alpha_j(0)$$

$$= \frac{1}{\delta}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx - \frac{1}{\delta}\sum_{j=1}^{l_\varepsilon}\int_0^{\varphi_{0,j}(\varepsilon)}\beta_j(\varepsilon,x)dx - 3M\sum_{j=1}^{l_\varepsilon}\alpha_j(0) \tag{23}$$

is obtained. From (6),(7), and (8)

$$\frac{1}{\delta}\beta_j(\varepsilon,x) = \frac{1}{\delta}\int_0^{b_j(\varepsilon,x)}[\alpha_j(x) - p(x)(\frac{\pi t}{\delta})^2]dt = \frac{1}{\delta}\alpha_j(x)b_j(\varepsilon,x) - \pi^2\frac{p(x)}{3\delta^3}b_j^3(\varepsilon,x)$$

$$= \frac{1}{\delta}b_j(\varepsilon,x)[\alpha_j(x) - \pi^2\frac{p(x)}{3\delta^2}b_j^2(\varepsilon,x)] = \frac{1}{\pi}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\Big[\alpha_j(x) - \pi^2\frac{p(x)}{3\delta^2}.$$

$$\frac{\delta^2(\alpha_j(x)-\varepsilon)}{\pi^2 p(x)}\Big] = \frac{1}{\pi}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\Big[\frac{2}{3}\alpha_j(x) + \frac{\varepsilon}{3}\Big] < const.\alpha_j^{\frac{3}{2}}(x)dx \tag{24}$$

is found. From (9) and (24),

$$\frac{1}{\delta}\sum_{j=1}^{l_\varepsilon}\int_0^{\varphi_{0,j}(\varepsilon)}\beta_j(\varepsilon,x)dx < const.\sum_{j=1}^{l_\varepsilon}\int_0^{\delta}\alpha_j^{\frac{3}{2}}(x)dx \tag{25}$$

is obtained. From (3),(23) and (25)

$$\sum_{i=1}^{N(\varepsilon)}\lambda_i > \frac{1}{\delta}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx - const.\sum_{j=1}^{l_\varepsilon}\int_0^{\delta}\alpha_j^{\frac{3}{2}}(x)dx - const.\psi_1^k(\varepsilon)\sum_{j=1}^{l_\varepsilon}\alpha_j(0)$$

□

Let $-\mu'_{i(1)1} \leq -\mu'_{i(1)2} \leq -\mu'_{i(1)3} \leq \ldots$ be eigenvalues of the operator $L'_{i(1)}$ and $n'_{i(1)}(\lambda)$ be number of the eigenvalues smaller than $-\lambda$ ($\lambda > 0$) of the operator $L'_{i(1)}$. Moreover, we will simply write $n'_{i(1)}$ instead of $n'_{i(1)}(\varepsilon)$.

Proof. [Proof of Lemma 3] The eigenvalues of the operator $L'_{i(1)}$ are the form $p(x_{i-1})\Big[\frac{(m-1)\pi}{x_i-x_{i-1}}\Big]^2 - \alpha_j(x_{i-1})$ ($m = 1,2,\ldots; j = 1,2,\ldots$). Therefore, $n'_{i(1)}$ is the number of pairs (m,j) ($m, j \geq 1$) satisfying the inequality

$$p(x_{i-1})\Big[\frac{(m-1)\pi}{x_i-x_{i-1}}\Big]^2 - \alpha_j(x_{i-1}) < -\varepsilon \tag{26}$$

