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Study of Weakly Ricci-Symmetric Spacetimes under Gray's Decomposition and $f(\mathcal{R}, T)$-Gravity

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Abstract. In this paper we characterize weakly Ricci-symmetric spacetimes $(WRS)_n$ endowed with the Gray's Decomposition. We provide, several interesting results of $(WRS)_n$ in Gray's Decomposition. In addition we discuss some results based on weakly Ricci-symmetric Generalized Robertson Walker (GRW) spacetimes. Moreover, we study $(WRS)_n$ spacetimes which satisfy the $f(\mathcal{R}, T)$-gravity equation.

1. Introduction

Einstein's field equations

$$Ric(V,W) - \frac{\mathcal{R}}{2}g(V,W) = \kappa^2\tilde{T}(V,W), \tag{1}$$

indicate that the energymomentum tensor $\tilde{T}$ is divergence free. This requirement is fulfilled if $\nabla\tilde{T} = 0$, where ∇ denotes the semi-Riemannian connection. In (1), κ^2 is the gravitational constant, $\mathcal{R}$ is the Ricci scalar and Ric denotes the Ricci tensor. Chaki and Ray[4] asserted that $\nabla\tilde{T} = 0$ implies $\nabla Ric = 0$. Tamassy and Binh[32] initiated the notion of wakly Ricci symmetric manifold denoted by $(WRS)_n$ which generalizes $\nabla Ric = 0$. A non-flat semi-Riemannian manifold is called $(WRS)_n$ if the Ricci tensor Ric fulfills the condition

$$(\nabla_U Ric)(V,W) = \alpha(U)Ric(V,W) + \beta(V)Ric(U,W) + \gamma(W)Ric(V,U), \tag{2}$$

α, β, γ being three non-zero 1-forms.

A Lorentzian manifold admitting a globally timelike vector field is known as a spacetime. A Lorentzian manifold is named weakly Ricci symmetric spacetime if the Ricci tensor satisfies (2).

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In a Lorentzian manifold (M^n, g) $(n > 3)$ the conformal curvature tensor is given by

$$C(U, V)W = R(U, V)W - \frac{1}{n-2}[g(V, W)QU - g(U, W)QV + Ric(V, W)U - Ric(U, W)V] + \frac{\mathcal{R}}{(n-1)(n-2)}[g(V, W)U - g(U, W)V], \tag{3}$$

$\mathcal{R}$ being the Ricci scalar, Q is the Ricci operator defined by $g(QY, Z) = Ric(Y, Z)$.

De and Mallick[10] proved that a conformally flat $(WRS)_n$ with non-zero Ricci scalar is a Robertson-Walker spacetime. Recently, Mantica and Molinari[21] cultivated some properties for $(WRS)_n$ with the help of Lovelock's identity. Also several researchers([3], [7], [12], [13], [33]) have explored geometrical and physical consequences of manifolds and perfect fluid spacetimes in different ways.

Definition 1.1. *{[1], [22]} A Lorentzian manifold of dimension n is called a* generalized Robertson-Walker (GRW) spacetime *if the metric takes the local form*

$$ds^2 = -(dt)^2 + r(t)^2 g^*_{ij} dx^i dx^j,$$

where $g^*_{ij}(x^k)$ are functions of x^k only $(i, j, k = 2, 3, \cdots, n)$ and r is a function of t only. If g^*_{ij} is of dimension 3 and of constant curvature, then the GRW spacetime reduces to RW spacetime.

Definition 1.2. *[11] A Lorentzian manifold of dimension n whose Ricci tensor Ric satisfies the condition*

$$Ric = ag + b\eta \otimes \eta, \tag{4}$$

is often called perfect fluid spacetime; a, b being the scalar fields and ρ is unit timelike vector field, called velocity vector field defined by $g(U, \rho) = \eta(U)$.

In a perfect fluid spacetime, the energymomentum tensor $\tilde{T}$ is of the form [25]

$$\tilde{T}(U, V) = pg(U, V) + (\sigma + p)\eta(U)\eta(V) \tag{5}$$

σ, p being the energy density and isotropic pressure of the perfect fluid respectively.
If $\sigma = p$, the perfect fluid is named *stiff matter* (for more details we refer to [31]).

