Filomat 36:7 (2022), 2345–2355
https://doi.org/10.2298/FIL2207345N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Three-Factor Mean Reverting Ornstein-Uhlenbeck Process with Stochastic Drift Term Innovations: Nonlinear Autoregressive Approach with Dependent Error

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Abstract. This paper introduces a novel approach, withen the context of energy market, by employing a three-factor mean reverting Ornstein-Uhlenbeck process with a stochastic nonlinear autoregressive drift term having a dependent error. Initially the unique solvability for the given nonlinear system is investigated. Then, to estimate the nonlinear regression function, a semiparametric method, based on the conditional least square estimator for the parametric approach, and the nonparametric kernel method for autoregressive modification estimation have been presented . A maximum likelihood estimator has been used for parameter estimation of the Ornstein-Uhlenbeck process. Finally, some numerical simulations and real data studies have been provided to support the main conclusions of the study.

1. Introduction

The Ornstein-Uhlenbech processes (OU processes) have been employed as the main structure of the Barndroff-Nielsen-Shephard stochastic volatility model [1]. The modeling, based on the mean-reverting price processes of the diffusion type applicable to energy prices, has attracted a huge interest in the latest literature. Generally, there are one to three-factor models that motivate the concept of mean reversion. Pilipovic (1997) presented a one-factor mean-reverting OU process for the spot price of energy markets [2]. Tifenbach (2000) generalized the proposed model of Pilipovic to the case where the mean-reverting parameter was made time-dependent to capture seasonality [3]. Two-factor models have been used through either permitting the long-run mean or the volatility ruled via a stochastic differential equation. Some numerical implementation of these models can be found in [3–5]. Modeling electricity price processes is presently an energetic space of educational analysis. Here capturing the spikes occurring in the price is the main issue. Different groups of dynamic of these pricing processes are identified using the two factors mean-reverting OU processes. The additional flexibility is provided by the three-factor models to reproducing spikes. This paper aims at introducing a three-factor model. An OU process with an intermediate model of stochastic mean l_t and the second level of stochastic mean reversion M_t, incorporating stronger local shocks

2020 *Mathematics Subject Classification*. 60H10; 37M10; 62F10
Keywords. Energy markets, Mean reverting Ornstein Uhlenbeck process, Nonlinear autoregressive models, Semiparametric estimation
Received: 19 June 2021; Revised: 27 August 2021; Accepted: 10 January 2022
Communicated by Miljana Jovanović
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to S_t via l_t, is the first three-factor mean-reverting model that suits modeling electricity price spikes is as follows [4].

$$\begin{cases} dS_t = \alpha(l_t - S_t)dt + \sigma S_t^{\gamma} dZ_t \\ dl_t = \mu(M_t - l_t)dt + \xi l_t dW_t \\ dM_t = \beta M_t dt + \lambda M_t^{\gamma} dV_t \quad \gamma = 0 \text{ or } 1, \end{cases}$$

where the parameters are constants. Another three-factor mean-reverting model is an OU process with both stochastic mean and volatility. Local price shocks are caused by combined nonlinear effects of stochastic mean and volatility.

$$\begin{cases} dS_t = \alpha(l_t - S_t)dt + \sigma_t S_t^{\gamma} dZ_t \\ dl_t = \mu l_t dt + \xi l_t^{\gamma} dW_t \\ d\sigma_t = \beta(\sigma_0 - \sigma_t)dt + \lambda \sigma_t^{\gamma} dV_t \quad \gamma = 0 \text{ or } 1, \end{cases}$$