From (6),(7) and (26)

$$\sum_{m=1}^{n'_{i(1)}} \mu'_{i(1)m} = \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \sum_{\substack{m \\ \alpha_j(x_{i-1},m-1)>\varepsilon}} \alpha_j(x_{i-1}, m-1)$$

$$= \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \sum_{m=1}^{[b_j(\varepsilon,x_{i-1})]+1} \alpha_j(x_{i-1}, m-1) \tag{27}$$

is found. It is easy to see that

$$\sum_{m=1}^{[b_j(\varepsilon,x_{i+1})]+1} \alpha_j(x_{i-1}, m-1) \leq \alpha_j(x_{i-1}) + \int_0^{b_j(\varepsilon,x_{i-1})} \alpha_j(x_{i-1}, t)dt$$

$$= \alpha_j(x_{i-1}) + \beta_j(\varepsilon, x_{i-1}) \tag{28}$$

Since the functions $\beta_j(\varepsilon, x)$ $(j = 1, 2, \dots)$ are monotone decreasing by (27) and (28),

$$\sum_{m=1}^{n'_{i(1)}} \mu'_{i(1)m} \leq \sum_{j=1}^{l_\varepsilon} \alpha_j(0) + \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x_{i-1})dx$$

$$< \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx + \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \quad (i = 2, 3, \dots)$$

is obtained. □

Let $n'_i(\lambda)$ be number of the eigenvalues smaller than $-\lambda$ $(\lambda > 0)$ of the operator L'_i, $-\mu'_1 \leq -\mu'_2 \leq -\mu'_3 \leq \dots$ be eigenvalues of the operator L'_1 and $n'_i(\varepsilon) = n'_i$.

Proof. [Proof of Theorem 4] We can easily show that $L'_i > L'_{i(1)}$. In this case we have

$$n'_i(\lambda) \leq n'_{i(1)}(\lambda) \tag{29}$$

[17]. On the other hand, from variation principles of R.Courant [18], we have

$$N'(\lambda) \leq \sum_{i=1}^{M} n'_i(\lambda) \tag{30}$$

From (29) and (30),

$$N'(\lambda) \leq \sum_{i=2}^{M} n'_{i(1)}(\lambda) + n'_1(\lambda) \tag{31}$$

is obtained. From (5) and (31)

$$N(\lambda) \leq \sum_{i=2}^{M} n'_{i(1)}(\lambda) + n'_1(\lambda) \quad (\forall \lambda \geq \varepsilon) \tag{32}$$

is found. By using (32), we have

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \le \sum_{i=2}^{M} \sum_{m=1}^{n'_{i(1)}} \mu'_{i(1)m} + \sum_{m=1}^{n'_1} \mu'_m \tag{33}$$

By using Lemma 1 and (33)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \le \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{i=2}^{M} \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx + M \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

$$= \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \sum_{i\ge 2} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx + M \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{34}$$

is found. Since the functions $\alpha_j(x) \quad (j = 1, 2, \dots)$ are monotone decreasing, we have

$$\sum_{\substack{j \\ \alpha_j(x_{i-1})>\varepsilon}} \sum_{i\ge 2} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx = \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i\ge 2 \\ \alpha_j(x_{i-1})>\varepsilon}} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx. \tag{35}$$

From (34) and (35)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i < \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \sum_{\substack{i\ge 2 \\ \alpha_j(x_{i-1})>\varepsilon}} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx + M \sum_{j=1}^{l_\varepsilon} \alpha_j(0) = \sum_{m=1}^{n'_1} \mu'_m$$

$$+\frac{1}{\delta} \sum_{\substack{j \\ \alpha_j(x_1)>\varepsilon}} \Big[\int_0^{x_1} \beta_j(\varepsilon, x)dx + \int_{x_1}^{x_2} \beta_j(\varepsilon, x)dx \dots + \int_{x_{i_0-1}}^{x_{i_0}} \beta_j(\varepsilon, x)dx \Big] + M \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

is obtained. Here, i_0 is a natural number satisfying the conditions

$$\alpha_j(x_{i_0}) > \varepsilon \qquad \alpha_j(x_{i_0+1}) \le \varepsilon. \tag{36}$$