Recent observations of the late-time acceleration of the Universe posed a fundamental theoretical challenge to gravitational theories. Various extended gravity theories, where we replace the Ricci scalar in the Einstein-Hilbert action by some arbitrary yet observationally and theoretically restricted function of the Ricci scalar, or other scalar or tensor field, or other geometric quantities, to some extent can explain the presence of a late-time cosmic acceleration of the Universe. $f(\mathcal{R}, T)$-gravity was introduced by Harko et al.[17] which modifies general relativity as well as $f(\mathcal{R})$-gravity[30]. In $f(\mathcal{R}, T)$-gravity they considered gravitational Lagrangian as an arbitrary function of $\mathcal{R}$ and T, T being the trace of $\tilde{T}$. With the help of $f(\mathcal{R}, T)$ modified theory of gravity, the changes in Earth's atmospheric models have been investigated by Ordines et al.[26]. Several authors (see [28], [5]) have studied various properties of $f(\mathcal{R}, T)$-gravity in different points of view.
Also De and his co-authors explored in ([9], [10]) about weakly Ricci-symmetric spacetimes. Related properties of symmetric spacetimes were also analyzed by Mantica et al. in ([22], [23]), specifically GRW spacetimes. Furthermore, certain features of $(WRS)_n$ spacetimes were also addressed by De and Majhi in [8]. Recently, Mantica et al. also studied in [24] perfect fluid GRW-spacetime with Gray's decomposition. Influenced by the above studies, we aim in this paper to explore the weakly Ricci-symmetric spacetimes $(WRS)_n$ via Gray's decomposition. In addition, we like to investigate $(WRS)_4$ spacetimes with $f(\mathcal{R}, T)$-gravity. The importance of $(WRS)_n$ spacetimes lies in the fact that such a spacetime illustrates stiff matter.

2. Gray's decomposition and weakly Ricci-symmetric spacetimes

For this section we have required the following results which are essentials to the further results.

Theorem A. [23] *Let (M, g) be an n-dimensional Lorentzian manifold with $n > 3$. If the Ricci tensor has the form $Ric(U, V) = -\mathcal{R}\mu(U)\mu(V)$ and the conformal curvature tensor is divergence free, that is, $div\, C = 0$, then (M, g) is a GRW spacetime.*

Theorem B. [6] *A Lorentzian manifold (M, g) of dimension $n > 3$ is a GRW spacetime if and only if it admits a timelike vector field Z such that $\nabla_U Z = fU$ for some function f on M.*

Theorem C. [8] The expression of the Ricci tensor Ric of a $(WRS)_n$ spacetime is given by

$$Ric(U, V) = -\mathcal{R}\mu(U)\mu(V). \tag{6}$$

Moreover the Ricci scalar $\mathcal{R}$ is not zero, $\mu = \beta - \gamma$ and ρ is a unit timelike vector field such that $g(U, \rho) = \mu(U)$ for all vector fields U.

Remark 2.1. *The expression (6) implies that the spacetime is* Ricci simple (for more details see [23]). *In fact $(WRS)_n$ obeying Einstein's field equations represents* stiff matter fluid *and massless scalar field spacetime with timelike gradient vector.*

Now, by above remark and Theorem C we conclude the following:

Proposition 2.2. *A $(WRS)_n$ spacetime obeying Einstein's field equations represents perfect fluid spacetime with $p = \sigma$.*