Considering an OU process with a stochastic mean governed by a first order nonlinear autoregressive (AR) model with a dependent error is the innovation presented in this paper. When the AR function in a model is a nonlinear function of previous data, linear analysis can not be appropriate, [6]. There are many approaches to the estimation of nonlinear AR functions of the model. Zhuoxi et al. (2009) suggested a semiparametric (SP) methodology for the estimation of nonlinear autoregression functions in an AR model [7]. Farnoosh and Mortazavi (2011) followed the method of Zhuoxi et al. (2009) for the AR models by considering a dependent error and explored the asymptotic manner of the SP estimators [8]. The standard way of analyzing an AR model is based on the Gaussian hypothesis of errors, whereas in some actual conditions, the data contradict this hypothesis [9, 10]. There are researches permitting changing the Gaussian error by a non-Gaussian error to make and study an AR model (for more details see [11], [12], [13], [14, 15]). SP estimation of the AR parameter in non-Gaussian OU processes has been explored by Jammalamadaka and Taufer (2019), [17]. Nabati (2021), has also investigated the nonlinear AR model with OU processes driven with white noise [16]. For many years, the parameter estimation of SDEs has been discussed in the literature. The nonparametric drift and diffusion function estimators by using the kernel regression method have been proposed by Fan and Yao (1998), Jacod (2000), and Fan and Zhang (2003), [18–20]. The performance of this approach is determined by the kernel function and its bandwidth; more information can be founded in Fan and Gijbles (1996), [21]. A functional estimation procedure for homogeneous SDE based on a discrete sample of observation has been studied by Bandi and Philips [22]. Hernandez et al. (2012) developed a moment method algorithm for the estimation of the parameters of both the observable process and the unobservable stochastic mean [5]. A maximum likelihood (ML) approach to estimate the parameters of a one-dimensional stationary process of the Ornstein-Uhlenbeck type has been proposed by Valdivieso et al. (2009), [23]. In this paper, we consider a three-factor OU process similar to the work of Lari Lavassani (2001)with the difference that we assumed a nonlinear AR drift term in the model and allowed the dependent errors as the AR(1) model instead of a sequence of independent and identically distributed random variables. The estimation of the parameters through the SP methodology and the Least square estimators (LSE) of model parameters have been emphasized throughout the paper. Also, the closed iterative form for the LSE of parameters has been obtained. This paper has been organized as follows. In section 2, the mathematical modeling for the three-factor mean-reverting OU process with nonlinear AR drift term by considering dependent error is presented. The existence of a unique positive solution for this system with initial values is proved in this section. The SP estimation for the nonlinear AR function is discussed in section 3. Parameters estimation for the OU process is carried out in section 4 using the ML estimator. Section 5 provides numerical simulations and a computational real case illustrating the use of the presented method. Our conclusions are drawn in the final section.

2. The OU process with nonlinear AR drift term and a dependent error

To illustrate the stochastic price process in commodities markets, various models have been proposed. In theory, by considering the dominating volatility and market deregulation, these models should be able to provide a reliable representation of the evolution of electricity prices. One of these models, is the three-factor model for energy and commodity spot price [5]. Now consider the following three-factor model:

$$\begin{cases} dS_t = \beta(l_t - S_t)dt + \sigma S_t^{\gamma} dB_t \\ l_t = g(l_{t-1}) + a_t \\ a_t = \rho a_{t-1} + z_t, \quad \gamma = 0 \text{ or } 1, \end{cases} \tag{1}$$

where β, σ are constants and $|\rho| < 1$. $g(.)$ is the unknown nonlinear AR function and z_t is a sequence of i.i.d random variables with mean zero and variance $\sigma_{z_t}^2$. We can get

$$a_t = l_t - g(l_{t-1}) \rightarrow a_{t-1} = l_{t-1} - g(l_{t-2}), \tag{2}$$

and therefore, the model in (1) can be written as follows

$$\begin{cases} dS_t = \beta(l_t - S_t)dt + \sigma S_t^{\gamma} dB_t \\ l_t = g(l_{t-1}) + \rho(l_{t-1} - g(l_{t-2})) + z_t. \end{cases} \tag{3}$$

This model allows for price to fluctuate around a level that is stochastic and also has the dependent error in form of AR(1). To investigate this model, at first, we prove the existence of a unique global positive solution for this system. Let $\gamma = 1$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathcal{F}_t (t \geq 0)$.

Theorem 2.1. *For any initial value $(S_0, l_0) \in \mathbb{R}_+^2$ and every $t \geq 0$, the system (3) have the unique solution. This solution will remain positive with probability one namely, $(S(t), l(t)) \in \mathbb{R}_+^2$.*