From (2.1)

$$x_{i_0} \le \psi_j(\varepsilon). \tag{37}$$

From (36) and (37),

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i < \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx + \frac{\psi_1(\varepsilon)}{\delta} \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

is found. □

Let

$$\delta_i = \frac{\delta_{i-1}}{[\delta_{i-1}\psi_1^{(i+1)k-1}] + 1}, \quad (i = 1, 2, \dots; \delta_0 = \delta) \tag{38}$$

$$a_{j(i)}(x,t) = \alpha_j(x) - p(x)\Big(\frac{\pi t}{\delta_i}\Big)^2, \quad b_{j(i)}(\varepsilon, x) = \frac{\delta_i}{\pi}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}},$$

$$\beta_{j(i)}(\varepsilon, x) = \int\limits_0^{b_{j(i)}(\varepsilon,x)} a_{j(i)}(x,t)dt$$

$$\varphi_j(\delta_i, \varepsilon) = min\{\delta_i, \psi_j(\varepsilon)\}. \quad (i = 0, 1, 2, \dots) \tag{39}$$

Let $L_{(i)}$ be operator in the space $L_2(0, \delta_i; H)$ which is formed by the expression (1) and with the boundary conditions

$$y'(0) = y'(\delta_i) = 0. \tag{40}$$

Moreover, let $L_{(i)}^{(0)}$ be operator which is formed by the expression $-p(0)y''(x) - Q(0)y(x)$ and with the boundary conditions (40). Let $-\mu_{(i)1} \leq -\mu_{(i)2} \leq \dots$ and $-\mu_{(i)1}^{(0)} \leq -\mu_{(i)2}^{(0)} \leq \dots$ be eigenvalues smaller than $-\lambda$ ($\lambda > 0$) of the operators $L_{(i)}$ and $L_{(i)}^{(0)}$,respectively.
Let $n_i(\lambda)$ and $n_{(i)}^{(0)}(\lambda)$ be numbers of the eigenvalues smaller than $-\lambda$ ($\lambda > 0$) of the operators $L_{(i)}$ and $L_{(i)}^{(0)}$, respectively.
Since $L_{(i)} \geq L_{(i)}^{(0)}$, we have

$$n_{(i)}(\lambda) \leq n_{(i)}^{(0)}(\lambda), \tag{41}$$

[17]. By using (41), we can show that

$$\sum_{m=1}^{n_i} \mu_{(i)m} \leq \sum_{m=1}^{n_i^{(0)}} \mu_{(i)m}^{(0)}. \tag{42}$$

Here, $n_{(i)} = n_{(i)}(\varepsilon)$,$n_{(i)}^{(0)} = n_{(i)}^{(0)}(\varepsilon)$. Since $\delta_{-1} = \psi_1(\varepsilon)$ and from the formula (39)

$$\frac{\delta_{i-1}}{\delta_i} = [\delta_{i-1}\psi_1^{(i+1)k-1}(\varepsilon)] + 1 \leq \delta_{i-1}\psi_1^{(i+1)k-1}(\varepsilon) + 1$$

$$= \frac{\delta_{i-2}}{[\delta_{i-2}\psi_1^{ik-1}(\varepsilon)] + 1}\psi_1^{(i+1).k-1}(\varepsilon) + 1$$

$$< \frac{\delta_{i-2}}{\delta_{i-2}\psi_1^{ik-1}(\varepsilon)}\psi_1^{(i+1).k-1}(\varepsilon) + 1 = \psi_1^k(\varepsilon) + 1 \ (i = 1, 2, \dots)$$

is obtained. From the last relation, we find

$$\frac{\delta_{i-1}}{\delta_i} < 2\psi_1^k(\varepsilon), \quad (i = 1, 2, \dots) \tag{43}$$

for the values of ε satisfying the inequality $\psi_1^k(\varepsilon) \geq 2$.