Gary[14] proposed that the gradient of the Ricci tensor, ∇Ric, can be decomposed into $O(n)$-invariant terms (for more details, see [2], [16], [19]). It follows from [14] that the gradient of the Ricci tensor $(\nabla_U Ric)(V, W)$ can be decomposed into $O(n)$-invariant term as follows [24]:

$$(\nabla_U Ric)(V, W) = \tilde{R}(U, V)W + \xi(U)g(V, W) + \omega(V)g(U, W) + \omega(W)g(V, U), \tag{7}$$

where

$$\xi(U) = \frac{n}{(n-1)(n+2)}\nabla_U \mathcal{R}, \quad \omega(U) = \frac{n}{(n-1)(n+2)}\nabla_U \mathcal{R}, \tag{8}$$

and $\tilde{R}(U, V)W = \tilde{R}(U, W)V$ is trace-less which can be decomposed as a sum of orthogonal components

$$\begin{aligned}\tilde{R}(U, V)W = {} & \frac{1}{3}[\tilde{R}(U, V)W + \tilde{R}(V, W)U + \tilde{R}(W, U)V] \\ & + \frac{1}{3}[\tilde{R}(U, V)W - \tilde{R}(V, U)W] + \frac{1}{3}[\tilde{R}(U, V)W - \tilde{R}((W, U)V].\end{aligned} \tag{9}$$

The decompositions (7) and (9) provide $O(n)$-invariant subspace, characterized by invariant equations that are linear in $(\nabla_U Ric)(V, W)$.

The gradient of the Ricci tensor and the divergence of the conformal curvature tensor C are connected by the following relation (see [24])

$$(div\, C)(U, V)W = \frac{n-3}{n-2}[\tilde{R}(U, V)W - \tilde{R}(V, U)W]. \tag{10}$$

In Gray's decomposition we have the following subspaces:

(i) The **trivial subspace** $\nabla Ric = 0$.

(ii) The **subspace** $\mathcal{I}$ is characterized by $\tilde{R}(U,V)W = 0$, i.e.,

$$(\nabla_U Ric)(V,W) = \xi(U)g(Y,Z) + \omega(V)g(U,W) + \omega(W)g(V,U). \tag{11}$$

It is to be noted that the manifold fulfilling the above equation (11) is named *Sinyukov manifold*[29].

(iii) The orthogonal complements $\mathcal{I}'$ is characterized by

$$(\nabla_U Ric)(V,W) + (\nabla_V Ric)(U,W) + (\nabla_W Ric)(U,V) = 0. \tag{12}$$

This implies that the Ricci scalar $\mathcal{R} = constant$. Also this equation reflects that the Ricci tensor is Killing[35].

(iv) In B and B' the Ricci tensor is a Codazzi tensor i.e.,

$$(\nabla_U Ric)(V,W) = (\nabla_V Ric)(U,W). \tag{13}$$

(v) The subspace $\mathcal{I} \oplus A$ contains tensors that satisfies the cyclic condition

$$\begin{aligned}(\nabla_U Ric)(V,W) &+ (\nabla_V Ric)(U,W) + (\nabla_W Ric)(U,V)\\ &= 2\frac{d\mathcal{R}(U)}{(n+2)}g(V,W) + 2\frac{d\mathcal{R}(V)}{(n+2)}g(U,W) + 2\frac{d\mathcal{R}(W)}{(n+2)}g(U,V).\end{aligned} \tag{14}$$

(vi) The subspace $\mathcal{I} \oplus B$ contains tensors that satisfy the Codazzi condition

$$\nabla_W[Ric(U,V) - \frac{\mathcal{R}}{2(n-1)}g(U,V)] = \nabla_U[Ric(V,W) - \frac{\mathcal{R}}{2(n-1)}g(V,W)], \tag{15}$$

which implies that $div\, C = 0$.

Now, we consider these six cases separately.

Case (i) $\nabla Ric = 0$, which implies $\mathcal{R}$ = constant. Also in $(WRS)_n$ we have

$$Ric(U,V) = -\mathcal{R}\mu(U)\mu(V), \tag{16}$$

where ρ is a unit timelike vector field associated with the 1-form μ. Taking covariant derivative of (16) and using $\mathcal{R} = constant$, we derive

$$\mathcal{R}[(\nabla_W \mu)U\mu(V) + \mu(U)(\nabla_W \mu)V] = 0.$$

Since $\mathcal{R} \neq 0$ in $(WRS)_n$, therefore the foregoing equation becomes

$$(\nabla_W \mu)U\mu(V) + \mu(U)(\nabla_W \mu)V = 0. \tag{17}$$

Since ρ is a unit timelike vector field, we obtain that $g(\nabla_U \rho, \rho) = 0$ and hence $(\nabla_W \mu)\rho = 0$. Replacing V by ρ in (17) gives $(\nabla_W \mu)U = 0$, that is, the 1-form μ is closed.