Proof. The coefficient of system (3) are locally lipschitz continuous. Hence, this system has unique solution $(S(t), l(t))$ on $[0, \tau_e)$ where τ_e is the explosion time [24]. Now, we show that $\tau_e = \infty$ almost surely (a.s.), and as a result the solution is global. Suppose $\omega_0 \geq 1$ is sufficiently large such that the initial values $S(0), l(0) \in [\frac{1}{\omega_0}, \omega_0]$. Define the stopping time

$$\tau_\omega = \inf\{t \in [0, \tau_e) : S(t) \notin (\frac{1}{\omega}, \omega) \text{ or } l(t) \notin (\frac{1}{\omega}, \omega)\}$$

for any $\omega \geq \omega_0$. Set $\inf \phi = \infty$ where ϕ shows the empty set. Then τ_ω is increasing when $\omega \to \infty$. Let $\tau_\infty = \lim_{\omega \to \infty} \tau_\omega$, so $\tau_\infty \leq \tau_0$ a.s. If we can prove $\tau_\infty = \infty$ a.s. then $\tau_e = \infty$ a.s. Let $\tau_\infty \neq \infty$, therefore, there exists two constants $\hat{\delta} > 0$ and $\tilde{\epsilon} \in (0, 1)$ such that $P(\tau_\infty \leq \hat{\delta}) \geq \tilde{\epsilon}$. So

$$\exists \omega_1 \in \mathbb{Z}, \omega_1 > \omega_0 \quad s.t. \quad P(\tau_\omega \leq \hat{\delta}) \geq \tilde{\epsilon} \ \ \forall \ \omega \geq \omega_1 \tag{4}$$

Consider the twice differentiable function $\Psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ with the definition,

$$\Psi(S, l) = (S - 1 - \log S) + (l - 1 - \log l)$$

since $\log u \leq u - 1$ for every $u \geq 0$, hence Ψ is nonnegtive. Using the system (3) and Ito formula,

$$\begin{aligned} d\Psi(S(t), l(t)) \ &= \ \mathcal{L}\Psi(S(t), l(t))dt + (1 - \tfrac{1}{S(t)})\sigma S(t) dB(t) \\ &+ \ (1 - \tfrac{1}{l(t)})dl(t) + \tfrac{1}{2l^2(t)}(dl(t))^2 \end{aligned} \tag{5}$$

Where

$$\mathcal{L}\Psi(S(t), l(t)) = (1 - \frac{1}{S(t)})\Big(\beta(l(t) - S(t))\Big) + \frac{\sigma^2}{2} = \mathcal{J}$$

which is bounded for $S(t) \geq 1$ and $\mathcal{J} \in \mathbb{R}_+$. Integrating both sides of (5) from 0 to $\tau_\omega \wedge \hat{\delta}$ and taking the expectations, concluded that

$$\begin{aligned} \mathbb{E}\Big(\Psi(S(t), l(t)\Big) \ &\leq \ \mathbb{E}\Big(\Psi(S(0), l(0)\Big) + \mathcal{J}\mathbb{E}(.) \\ &\leq \ \mathbb{E}\Big(\Psi(S(0), l(0))\Big) + \mathcal{J}\hat{\delta} \end{aligned} \tag{6}$$

where $\mathbb{E}$ shows the mathematical expectation. Let $\Omega_\omega := \tau_\omega \leq \hat{\delta}$ for $\omega \geq \omega_1$. From equation(4), we have $P(\Omega_\omega) \geq \tilde{\epsilon}$. Define

$$\Gamma_{\tau_\omega} := \Psi\Big((S(\tau_\omega), l(\tau_\omega))\Big)$$

then

$$\Gamma_{\tau_\omega} \geq (\omega - 1 - \log\omega) \wedge (\frac{1}{\omega} - 1 + \log\omega).$$

Hence equations (5) and (6) results:

$$\begin{aligned}\mathbb{E}(\Gamma_0) + \mathcal{J}\hat{\delta} &\geq \mathbb{E}\Big(I_{\Omega_\omega}\Gamma_{\tau_\omega}\Big)\\ &\geq \tilde{\epsilon}[(\omega - 1 - \log\omega) \wedge (\tfrac{1}{\omega} - 1 + \log\omega)]\end{aligned}$$

where I_{Ω_ω} is the indicator of set Ω_ω. If $\omega \to \infty$ then $\infty > \mathbb{E}(\Gamma_0) + \mathcal{J}_{\hat{\delta}} = \infty$ which is contradiction. So the hypothesis $P(t_\infty \leq \hat{\delta}) > \tilde{\epsilon}$ is wrong and $t_\infty = \infty$ a.s. □