Proof. [Proof of Theorem 5] By the similar way to the proof of Theorem 4, the following inequality

$$\sum_{m=1}^{n_1'} \mu_m' < \sum_{m=1}^{n_1} \mu_{(1)m} + \frac{1}{\delta_1}\sum_{\psi_j(\varepsilon)<\delta_0} \int\limits_0^{\psi_j(\varepsilon)} \beta_{j(1)}(\varepsilon, x)dx + \frac{1}{\delta_1}\sum_{\psi_j(\varepsilon)\geq\delta_0} \int\limits_0^{\delta_0} \beta_{j(1)}(\varepsilon, x)dx + \frac{\delta_0}{\delta_1}\sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{44}$$

can be proved. If we use the equation (39) in (44), then we have

$$\sum_{m=1}^{n_1^{'}} \mu_m^{'} < \sum_{m=1}^{n_1} \mu_{(1)m} + \frac{1}{\delta_1} \sum_{j=1}^{l_\varepsilon} \sum_{0}^{\varphi_j(\delta_0,\varepsilon)} \beta_j(\varepsilon, x) dx + \frac{\delta_0}{\delta_1} \sum_{j=1}^{l_\varepsilon} \alpha_j(0). \tag{45}$$

If we apply the inequality (45) for the eigenvalues of the operator $L_{(i)}$, then

$$\sum_{m=1}^{n_{(i)}} \mu_{(i)m} < \sum_{m=1}^{n_{(i+1)}} \mu_{(i+1)m} + \frac{1}{\delta_{i+1}} \sum_{j=1}^{l_\varepsilon} \int_0^{\varphi_j(\delta_i,\varepsilon)} \beta_{j(i+1)}(\varepsilon, x) dx + \frac{\delta_i}{\delta_{i+1}} \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{46}$$

is obtained. From (43) and (46)

$$\sum_{m=1}^{n_{(i)}} \mu_{(i)m} < \sum_{m=1}^{n_{(i+1)}} \mu_{(i+1)m} + \frac{1}{\delta_{i+1}} \sum_{j=1}^{l_\varepsilon} \int_0^{\varphi_j(\delta_i,\varepsilon)} \beta_{j(i+1)}(\varepsilon, x) dx + 2\psi_1^k(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{47}$$

is found. By using (27) and (28)

$$\sum_{m=1}^{n_{(i+1)}^{(0)}} \mu_{(i+1)m}^{(0)} \leq \sum_{j=1}^{l_\varepsilon} \left(\alpha_j(0) + \beta_{j(i+1)}(\varepsilon, 0)\right) \tag{48}$$

is obtained. Moreover, if we use the equation (24), then we get

$$\beta_{j(i+1)}(\varepsilon, x) \leq const.\delta_{i+1} \alpha_j^{\frac{3}{2}}(x). \tag{49}$$

From (42), (48) and (49),

$$\sum_{m=1}^{n_{(i+1)}} \mu_{(i+1)m} \leq \sum_{j=1}^{l_\varepsilon} \alpha_j(0) + const.\delta_{i+1} \sum_{j=1}^{l_\varepsilon} \alpha_j^{\frac{3}{2}}(0) \tag{50}$$

is obtained. By using inequality (38), we find

$$\delta_{i_0+1} \leq 1. \tag{51}$$

Here, $i_0 \in N$ is a constant satisfying the condition $i_0 \geq \frac{1}{k} - 2$. From (50) and (51), we get

$$\sum_{m=1}^{n_{(i_0+1)}} \mu_{(i_0+1)m} \leq const. \sum_{j=1}^{l_\varepsilon} \alpha_j(0). \tag{52}$$

