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Some Mean and Uniform Ergodic Type Theorems

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Abstract. Let X be a Banach space and $T \in B(X)$. Cohen determined a class of regular infinite matrices $A = (a_{nk})$ for which $L_n := \sum_{k=1}^{\infty} a_{nk} T^k$ converges strongly to an element invariant under T. In the present paper we study A-mean and A-uniform ergodic type results when $A = (a_{nk})$ is a regular infinite matrix satisfying Cohen's uniformity condition $\lim_j \sum_{k=j}^{\infty} |a_{n,k+1} - a_{nk}| = 0$, uniformly in n.

1. Introduction

The ergodic theorem asserts that if $T : X \to X$ is a bounded linear operator on a reflexive Banach space whose iterates T^j form a bounded sequence of bounded linear operators, then its Cesàro averages

$$M_n(T) := \frac{1}{n} \sum_{k=1}^{n} T^k$$

form a sequence of bounded linear operators that converge strongly to a projection onto the kernel of the operator $I - T$.

Now let X be a Banacah space and $T \in B(X)$. An operator $T \in B(X)$ is called mean ergodic, respectively uniformly ergodic, if $\{M_n(T)\}$ is strongly, respectively uniformly convergent in $B(X)$. Cohen [2] determined a class of regular infinite matrices $A = (a_{nk})$ for which

$$L_n := \sum_{k=1}^{\infty} a_{nk} T^k$$

converges strongly to an element invariant under T. He proved that such a sequence $\{L_n\}$ is strongly convergent provided that $\{L_n x : n \in \mathbb{N}\}$ is weakly compact and $\lim_j \sum_{k=j}^{\infty} |a_{n,k+1} - a_{nk}| = 0$ uniformly in n. Recall that, if the matrix $A = (a_{nk})$ maps convergent sequences into the convergent sequences leaving the limit

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invariant, then A is called a regular matrix. It is well known that A is regular if and only if (i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$, (ii) For all k, $\lim_n a_{nk} = 0$, (iii) $\lim_n \sum_{k=1}^{\infty} a_{nk} = 1$ (see, e.g, [1]).

It seems that Cohen's result provides a generalization of the mean ergodic theorems given by J. von Neumann [11], F. Riesz [10] and K. Yosida [13].

Throughout the paper we will call an operator $T \in B(X)$ an A-mean ergodic operator, respectively A-uniformly ergodic operator, if the limit of $\{L_n x\}$, respectively the limit of $\{L_n\}$ exists. In the present paper we first study A-mean ergodic type results in a Banach space. In particular we get an ergodic decomposition. Using this we also give necessary and sufficient conditions in order that $\{T^n x\}$ is convergent. This is an extension of Lemma 4.2 in [8]. Next we examine A-uniform ergodic type results which may be considered as an extension of Lin's result [4].

Inspired by the uniformity condition of Cohen, Yoshimoto [12] introduced a URS-method and gave some sufficent conditions for ergodic theorems in Banach Spaces. Furthermore, Jardas and Sarapa [3] studied this result in locally convex vector spaces. Then Oğuz and Orhan [9] gave a version of Cohen's result with the help of the sequence of infinite matrices.

An infinite matrix A is called a URS-method, if the matrix satisfies the regularity conditions and $\lim_{j\to\infty} \sum_{k=j}^{\infty} |a_{n,k+1} - a_{nk}| = 0$, (uniformly in n).

An operator $T \in B(X)$ is called power bounded if $\sup_n \|T^n\| < \infty$.

2. Mean Ergodic Type Theorems

In this section, using a URS-method we give some extensions of the mean ergodic type theorems. Our first proposition concerns an inequality for a URS-method. We should recall that our result is motivated by that of Proposition 2.1 of [7] proved for the Cesàro matrix.