Now, $(\nabla_W \mu)U = 0$ implies $g(U, \nabla_W \rho) = 0$. Putting $Z = \rho$ provides $\nabla_\rho \rho = 0$. Hence the integral curves of the velocity vector filed ρ are geodesic. Consequently, we have the following result.

Theorem 2.3. *If a $(WRS)_n$ spacetime belongs to the trivial subspace, then the 1-form μ is closed and the integral curves of the velocity vector filed ρ are geodesic.*

Case (ii). From (7) we find

$$(\nabla_U Ric)(V, W) = \tilde{R}(U, V)W + \alpha(U)g(V, W) + \beta(V)g(U, W) + \beta(W)g(V, U).$$

The subspace $\mathcal{I}$ is characterized by $\tilde{R}(U, V)W = 0$, i.e.,

$$(\nabla_U Ric)(V, W) = \alpha(U)g(V, W) + \beta(V)g(U, W) + \beta(W)g(V, U).$$

Once again, from (9) we have

$$(div\, C)(U, V)W = \frac{n-3}{n-2}[\tilde{R}(U, V)W - \tilde{R}(V, X)W].$$

In subspace $\mathcal{I}$ Ricci tensor fulfills the condition $\tilde{R}(U, V)W = 0$ and hence from the above result we obtain $div\, C = 0$. Also in a $(WRS)_n$ spacetime, $Ric(U, V) = -\mathcal{R}\mu(U)\mu(V)$. Hence in the light of Theorem A, we conclude the following:

Theorem 2.4. *If a $(WRS)_n$ spacetime belongs to the subspace $\mathcal{I}$, then the spacetime is a GRW spacetime.*

Case (iii). To discuss the case we first define Killing tensor and then prove a lemma.

Definition 2.5. [35] *A second order symmetric tensor K is said to be a Killing tensor if*

$$(\nabla_U K)(V, W) + (\nabla_V K)(W, U) + (\nabla_W K)(U, V) = 0.$$

Lemma 2.6. *In a spacetime the energy momentum tensor is Killing if and only if the Ricci tensor is Killing.*

Proof. Taking covariant derivative of the Einstein's field equations (1), we get

$$(\nabla_Z Ric)(U, V) - \frac{d\mathcal{R}(Z)}{2}g(U, V) = \kappa(\nabla_Z \tilde{T})(U, V),$$

which implies

$$\begin{aligned}
&(\nabla_Z Ric)(U, V) + (\nabla_V Ric)(Z, U) + (\nabla_U Ric)(V, Z) - \frac{d\mathcal{R}(Z)}{2}g(U, V) \\
&-\frac{d\mathcal{R}(V)}{2}g(Z, U) - \frac{d\mathcal{R}(U)}{2}g(V, Z) = \kappa\{(\nabla_Z \tilde{T})(U, V) + (\nabla_V \tilde{T})(Z, U) \\
&\qquad\qquad + (\nabla_U \tilde{T})(V, Z)\}.
\end{aligned} \tag{18}$$

First suppose that $\tilde{T}$ is Killing . Then (18) infers

$$\begin{aligned}
&(\nabla_Z Ric)(U, V) + (\nabla_V Ric)(Z, U) + (\nabla_U Ric)(V, Z) \\
&= \frac{d\mathcal{R}(Z)}{2}g(U, V) + \frac{d\mathcal{R}(V)}{2}g(Z, U) + \frac{d\mathcal{R}(U)}{2}g(V, Z).
\end{aligned} \tag{19}$$

Contracting U and V gives $\mathcal{R} = constant$.
Hence equation (19) reflects that the Ricci tensor is Killing. Conversely, if the Ricci tensor is Killing, then $\mathcal{R} = constant$ and hence from (18) it follows that the energy momentum tensor is Killing. This finishes the proof.