From (43),(45),(47) and (52),

$$\sum_{m=1}^{n_1^{'}} \mu_m^{'} \leq const. \sum_{j=1}^{l_\varepsilon} \alpha_j(0) + \sum_{i=0}^{i_0} \frac{1}{\delta_{i+1}} \int_0^{\varphi_j(\delta_i,\varepsilon)} \beta_{j(i+1)}(\varepsilon, x) dx + 2(i_0+1)\psi_1^k(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{53}$$

is found. From (39),(49) and (53),

$$\sum_{m=1}^{n_1^{'}} \mu_m^{'} < const. \sum_{j=1}^{l_\varepsilon} \int_0^{\delta} \alpha_j^{\frac{3}{2}}(x) dx + const.\psi_1^k(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \tag{54}$$

is obtained. By the Theorem 4 and (54), we have

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i < \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx + const. \sum_{j=1}^{l_\varepsilon} \int_0^{\delta} \alpha_j^{\frac{3}{2}}(x)dx + const.\psi_1^k(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_i(0)$$

is obtained. □

Proof. [Proof of Theorem 6] By using Theorem 2 and Lemma 3, we have

$$\Big|\sum_{i=1}^{N(\varepsilon)} \lambda_j - \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx\Big| < const.l_\varepsilon(\delta + \psi_1^k(\varepsilon))$$

for the small positive values of ε . If we take $k = \frac{1}{2}$ and consider (3)

$$\Big|\sum_{i=1}^{N(\varepsilon)} \lambda_j - \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx\Big| < const.l_\varepsilon \psi_1^{\frac{1}{2}}(\varepsilon) \tag{55}$$

is found. Let us take $f(\varepsilon) = \psi_1(\varepsilon)[ln\psi_1(\varepsilon)]^{-1}$ By using the function $p(x)$ which satisfies the condition $(p1)$ and the inequality (24)

$$\frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx > \frac{1}{\delta} \int_0^{\psi_1(\varepsilon)} \beta_1(\varepsilon, x)dx = \frac{1}{3\pi} \int_0^{\psi_1(\varepsilon)} \sqrt{\frac{\alpha_1(x)-\varepsilon}{p(x)}}(2\alpha_1(x) + \varepsilon)dx$$

$$> \frac{1}{3\pi} \int_{\frac{1}{2}f(\varepsilon)}^{f(\varepsilon)} \sqrt{\frac{\alpha_1(x) - \varepsilon}{p(x)}}(2\alpha_1(x) + \varepsilon)dx > const.f(\varepsilon)(\alpha_1(f(\varepsilon)) - \varepsilon)^{\frac{3}{2}} \tag{56}$$

is obtained. It is proved that

$$\alpha_1(f(\varepsilon)) - \varepsilon > (ln\psi_1(\varepsilon))^{-(\xi+1)(n+1)} \tag{57}$$

for the small values of $\varepsilon > 0$, [13]. From (56) and (57)

$$\frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx > const.\frac{\psi_1(\varepsilon)}{ln\psi_1(\varepsilon)}(ln\psi_1(\varepsilon))^{(\frac{-3}{2})(\xi+1)(n+1)}$$

$$> const.\psi_1^{\frac{3}{4}}(\varepsilon) \tag{58}$$

is found. From (55) and (58)

$$\left|\frac{\sum_{i=1}^{N(\varepsilon)} \lambda_i}{\delta^{-1} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x)dx} - 1\right| < const.l_\varepsilon \psi_1^{\frac{-1}{4}}(\varepsilon) \tag{59}$$

is obtained. Since the series $\sum_{m=1}^{\infty} [\alpha_j(0)]^m$ is convergent, we have

$$const > \sum_{\alpha_j(0)\geq\varepsilon} [\alpha_j(0)]^m \geq \sum_{\alpha_j(0)\geq\varepsilon} \varepsilon^m = \varepsilon^m l_\varepsilon.$$