Proposition 2.1. *Let X be a Banach space and $T \in B(X)$ be a power bounded operator. Suppose that $A = (a_{nk})$ is a URS-method. Given that $Y := \overline{(I-T)X}$ and $C_T := \sup_n \|T^n\|$, then we have for all $x \in X$, that*

$$\operatorname{dist}(x, Y) \le \varliminf_{n\to\infty} \left\| \sum_{k=1}^{\infty} a_{nk} T^k x \right\| \le \varlimsup_{n\to\infty} \left\| \sum_{k=1}^{\infty} a_{nk} T^k x \right\| \le \|A\| C_T \operatorname{dist}(x, Y),$$

where $\operatorname{dist}(x, Y)$ *stands for the distance of x to the set Y.*

Proof. Take an $x \in X$ such that $y = (I-T)x$. Since T is a power bounded operator and $A = (a_{nk})$ is a URS-method, the technique used in [2, pg 507], we have

$$\left\| \sum_{k=1}^{\infty} a_{nk} T^k (x - Tx) \right\| \to 0, \qquad (n \to \infty). \tag{1}$$

Since T is a power bounded operator, by (1) we obtain, for all $y \in Y$, that

$$\left\| \sum_{k=1}^{\infty} a_{nk} T^k y \right\| \to 0, \qquad (n \to \infty).$$

Then, for $y \in Y$ we find the following inequality

$$\left\| \sum_{k=1}^{\infty} a_{nk} T^k x - \sum_{k=1}^{\infty} a_{nk} T^k y \right\| \le \|A\| C_T \|x - y\|$$

which yields

$$\overline{\lim_{n\to\infty}} \left\| \sum_{k=1}^{\infty} a_{nk} T^k x \right\| \leq \|A\| C_T \operatorname{dist}(x, Y).$$

On the other hand, let us take $\varphi \in X'$ such that $T^*\varphi = \varphi$ and $\|\varphi\| \leq 1$. This clearly gives $\varphi(T^k x) = \varphi(x)$. Then for all $x \in X$, we can write

$$\sum_{k=1}^{\infty} a_{nk}\varphi(x) = \varphi\left(\sum_{k=1}^{\infty} a_{nk} T^k x\right).$$

Using the regularity of A and $\|\varphi\| \leq 1$, one can obtain, for all $x \in X$, that

$$|\varphi(x)| \leq \underline{\lim}_{n\to\infty} \left\| \sum_{k=1}^{\infty} a_{nk} T^k x \right\|.$$

Finally, the equality $\operatorname{dist}(x, Y) = \sup\{|\varphi(x)| : T^*\varphi = \varphi, \|\varphi\| \leq 1\}$ given in [6] implies that

$$\operatorname{dist}(x, Y) \leq \underline{\lim}_{n\to\infty} \left\| \sum_{k=1}^{\infty} a_{nk} T^k x \right\|$$

which concludes the proof. □

The following corollary is a direct consequence of Proposition 2.1.

Corollary 2.2. *Y is characterized by the following set:*

$$Y := \{x \in X : \lim_{n\to\infty} \left\| \sum_{k=1}^{\infty} a_{nk} T^k x \right\| = 0\}.$$

Motivated by that of Jardas and Sarapa [3] we will give a proof of the ergodic theorem. In order to do this, we shall use Corollary 2.2.

Theorem 2.3. *Let X be a Banach space and $T \in B(X)$ be a power bounded operator. Assume that $A = (a_{nk})$ is a URS-method and define L_n by*

$$L_n := \sum_{k=1}^{\infty} a_{nk} T^k \qquad (n = 1, 2, ...).$$

Let E denote the set of all $x \in X$ such that $\lim_{n\to\infty} L_n x$ exists. Then

$$E = \ker(I - T) \oplus \overline{(I - T)X}.$$

Proof. Take $x \in E$ and define $Px := \lim_{n\to\infty} L_n x$. Then it is obvious that $P : E \to E$ is a bounded linear operator. Since $x \in E$, we have

$$(I - T)Px = (I - T)\lim_{n\to\infty} L_n x = \lim_{n\to\infty} L_n(x - Tx).$$

Following [2, pg 507] we already know that

$$\lim_{n\to\infty} L_n(x - Tx) = 0.$$

Thus, we obtain $P = TP$ which immediately yields $P = T^nP$ for all $n \in \mathbb{N}$, and we get

$$\lim_{n\to\infty} L_n(Px) = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}T^kPx = Px.$$