In [27] Sharma and Ghosh obtained an interesting result as follows:

Theorem D. [27] *Let (M, g) be a perfect fluid spacetime such that its energy momentum tensor is Killing. Then (i) M is expansion-free and shear-free and its flow is geodesic, however, not necessarily vorticity-free, and (ii) its energy density and isotropic pressure are constant on M.*

Now invirtue of Lemma (2.6) and Theorem D, we state the following:

Theorem 2.7. *If a $(WRS)_n$ perfect fluid spcetime belongs to the class $\mathcal{I}'$, then (i) the spacetime is expansion-free and shear-free and its flow is geodesic, however, not necessarily vorticity-free, and (ii) its energy density and isotropic pressure are constant.*

Case (iv). If $(WRS)_n$ belongs to B and B', then

$$(\nabla_U Ric)(V,W) = (\nabla_V Ric)(U,W),$$

which implies $\mathcal{R}$ = constant.

Note that Guilfoyle and Nolan [15] named a *Yang pure space* as a four dimensional Lorentzian manifold whose metric solves Yang's equation:

$$(\nabla_U Ric)(V,W) = (\nabla_V Ric)(U,W). \tag{20}$$

In any dimension, they are equivalent to the condition: $div\, C = 0$ with constant scalar curvature.

For $n \geq 4$, Mantica and Molinari [23] stated the following:

Proposition 2.8. *An n-dimensional perfect fluid Yang pure space with $p + \sigma \neq 0$ is a GRW spacetime.*

Since $(WRS)_n$ is a perfect fluid spacetime satisfying $p = \sigma$ and (20), we have the following result from Proposition 2.8.

Theorem 2.9. *If a $(WRS)_n$ spacetime belongs to class B and B', then the spacetime is a GRW spacetime.*

Case (v). In this case, we have

$$\begin{aligned}(\nabla_U Ric)(V,W) + (\nabla_V Ric)(U,W) + (\nabla_W Ric)(U,V) &= 2\frac{d\mathcal{R}(U)}{(n+2)}g(V,W) \\ &+2\frac{d\mathcal{R}(V)}{(n+2)}g(U,W) + 2\frac{d\mathcal{R}(W)}{(n+2)}g(U,V).\end{aligned} \tag{21}$$

Applying (2) in the above equation, we find

$$\begin{aligned}&F(U)Ric(V,W) + F(V)Ric(U,W) + F(W)Ric(U,V) \\ &\quad = 2\frac{d\mathcal{R}(U)}{(n+2)}g(V,W) + 2\frac{d\mathcal{R}(V)}{(n+2)}g(U,W) + 2\frac{d\mathcal{R}(W)}{(n+2)}g(U,V),\end{aligned}$$

where $F(U) = \alpha(U) + \beta(U) + \gamma(U)$. Now contracting V and W in the foregoing equation, we obtain

$$F(U)\mathcal{R} + Ric(U,\tilde{\rho}) + Ric(U,\tilde{\rho}) = 2d\mathcal{R}(U),$$

where $g(U,\tilde{\rho}) = F(U)$ for all U. Hence, we get

$$F(U)\mathcal{R} + 2Ric(U,\tilde{\rho}) = 2d\mathcal{R}(U).$$

Contracting equation (21) we get $\mathcal{R} = constant$.
Therefore, we have the following:

Theorem 2.10. *If a $(WRS)_n$ spacetime belongs to the subspace $\mathcal{I} \oplus A$, then $\tilde{\rho}$ is an eigenvector corresponding to the eigenvalue $-\frac{\mathcal{R}}{2}$.*

Case (vi). Let the $(WRS)_n$ belongs to $\mathcal{I} \oplus B$. In this case, we get $div\, C = 0$. Also in a $(WRS)_n$ we have $Ric(U,V) = -\mathcal{R}\mu(U)\mu(V)$. Hence from Theorem A, we obtain

Theorem 2.11. *If a $(WRS)_n$ spacetime belongs to the subspace $\mathcal{I} \oplus B$, then it is a GRW spacetime.*