3. Semiparametric estimation

The unknown function $g(.)$ is estimated using an SP technique based on Farnoosh and Mortazavi's (2011) work. We assume that $g(.)$ has a parametric form $\zeta(x,\theta), \theta \in \Theta$ which is a known and previous function selection where $\Theta \in \mathbf{R}^p$ is the parameter space. In this scenario, $\hat{g}(x) = \zeta(x,\hat{\theta})$ is used to estimate the regression function $g(.)$, where $\hat{\theta}$ is an estimator of θ. The exact value of θ is denoted by θ_0 that is defined as follows:

$$\theta_0 \quad = argmin_{\theta\in\Theta,|\rho|<1} E(l_t - E_\theta(l_t|l_{t-1}) - \rho(l_{t-2} - E_\theta(l_t|l_{t-2}))^2.$$

It is obvious that

$$E_\theta(l_t|l_{t-1}, l_{t-2}) = g(l_{t-1}) + \rho(l_{t-1} - g(l_{t-2})).$$

In model (3), the estimators of θ and ρ are obtanied via conditional nonlinear least square errors method as follows:

$$A_n(\theta,\rho) = \sum_{t=2}^{n}\{\Big(l_t - g(l_{t-1}) + \rho(l_{t-1} - g(l_{t-2}))\Big)^2\},$$

therefore

$$(\hat{\theta_n}, \hat{\rho_n}) = argmin A_n(\theta,\rho).$$

The strong consistency of $(\hat{\theta_n}, \hat{\rho_n})$ under a variaty of conditions has been studied in work of Farnoosh and Mortazavi (2011) [8]. Because $\zeta(x,\theta)$ is a rough approximation of $g(x)$, the SP form $\zeta(x,\theta)\chi(x)$ is used to correct the previous estimate, where $\chi(x)$ is the adjustment factor. For determining $\chi(x)$, the local L2-fitting criterion is used as follows:

$$\begin{aligned}B(x,\chi) &= \frac{1}{b_n}\sum_{t=2}^{n} K(\frac{l_{t-1}-x}{b_n})\{g(l_{t-1}) - \zeta(l_{t-1},\hat{\theta}).\chi(x)\}^2\\ &+ \frac{1}{b_n}\sum_{t=2}^{n} K(\frac{l_{t-2}-x}{b_n})\{g(l_{t-2}) - \zeta(l_{t-2},\hat{\theta}).\chi(x)\}^2, \end{aligned} \tag{7}$$

where K and b_n are the kernel and bandwidth respectivly. The estimator $\hat{\chi}(x)$ is obtained by minimizing the criterion in equation (7) with regard to $\chi(x)$.

$$\hat{\chi}(x) = \frac{\sum_{t=2}^{n} g(l_{t-1})K(\frac{l_{t-1}-x}{b_n})\zeta(l_{t-1},\hat{\theta_n}) + g(l_{t-2})K(\frac{l_{t-2}-x}{b_n})\zeta(l_{t-2},\hat{\theta_n})}{\sum_{t=2}^{n} K(\frac{l_{t-1}-x}{b_n})\zeta^2(l_{t-1},\hat{\theta_n}) + K(\frac{l_{t-2}-x}{b_n})\zeta^2(l_{t-2},\hat{\theta_n})}. \tag{8}$$

Unfortunately, the equation(8) containe the unknown function $g(x)$, therefore by using $a_t = l_t - g(l_{t-1})$ and $a_{t-1} = l_{t-1} - g(l_{t-2})$ and with considering the fact that $E(a_t) = 0$ and $E(a_{t-1}) = 0$, the equation (8) can be written as,

$$\tilde{\chi}(x) = \frac{\sum_{t=2}^{n} l_t K(\frac{l_{t-1}-x}{b_n})\zeta(l_{t-1},\hat{\theta}_n) + l_{t-1}K(\frac{l_{t-2}-x}{b_n})\zeta(l_{t-2},\hat{\theta}_n)}{\sum_{t=2}^{n} K(\frac{l_{t-1}-x}{b_n})\zeta^2(l_{t-1},\hat{\theta}_n) + K(\frac{l_{t-2}-x}{b_n})\zeta^2(l_{t-2},\hat{\theta}_n)}.$$