Hence, for all $x \in E$ we have the following

$$P^2x = P(Px) = \lim_{n\to\infty} L_n(Px) = Px.$$

This directly gives us that

$$P^2 = P. \tag{2}$$

Now, we need to show that

$$P(E) = \ker(I - T). \tag{3}$$

Let $x \in \ker(I - T)$, then $Tx = x$ and $T^nx = x$ for all $n \in \mathbb{N}$. Thus,

$$\lim_{n\to\infty} L_nx = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}T^kx = x.$$

Since $Px = \lim L_nx = x$, we get $x \in P(E)$. Now take $x \in P(E) \subset E$. Then there exist a $z \in E$ such that $x = Pz$. Using (2) we find $x = Pz = P^2z = P(Pz) = Px$. Hence, we have

$$Tx = TPx = Px = x$$

which yields that $x \in \ker(I - T)$.

On the other hand, we immediately have

$$\overline{(I - T)X} = \ker P \tag{4}$$

because of Corollary 2.2. By (2), (3) and (4) we obtain

$$E = \ker(I - T) \oplus \overline{(I - T)X}$$

which completes the proof. □

The next theorem is an extension of Lemma 4.2 in [8].

Theorem 2.4. *Let X be a Banach space and let $T \in B(X)$ be a power bounded operator. Assume that $A = (a_{nk})$ is a URS-method and $x \in X$. Then the sequence $\{T^nx\}$ is convergent if and only if $\lim_{n\to\infty} \|T^{n+1}x - T^nx\| = 0$ and the sequence $\left\{\sum_{k=1}^{\infty} a_{nk}T^kx\right\}_{n=1}^{\infty}$ is convergent.*

Proof. The necessity is clear from the regularity of A.

For the sufficiency, let us assume that $\left\{\sum_{k=1}^{\infty} a_{nk}T^kx\right\}_{n=1}^{\infty}$ is convergent sequence. Now let

$$Z := \{y \in X : \lim_{n\to\infty} \|T^{n+1}y - T^ny\| = 0\}.$$

The subspace Z is T-invariant and closed. Since $\lim_{n\to\infty} \|T^{n+1}y - T^ny\| = 0$ for all $y \in Z$ and T is a power bounded operator, we obtain

$$\lim_{n\to\infty} \|T^ny\| = 0, \qquad \text{for all } y \in \overline{(I - T)Z}. \tag{5}$$

Now consider the set $E := \{x \in Z : \lim_{n\to\infty} L_n x \quad \text{exists}\}$. By Theorem 2.3, $x \in E$ can be written as $x = x_0 + y_0$ such that $x_0 \in \ker(I-T)$ and $y_0 \in \overline{(I-T)Z}$. Hence we have $T^n x = T^n x_0 + T^n y_0$ which yields $T^n x = x_0 + T^n y_0$. This implies, by (5), that

$$\|T^n x - x_0\| \to 0, \quad (n \to \infty).$$

This concludes the proof. □

3. Uniform Ergodic Type Theorems

In this section, we give some extensions of the uniform ergodic theorems ([4],[5]).

Firstly, we present a proposition which will be used in the proof of the main theorem.

Proposition 3.1. *Let X be a Banach space and $T \in B(X)$ be a power bounded operator. Suppose that $A = (a_{nk})$ is a URS-method and $\sum_{k=1}^{\infty} a_{nk} = 1$, (for all $\quad n \in \mathbb{N}$). Let $\ker(I-T) = \{0\}$. Then the following assertions are equivalent:*

(i) $I - L_n$ *is surjective, (for all* $\quad n \in \mathbb{N}$*).*
(ii) $I - T$ *is surjective.*
(iii) $\lim_{n\to\infty} \|L_n\| = 0$.

Proof. (*iii*) ⇒ (*i*) Since $\lim_{n\to\infty} \|L_n\| = 0$, we can write $\|L_{n_0}\| \le 1$ for an $n_0 \in \mathbb{N}$. Thus, we have that $I - L_n$ is invertible which yields $I - L_n$ is surjective.

(*i*) ⇒ (*ii*) Take $y \in X$. There exists an $x \in X$ such that $(I - L_n)x = y$ by (*i*). Hence,

$$y = (I - L_n)x = (I - T)\sum_{j=1}^{\infty} a_{nj} \sum_{k=1}^{j} T^{k-1}x.$$

This gives us that $(I - T)$ is surjective.