3. Weakly Ricci-symmetric Generalized Robertson walker spacetimes

In this section we characterize $(WRS)_n$ *GRW* spacetimes. Mantica et al. proved

Proposition 3.1. ([23]) *A perfect fluid spacetime with* $div\, C = 0$ *is a GRW-spacetime with Einstein fibers provided the velocity vector field is irrotational.*

By combining the above result and Theorem C we state

Remark 3.2. *A* $(WRS)_n$ *spacetime satisfying* $div\, C = 0$ *is a GRW-spacetime provided that the velocity vector field is irrotational.*

For *GRW* spacetimes, let us assume that ρ is a concircular vector field according to Theorem B. Then we get $\nabla_U \rho = fU$, which implies

$$R(U, V)\rho = (Uf)V - (Vf)U. \tag{22}$$

Hence

$$Ric(V, \rho) = (1 - n)(Vf). \tag{23}$$

For $(WRS)_n$ we have $Ric(U, V) = -\mathcal{R}\mu(U)\mu(V)$, which gives

$$Ric(V, \rho) = \mathcal{R}\mu(V). \tag{24}$$

Equation (23) and (24) together yield

$$Vf = \frac{\mathcal{R}}{1 - n}\mu(V). \tag{25}$$

Using (25) in (22) we turn up

$$R(U, V)\rho = \frac{\mathcal{R}}{(1 - n)}[\mu(U)V - \mu(V)U]. \tag{26}$$

Now, adopting (3) we obtain

$$\begin{aligned} C(U, V, W, \rho) &= R(U, V, W, \rho) - \frac{1}{(n-2)}[Ric(V, W)g(U, \rho) \\ &\quad - Ric(U, W)g(V, \rho) + Ric(U, \rho)g(V, W) - Ric(V, \rho)g(U, W)] \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)}[g(Y, W)g(U, \rho) - g(U, W)g(V, \rho)] \\ &= \frac{\mathcal{R}}{(1-n)}[\mu(V)g(U, W) - \mu(U)g(V, W)] \\ &\quad - \frac{1}{(n-2)}[Ric(V, W)\mu(U) - Ric(U, W)\mu(V) \\ &\quad + \mathcal{R}\mu(U)g(V, W) - \mathcal{R}\mu(V)g(U, W)] \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)}[g(V, W)\mu(U) - g(U, W)\mu(V)]. \end{aligned} \tag{27}$$

Using (24), (26), (27) and (6), we finally obtain $C(U, V, W, \rho) = 0$. Therefore, we get $C(U, V)\rho = 0$, which implies that the conformal curvature tensor is purely electric [18]. Further it is known that in a *GRW* spacetime $div\, C = 0$ if and only if $C(U, V)\rho = 0$ [22].

Consenquetly, we obtain:

Theorem 3.3. *In a $(WRS)_n$ GRW spacetime, the conformal curvature tensor is purely electric and it is also divergence free.*

For dimension $n = 4$, the condition $C(U, V)\rho = 0$ is equivalent to

$$\mu(Y)C(U, V, Z, W) + \mu(U)C(V, Y, Z, W) + \mu(V)C(Y, U, Z, W) = 0,$$

where $\mu(Y) = g(Y, \rho)$ for all vector fields Y. Now, replacing Y by ρ in the above expression, we obtain $C(U, V, Z, W) = 0$ [20].

In [34], Vazquez et al. proved that "A *GRW* spacetime M is conformally flat if and only if M is a Robertson-Walker spacetime". Hence we have the following two corollaries.