From last inequality

$$l_\varepsilon < const.\varepsilon^{-m} \tag{60}$$

is found. Since the function $\alpha_1(x)$ satisfy the condition $\alpha 1$), we have
$\varepsilon = \alpha_1(\psi_1(\varepsilon)) \geq (ln_n\psi_1(\varepsilon))^{-\xi} \geq (ln\psi_1(\varepsilon))^{-\xi}$ for the small values of $\varepsilon > 0$. From the last inequality,

$$\psi_1(\varepsilon) > \varepsilon^{\frac{-1}{\xi}} \tag{61}$$

is obtained. From (59),(60) and (61)

$$\Bigg|\frac{\sum\limits_{i=1}^{N(\varepsilon)} \lambda_i}{\delta^{-1}\sum\limits_{j=1}^{l_\varepsilon}\int\limits_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon,x)dx} - 1\Bigg| < const.\varepsilon^{-m}e^{\frac{-1}{4}\varepsilon^{\frac{-1}{\xi}}} < const.e^{-\varepsilon^{-\beta}} \tag{62}$$

is found. We can rewrite inequality (62)

$$\frac{\sum\limits_{i=1}^{N(\varepsilon)} \lambda_i}{\delta^{-1}\sum\limits_{j=1}^{l_\varepsilon}\int\limits_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon,x)dx} - 1 = O(e^{-\varepsilon^{-\beta}}) \tag{63}$$

as $\varepsilon \to 0$. From (3),(24) and (63)

$$\sum_{-\lambda_i<-\varepsilon} \lambda_i = \frac{1}{3\pi}[1 + O(e^{-\varepsilon^{-\beta}})]\sum_j \int\limits_{\alpha_j(x)\geq\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}(2\alpha_j(x)+\varepsilon))dx$$

as $\varepsilon \to 0$, is obtained. □

Proof. [Proof of Theorem 7] By Theorem 2 and Lemma 3, we have

$$\Big|\sum_{i=1}^{N(\varepsilon)} \lambda_i - \delta^{-1}\sum_{j=1}^{l_\varepsilon}\int\limits_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon,x)dx\Big| < const.l_\varepsilon\Big(\int\limits_0^{\delta} \alpha_1^{\frac{3}{2}}(x)dx + \psi_1^k(\varepsilon)\Big) \tag{64}$$

for small values of $\varepsilon > 0$. Since the function $\alpha_1(x)$ is decreasing,

$$\alpha_1(x) \geq \alpha_1(\psi(2\varepsilon)) = 2\varepsilon \tag{65}$$

in the interval $[0, \psi_1(2\varepsilon)]$. Since the function $p(x)$ satisfies the condition $p1$) and (24),(65) we find

$$\delta^{-1}\sum_{j=1}^{l_\varepsilon}\int\limits_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon,x)dx > \frac{1}{3\pi}\int\limits_0^{\psi_1(\varepsilon)} \sqrt{\frac{\alpha_1(x)-\varepsilon}{p(x)}}(2\alpha_1(x)+\varepsilon)dx$$

$$> const.\varepsilon^{\frac{3}{2}}\psi_1(2\varepsilon). \tag{66}$$

If we consider that the function $\alpha_1(x)$ satisfies the condition $\alpha 2$) and
$\lim\limits_{\varepsilon\to\infty} \psi_1(\varepsilon) = \infty$, then we have $\lim\limits_{\varepsilon\to\infty}[\alpha_1(\psi_1(2\varepsilon))(\psi_1(2\varepsilon))^{k_0+\eta}]^{-1} = 0$. From the last equality, we obtain

$$\psi_1(2\varepsilon) > (\varepsilon)^{\frac{-1}{k_0+\eta}} \tag{67}$$

for the small value of $\varepsilon > 0$. From (66) and (67)

$$\delta^{-1}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx > const.\varepsilon^{\frac{3k_0+3\eta-2}{2(k_0+\eta)}} \tag{68}$$

is found. We limit the integral $\int_0^\delta \alpha_1^{\frac{3}{2}}(x)dx$ at the right hand side of the inequality (64). Since the function $\alpha_1(x)$ satisfies the condition $\alpha 2)$, we have

$$\alpha_1(x) \le const.x^{\eta-k_0} \quad (\eta < k_0) \tag{69}$$

Therefore we have

$$\int_0^\delta \alpha_1^{\frac{3}{2}}(x)dx \le const.\int_0^\delta x^{\frac{3}{2}(\eta-k_0)}dx < const.\delta^{\frac{1}{2}(2-3k_0+3\eta)}. \tag{70}$$