(*ii*) ⇒ (*iii*) Because of the assumption that $\ker(I-T) = \{0\}$, $I - T$ is injective and onto by (*ii*). Furthermore it is obvious that $I - T$ is continuous. By the Open Mapping Theorem, the inverse operator $(I-T)^{-1}$ is continuous as well. Let us denote by B the closed unit ball in X. Then $C := (I-T)^{-1}B$ is bounded. Let $M = \sup_{x\in C} \|x\|$. Then we get

$$\|L_n\| = \sup_{z\in B} \|L_n z\| = \sup_{x\in C} \|L_n(I-T)x\| \le M\|L_n(I-T)\|.$$

From [2, pg 507] we know that $\|L_n(I-T)x\| \le 3C_T\varepsilon\|x\|$ and then by taking supremum all over x with $\|x\| = 1$, we get $\lim_{n\to\infty} \|L_n(I-T)\| = 0$, which immediately yields $\|L_n\| \to 0$, $(n \to \infty)$. □

We now present an extension of the Uniform Ergodic Theorem given by Lin [4] with the help of URS-method.

Theorem 3.2. *Let X be a Banach space and $T \in B(X)$ be a power bounded operator. Suppose that $A = (a_{nk})$ is a URS-method and $\sum_{k=1}^{\infty} a_{nk} = 1$, (for all $\quad n \in \mathbb{N}$). Then the following assertions are equivalent:*

(i) T *is A-uniformly ergodic operator.*
(ii) $(I-T)X$ *is closed and* $X = \ker(I-T) \oplus (I-T)X$.
(iii) $(I-T)^2 X$ *is closed.*
(iv) $(I-T)X$ *is closed.*

Proof. Throughout the proof we assume that $Y := \overline{(I-T)X}$.

$(i) \Rightarrow (ii)$ Since T is A-uniformly ergodic operator, there exists a $P \in B(X)$ such that $\|L_n - P\| \to 0, (n \to \infty)$. This gives that $\|L_n x - Px\| \to 0, (n \to \infty)$. Thus we have $X = \ker(I-T) \oplus \overline{(I-T)X}$ by Theorem 2.3. Now we show that the subspace $(I-T)X$ is closed. In order to do this, take $x \in X$. Hence $T(I-T)x = (I-T)Tx \in (I-T)X$. Thus we have,

$$T(Y) = T(\overline{(I-T)X}) \subseteq \overline{T(I-T)X} = \overline{(I-T)X} = Y.$$

Hence Y is T-invariant subspace and we can write $S := T_{|Y}$ and $S_n := L_n|_Y$. We also know that $\ker P = Y$ from Corollary 2.2 and so we get $\lim_{n\to\infty} \|S_n\| = 0$. Moreover it is clear that $ker(I-T) \cap \overline{(I-T)X} = \{0\}$ which yields that $Y \cap \overline{(I-T)X} = \{0\}$. So we get $\ker(I-S) = \{0\}$. Therefore, one can get by Proposition 3.1 that $(I-S)$ is onto. Thus,

$$(I-S)Y = Y = (I-T)Y \subseteq (I-T)X \subseteq Y.$$

This implies that $Y = (I-T)X$ which in turn yields that $(I-T)X$ is closed

$(ii) \Rightarrow (iii)$ To prove that $(I-T)^2X$ is closed, we need to prove that $Y = (I-T)^2X$. We have

$$(I-T)^2X = (I-T)\{(I-T)X\} \subseteq Y. \tag{6}$$

On the other hand, take $y \in Y$ and so there exists an $x \in X$ such that $y = (I-T)X$. By (ii), we can write $x = x_0 + x_1$ such that $x_0 \in (I-T)X$ and $x_1 \in \ker(I-T)$. Hence, we get

$$y = (I-T)x = (I-T)x_0 \in (I-T)Y = (I-T)^2X. \tag{7}$$

By (6) and (7) we obtain $Y = (I-T)^2X$.