Corollary 3.4. *In dimension $n = 4$, a weakly Ricci-symmetric GRW-spacetime is a RW spacetime.*

Corollary 3.5. *[20] A 4-dimensional weakly Ricci symmetric GRW-spacetime is of Petrov type O.*

4. $(WRS)_4$-spacetime satisfying $f(\mathcal{R}, T)$-gravity

In this Section we characterize $(WRS)_4$ spacetimes satifying $f(\mathcal{R}, T)$-gravity which is the generalization of $f(\mathcal{R})$-gravity. Harko et al[17] introduced this modified gravity theory. The corresponding field equations have been executed in metric formaslism for several particular cases of $f(\mathcal{R}, T)$ gravity. Also Harko et al[17] pursued posibility of reconstructing the FRW cosmologies by an approprite choice of $f(T)$ for the model $f(\mathcal{R}, T) = \mathcal{R} + 2f(T)$. Here we choose

$$f(\mathcal{R}, T) = \mathcal{R} + 2f(T), \tag{28}$$

$f(T)$ being an arbitrary function on the trace T of $\tilde{T}$ and the term $2f(T)$ in the gravitational action modifies the gravitational interaction between matter and curvature.

We assume a modified Einstein-Hilbert action term

$$\mathcal{H} = \frac{1}{16\pi} \int [f(\mathcal{R}, T) + \mathcal{L}_m] \sqrt{(-g)} d^4x, \tag{29}$$

where $f(\mathcal{R}, T)$ is an arbitrary function of scalar curvature $\mathcal{R}$ and the trace T of the energy-momentum tensor, and $\mathcal{L}_m$ is the matter Lagrangian of the scalar field. The stress energy tensor of the matter is given by

$$T_{ab} = \frac{-2\delta(\sqrt{-g})\mathcal{L}_m}{\sqrt{-g}\delta^{ab}}. \tag{30}$$

Let us consider that the matter Lagrangian of the scalar field depends only on the metric tensor g_{ab}, and not on its derivatives.

The variation of action (29) with respect to the metric tensor g_{ab} yields the field equations of $f(R, T)$-gravity

$$\begin{aligned} f_{\mathcal{R}}(\mathcal{R}, T)Ric_{ab} - \frac{1}{2} f(\mathcal{R}, T)g_{ab} + (g_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b) f_{\mathcal{R}}(\mathcal{R}, T) \\ = 8\pi T_{ab} - f_T(\mathcal{R}, T)T_{ab} - f_T(\mathcal{R}, T)\psi_{ab}, \end{aligned} \tag{31}$$

where $f_{\mathcal{R}}$ and f_T denote the partial derivatives of $f(\mathcal{R}, T)$ with respect to $\mathcal{R}$ and T, respectively. As per usual notation, ∇_a is the covariant derivative, $\square \equiv \nabla_c\nabla^c$ is the d'Alembert operator, and ψ_{ab} is defined by

$$\psi_{ab} = -2T_{ab} + g_{ab}\mathcal{L}_m - 2g^{lk} \frac{\partial^2 \mathcal{L}_m}{\partial g^{ab} \partial g^{lk}}. \tag{32}$$

If we consider $f(\mathcal{R}, T) = f(\mathcal{R})$, then (31) provides the field equations of $f(\mathcal{R})$-gravity.

Let consider the matter of a perfect fluid $(WRS)_n$ spacetime with isotropic pressure p, energy density σ, and velocity vector ρ is given by $g(X,\rho) = \mu(X)$ for all X. Also, we know that there is no unique value of Lagrangian, therefore we assume that $\mathcal{L}_m = -p$ and using (5) we turn up

$$\tilde{T}(U,V) = pg(U,V) + (\sigma + p)\eta(U)\eta(V), \tag{33}$$

where $g(\rho,\rho) = -1$. Using (33), we can easily obtain the variation of stress energy in the following form

$$\psi(U,V) = -pg(U,V) - 2\tilde{T}(U,V). \tag{34}$$

After adopting (28) and (31) we get the form

$$Ric(U,V) = \frac{\mathcal{R}}{2}g(U,V) - 2f^{'}(T)\tilde{T}(U,V) - 2f^{'}(T)\psi(U,V) + f(T)g(U,V) + 8\pi\tilde{T}(U,V). \tag{35}$$