Therefore, the AR estimator is obtained by

$$\tilde{g}(x) = \zeta(x,\hat{\theta}).\tilde{\chi}(x).$$

4. Parameter estimation

The ML estimation is a method of estimating the unknown parameters by maximizing a likelihood function when the observed data is most probable. A set of observations is a random sample from an unknown population in statistical terms. Let $\Pi_n = \{0, \Delta t, 2\Delta t, ..., n\Delta t\}$ is a partition for time interval $[0, T]$,where $\Delta t = \frac{T}{n}$. The Euler Maruyama scheme for model (3) is as follows,

$$S_{t+\Delta t} = S(t) + \beta(l_t - S_t)\Delta t + \sigma S_t \Delta B_t,$$

where $\Delta B_t \sim N(0, \Delta t)$. It is clear that

$$\begin{aligned} E(S_{t+\Delta t}|S_t) &= S_t + \beta\Big(E(l_t) - S_t\Big)\Delta t \\ &= S_t + \beta\Big(g(l_{t-1}) + \rho l_{t-1} - \rho g(l_{t-2}) - S_t\Big)\Delta t, \end{aligned}$$

The distribution of $S_{t+\Delta t}$ in the Euler scheme is given by,

$$f(S_{t+\Delta t}|S_t) = \frac{1}{\sqrt{2\pi\sigma^2 S_t^2 \Delta t}} \exp\{\frac{\Lambda}{2\sigma^2 S_t^2 \Delta t}\},$$

where

$$\Lambda = -[S_{t+\Delta t} - ((1-\beta)S_t + \beta\Big(g(l_{t-1}) + \rho l_{t-1} - \rho g(l_{t-2})\Big)\Delta t]^2$$

so the log-likelihood is as follows,

$$\log(L(\beta,\sigma)) = \log f_0(S_0|\beta,\sigma) + \sum_{t=1}^{n} \log f(S_t|S_{t-1},\beta,\sigma) \tag{9}$$

The purpose of ML estimation is to identify model parameter values that maximize the likelihood function throughout the parameter space, i.e. $(\beta,\sigma) = argmax(L(\beta,\sigma))$. The logarithm is a monotonic function, the maximum of L occurs at the same value of (β,σ) as does the maximum of $\log(L)$.

5. Simulation study and empirical application

We investigate the suggested framework in this part using both simulated and real data sets.

5.1. Simulation study

We consider a simulation study by generating the data sets from model (1) with two different AR drift term functions as follows,

$$g_1(x) = 5e^{-x^2} + 0.3\cos(x), \text{ by assuming } \zeta_1(x,\theta) = \theta_{11}e^{-x^2} \text{ and } \rho = 0.9$$
$$g_2(x) = e^{-3x} + 0.1\sin(x), \text{ by assuming } \zeta_2(x,\theta) = \theta_{21}e^{\theta_{22}x} \text{ and } \rho = 0.5$$

The use of nonparametric adjustment, $\hat{\chi}(x)$ requires the optimal selection of the smoothing parameter called the bandwidth, b_n, that is chosen by an opening the window technique, i.e. by trying several bandwidths and deciding for a good compromise which is neither too smooth nor too rough [13]. By trying several bandwidths, we get $b_n = 0.04$. Tables 1 and 2 show the estimation of the parameters and mean square error (MSE) for SP estimation with 1000 iteration with a different sample size of the simulation. Figures 1 and 2 show the curves of $g(x)$ and $\tilde{g}(x)$ for chosen bandwidth respectively. The solid line represents the function $g(x)$, whereas the broken line represents the function's estimator. The simulation results show that the SP estimator performs well.

Table 1: Parmeters estimation for AR functions with different sample size.

AR Function	Parameters	Sample Size		
		100	500	1000
	θ_{11}	5.1104	5.0655	5.0501
$g_1(x)$	ρ	0.9985	0.9995	0.9992
	θ_{21}	0.9986	0.9955	0.9993
$g_2(x)$	θ_{22}	-2.1146	-2.0861	-2.084
	ρ	0.9958	0.9937	0.9928

Table 2: MSE for semiparametric estimation of the AR functions.