$(iii) \Rightarrow (iv)$ To prove that $(I-T)X$ is closed, we must show that $Y = (I-T)X$. By (iii), one can get,

$$(I-T)Y = (I-T)\overline{(I-T)X} \subseteq \overline{(I-T)^2X} = (I-T)^2X. \tag{8}$$

Furhermore,

$$(I-T)^2X = (I-T)(I-T)X \subseteq (I-T)\overline{(I-T)X} = (I-T)Y. \tag{9}$$

By (8) and (9), we obtain

$$(I-T)Y = (I-T)^2X$$

from which we conclude that $(I-T)Y$ is closed.

Since $T(Y) \subseteq Y$, the operator $S := T_{|Y}$ is well defined. Thus for all $y \in (I-T)X$, we obtain $\lim_{n\to\infty} S_n y = 0$. Now take $y \in Y$. Then,

$$y - S_n y = (I-S)\sum_{j=1}^{\infty} a_{nj} \sum_{k=1}^{j} S^{k-1} y \in (I-S)Y$$

which implies $y = \lim_{n\to\infty}(y - S_n y) \in \overline{(I-S)Y}$. Hence we have that $(I-T)X \subseteq \overline{(I-S)Y}$ which leads to

$$(I-T)X \subseteq \overline{(I-S)Y} = (I-S)Y = (I-T)^2X. \tag{10}$$

By (10) and the fact that $(I-T)^2X \subseteq (I-T)X$, we conclude that $(I-T)X$ is closed.

$(iv) \Rightarrow (i)$ Because of (iv), by the Open Mapping Theorem we find that there exists a $K \geq 0$ such that for each $y \in Y$ there exists $z \in X$ with

$$(I-T)z = y \quad \text{and} \quad \|z\| \leq K\|y\|.$$

Thus, by the uniformity condition, given $\varepsilon > 0$ there is a k_ε such that for all $k > k_\varepsilon$, we get for all $y \in Y$, that

$$\|L_n y\| = \|L_n(I-T)z\| \leq C_T\|z\|\left(2\sum_{j=1}^{k_\varepsilon}|a_{nj}| + \varepsilon\right).$$

(see [2, pg 507], for details). Hence we have $\|L_n y\| \leq 3C_T K\varepsilon\|y\|$. Since $S_n := L_n|_Y$ taking supremum all over y with $\|y\| = 1$, we get $\lim_{n\to\infty}\|S_n\| = 0$. Therefore, we conclude that by Proposition 3.1 $(I-S)$ is surjective. This implies that

$$(I-T)X = Y = (I-S)Y = (I-T)^2X.$$

Hence for all $x \in X$ there exists a $y \in Y$ such that $(I-T)x = (I-T)y$. Note that $\ker(I-S) = \{0\}$. Hence $(I-S)$ is invertible. So we can write

$$y = (I-S)^{-1}((I-T)x). \tag{11}$$

Since $(I-S)^{-1}$ is also continuous, one can obtain

$$\|y\| \leq \|(I-S)^{-1}\|\|(I-T)x\|.$$

Since $(I-T)(x-y) = 0$, we observe that $T(x-y) = (x-y)$ and for all $m \in \mathbb{N}$, $T^m(x-y) = (x-y)$ which yields, for all $n \in \mathbb{N}$, that $L_n(x-y) = (x-y)$.

Now we define $P : X \to X$ by $Px = x - y$ such that y is the unique element defined by (11). It is easily checked that $(I-S)$ is well defined and continuous. Thus, to complete proof, we show that $\|L_n - P\| \to 0$, $(n \to \infty)$. To achive this, take $x \in X$ and define y by (11). Then we find

$$\begin{aligned}\|(L_n - P)x\| &= \|L_n x - Px\| = \|L_n x - (x-y)\| = \|L_n y\| \\ &= \|L_n(I-S)^{-1}((I-T)x)\| \leq \|(I-S)^{-1}\|\|L_n(I-T)x\| \\ &\leq 3C_T\varepsilon\|(I-S)^{-1}\|\|x\|.\end{aligned}$$

Hence taking supremum all over x with $\|x\| = 1$, we get that

$$\|L_n - P\| \leq 3C_T\varepsilon\|(I-S)^{-1}\|.$$

This concludes the proof. □

The following theorem is motivated by Proposion 2.8 in [5].