On the other hand, from (3)

$$\delta < \psi_1^{1-k}(\varepsilon) \tag{71}$$

is obtained. If we take $x = \psi_1(\varepsilon)$ in the inequality (69), then we find

$$\alpha_1(\psi_1(\varepsilon)) \le const.\psi_1^{\eta-k_0}(\varepsilon) \quad (\eta < k_0)$$

or

$$\psi_1(\varepsilon) \le const.\varepsilon^{\frac{-1}{k_0-\eta}}. \tag{72}$$

From (70),(71) and (72), we have

$$\int_0^\delta \alpha_1^{\frac{3}{2}}(x)dx \le const.\varepsilon^{-\frac{(1-k)(2-3k_0+3\eta)}{2(k_0-\eta)}}. \tag{73}$$

From (60),(72) and (73)

$$l_\varepsilon\int_0^\delta \alpha_1^{\frac{3}{2}}(x)dx < const.\varepsilon^{-m-\frac{(1-k)(2-3k_0+3\eta)}{2(k_0-\eta)}} \tag{74}$$

$$l_\varepsilon\psi_1^k(\varepsilon) < const.\varepsilon^{\frac{-m(k_0-\eta)+k}{(k_0-\eta)}} \tag{75}$$

are found. From (68),(74) and (75) we obtain

$$\frac{l_\varepsilon\int_0^\delta \alpha_1^{\frac{3}{2}}(x)dx}{\delta^{-1}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx} < const.\varepsilon^{F_1(\eta)} \tag{76}$$

$$\frac{l_\varepsilon\psi_1^k(\varepsilon)dx}{\delta^{-1}\sum_{j=1}^{l_\varepsilon}\int_0^{\psi_j(\varepsilon)}\beta_j(\varepsilon,x)dx} < const.\varepsilon^{F_2(\eta)}. \tag{77}$$

Here,

$$F_1(\eta) = -m - \frac{(1-k)(2-3k_0+3\eta)}{2(k_0-\eta)} - \frac{3k_0+3\eta-2}{2(k_0+\eta)}$$

$$F_2(\eta) = -\frac{m(k_0-\eta)+k}{(k_0-\eta)} - \frac{3k_0+3\eta-2}{2(k_0+\eta)}.$$

There is a number $\omega = \omega(t) > 0 \quad (0 < \eta < \omega)$ such that

$$F_1(\eta) > \frac{2k-2k_0m-3kk_0}{2k_0} - t, \tag{78}$$

$$F_2(\eta) > \frac{2-3k_0-2k_0m-2k}{2k_0} - t \tag{79}$$

for every $t > 0$. If we take

$$k = \frac{(2-3k_0)^2+6k_0^2m}{4(2-3k_0)}, \quad t = t_0 = \frac{1}{16k_0\Big((2-3k_0)^2+6k_0^2m-8k_0m\Big)}$$

in the inequalities (78) and (79), then we have

$$F_1(\eta) > t_0; \quad F_2(\eta) > t_0. \tag{80}$$

Since the number m satisfies the condition (10), we have $k \in (0,1)$ and $t_0 > 0$. From (64),(76), (77) and (80) we obtain

$$\left| \frac{\sum\limits_{i=1}^{N(\varepsilon)} \lambda_i}{\delta^{-1}\sum\limits_{j=1}^{l_\varepsilon} \int\limits_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon,x)dx} - 1 \right| < const.\varepsilon^{t_0}. \tag{81}$$

By (24),(78) and (81) we have the asymptotic formula

$$\sum_{-\lambda_i<-\varepsilon} \lambda_i = \frac{1}{3\pi}[1+O(\varepsilon^{t_0})] \sum_j \int\limits_{\alpha_j(x)\geq\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}(2\alpha_j(x)+\varepsilon)dx$$

as $\varepsilon \to 0$. This completes the proof. □

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