Throughout this study we consider $(WRS)_n$ spacetime solution of $f(\mathcal{R},T)$-gravity equation where the velocity vector $\eta = \mu$, so that equation (16) becomes

$$Ric(U,V) = -\mathcal{R}\eta(U)\eta(V). \tag{36}$$

Replacing V by ρ in (36) entails that

$$Ric(U,\rho) = \mathcal{R}\eta(U). \tag{37}$$

In light of (33), (34), and (35) we get

$$Ric(U,V) = \{\frac{1}{2}\mathcal{R} + f(T) + 4pf^{'}(T) + 8p\pi\}g(U,V) + \{(\sigma + p)(8\pi + 2f^{'}(T))\}\eta(U)\eta(V). \tag{38}$$

Contracting (38) we get

$$\mathcal{R} = (\sigma + p)(8\pi + 2f^{'}(T) - 4\{8p\pi + f(T) + 4pf^{'}(T)\}. \tag{39}$$

On the other hand using (37) in (38) we acquire

$$\mathcal{R} = -2(\sigma + p)\{8\pi + 2f^{'}(T)\} + 2\{8p\pi + f(T) + 4pf^{'}(T)\}. \tag{40}$$

Now equations (39) and (40) together yield

$$\sigma + p = -\frac{\mathcal{R}}{8\pi + 2f'(T)}. \tag{41}$$

Since in a $(WRS)_4$ spacetime $\mathcal{R} \neq 0$, consequently $\sigma + p \neq 0$.
Also (39) and (40) give

$$p = -\frac{\mathcal{R} + 2f(T)}{8\{2\pi + f'(T)\}}. \tag{42}$$

and equations (41) infers

$$\sigma = \frac{-2\mathcal{R}\{4\pi + 3f^{'}(T)\} + 4f(T)\{4\pi + f^{'}(T)\}}{8\{2\pi + f'(T)\}\{8\pi + 2f'(T)\}}. \tag{43}$$

Thus we obtain

Theorem 4.1. *In a perfect fluid $(WRS)_n$ spacetime obeying $f(\mathcal{R},T)$-gravity equation, if the four-velocity vector is identical with μ^a, then p and σ are given by in (42) and (43) which are not constant in this special situation.*

Since the scalar curvature $\mathcal{R}$ is non-zero in a $(WRS)_n$ spacetime, it follows from (41) that $(\sigma + p) \neq 0$.
Hence we write

Corollary 4.2. *Every perfect fluid $(WRS)_n$ spacetime obeying $f(\mathcal{R}, T)$-gravity equation cannot admit dark matter.*

Consider dust matter fluid i.e., $p = 0$, then we get from (41) and (43) that

$$f(T) = -\frac{\mathcal{R}}{2}, \tag{44}$$

provided $4\pi + f'(T) \neq 0$.
Therefore, we obtain

Corollary 4.3. *A perfect fluid $(WRS)_n$ spacetime unable to illustrate dust era for any viable $f(\mathcal{R}, T)$, provided $4\pi + f'(T) \neq 0$.*

In particular, if $f(T) = 0$, then $f(\mathcal{R}, T)$-gravity recovers $f(R)$-gravity. In this case we obtain from (42) and (43) that

$$p = -\frac{1}{16\pi}\mathcal{R} \quad \textit{and} \quad \sigma = -\frac{1}{16\pi}\mathcal{R}.$$

Hence we have

Corollary 4.4. *A perfect fluid $(WRS)_n$ spacetime obeying $f(\mathcal{R})$-gravity equation stands for stiff matter fluid.*

5. Discussion

In the present investigation, we study weakly Ricci-symmetric spacetimes $(WRS)_n$ endowed with the Gray's Decomposition. Specifically, it is observed that a $(WRS)_n$ spacetime becomes a GRW spacetime under certain condition. The *GRW* spacetime is investigated in this setting and obtained that, the conformal curvature tensor is purely electric and divergence free. Also, We consider, a perfect fluid $(WRS)_n$ spacetime obeying $f(\mathcal{R}, T)$-gravity equation and the four-velocity vector is identical with μ^a and observe the spacetime cannot admit dark matter era but stands for stiff matter fluid for $f(\mathcal{R})$-gravity.

6. Declarations

6.1. Conflicts of interest/Competing interests

The authors declare that they have no conflict of interest.

6.2. Availability of data and material

Not applicable.

6.3. Code availability

Not applicable.

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