Theorem 3.3. *Let X be a Banach space and $T \in B(X)$ be a power bounded operator. Suppose that $A = (a_{nk})$ is a URS-method and $\sum_{k=1}^{\infty} a_{nk} = 1$, (for all $n \in \mathbb{N}$). Then the following assertions are equivalent:*

(i) $\{T^n\}$ *converges uniformly.*
(ii) $\|T^{n+1} - T^n\| \to 0$ *and T is A-uniformly ergodic operator.*
(iii) $\|T^{n+1} - T^n\| \to 0$ *and $(I-T)X$ is closed.*

Proof. Clearly we have $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

We just prove $(iii) \Rightarrow (i)$. Since $(I-T)X$ is closed, following the technique used in the proof of Theorem 3.2 we conclude by the Open Mapping Theorem that $(I-S)$ is invertible on Y. Hence we observe that

$$(I-T)X = Y = (I-S)Y = (I-T)Y.$$

Thus, for all $x \in X$, there exists a $y \in Y$ such that $(I-T)x = (I-T)y$ then we may write $y = (I-S)^{-1}((I-T)x)$. Since $(I-S)$ is also continuous, we get

$$\|y\| \leq \|(I-S)^{-1}\|\|(I-T)x\|.$$

Since $(I-T)(x-y)=0$, we observe that $T(x-y)=(x-y)$ and for all $m\in\mathbb{N}$ $T^m(x-y)=(x-y)$. Let us define $P:X\to X$ with $Px=x-y$, where y is the unique element defined by (11). It is easily checked that $(I-S)$ is well defined and continuous. In order to complete the proof we show that $\|T^n-P\|\to 0$. Now take $x\in X$ and define y by (11). Then one can get

$$\begin{aligned}\|(T^n-P)x\| &= \|T^nx-Px\| = \|T^nx-(x-y)\| = \|T^nx-T^n(x-y)\| = \|T^ny\|\\ &= \|T^n(I-S)^{-1}((I-T)x)\| \le \|(I-S)^{-1}\|\|T^n(I-T)x\|\\ &\le \|(I-S)^{-1}\|\|T^{n+1}-T^n\|\|x\|.\end{aligned}$$

Hence taking supremum all over x with $\|x\|=1$, we observe that

$$\|T^n-P\|\le\|(I-S)^{-1}\|\|T^{n+1}-T^n\|$$

from which we get

$$\|T^n-P\|\to 0,\quad (n\to\infty)$$

by the hypothesis. So, the proof is completed. □

References

[1] J. Boos, Classical and Modern Methods in Summability, Oxford Science Publ., London, 2000.
[2] L. W. Cohen, On the mean ergodic theorem, Ann. Math. 41(3) (1940) 505-509.
[3] C. Jardas, N. Sarapa, A Summability Method in Some Strong Laws of Large Numbers, Math. Com. 2 (1997) 107-124.
[4] M. Lin, On the uniform ergodic theorem, Amer. Math. Soc. 43 (1974) 337-340.
[5] M. Lin, D. Shoikhet, L. Suciu, Remaks on uniform ergodic theorems, Acta Sci. Math. (Szeged) 81 (2015) 251-283.
[6] H. Mustafayev, Some convergence theorems in Fourier algebras, Bull. Austr. Math. Soc. 96 (2017) 487-495.
[7] H. Mustafayev, Convergence of iterates of convolution operators in L^p spaces, Bull. Sci. Math. 152 (2019) 61-92.
[8] H. Mustafayev, Some convergence theorems for operator sequences, Integr. Equ. Oper. Theory 92(4) (2020) 1-21.
[9] G. Oğuz, C. Orhan, Mean ergodic type theorems, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 68(2) (2019) 2264-2271.
[10] F. Riesz, Some mean ergodic theorems, J. Lond. Math. Soc. 13 (1938) 274.
[11] J. von Neumann, Proof of the quasi-ergodic hypothesis, Proc. Nat. Acad. Sci. USA 18 (1932) 70-82.
[12] T. Yoshimoto, Ergodic theorems and summability methods, Quart. J. Math. 38(3) (1987) 367-379.
[13] K. Yosida, Mean ergodic theorem in Banach space, Proc. Imp. Acad. Tokyo, 14 (1938) 292-294.