AR Function	Sample Size		
	100	500	1000
$g_1(x)$	0.0648	0.0594	0.0514
$g_2(x)$	0.0061	0.0057	0.0052

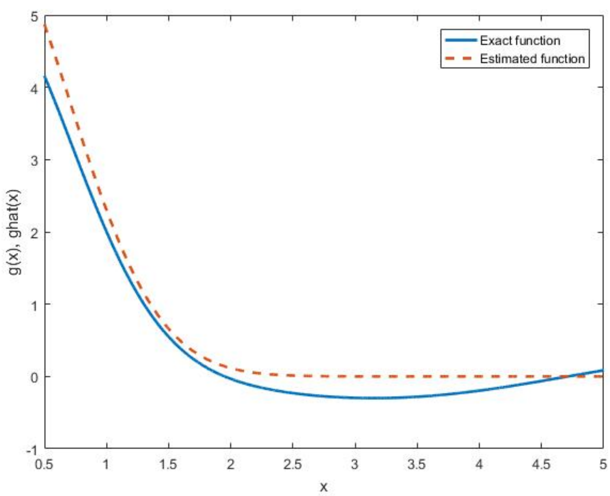


Figure 1: Exact and estimated functions for nonlinear AR function $g_1(x)$.

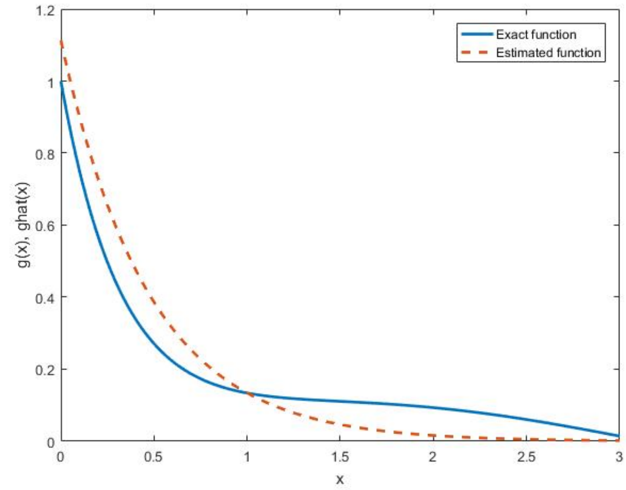


Figure 2: Exact and estimated functions for nonlinear AR function $g_2(x)$.

5.2. Empirical application

In this section, we provide a real-world example to illustrate the usefulness of the proposed model in applications. We consider a data set consisted of the Tabarrok stock price (TSP) downloaded from, http://www.tsetmc.com, for the period 2018-10-27 to 2019-03-03. The time series plot of TSP is shown in Fig. 3 (bottom) and also the top plot of Fig. 3 illustrates the linearity price of Tabarrok stock at time t, (S_t) against time $t-1$, (S_{t-1}). Table 3, shows descriptive statistics indices for this data. The pvalue of the Shapiro-Wilk indicates the normality of the model. Since the local variance of the time series was larger, we normalized the data sets. At first, according to the nature of observations that have the mean reverting properties, we consider three nonlinear AR functions for drift term of the data sets as follows:

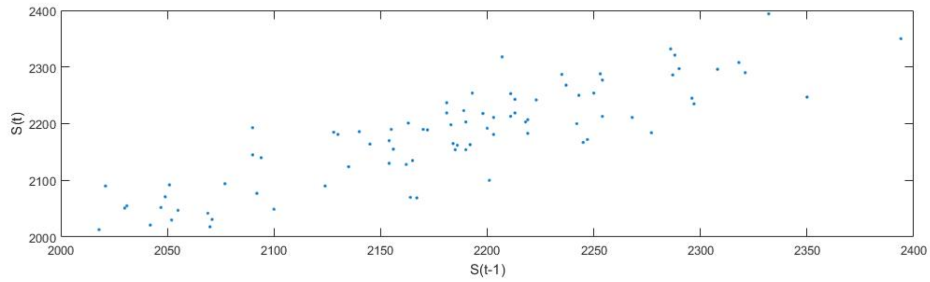
$$g_1(x,\alpha) = 0.5\exp(-x^2) + 0.3\cos x, \text{ by assuming } \zeta_1(x,\alpha) = \alpha_{11}\exp^{-x^2} \text{ and } \rho = 0.5$$
$$g_2(x,\alpha) = \exp(-3x) + 0.1\sin x, \text{ by assuming } \zeta_2(x,\alpha) = \alpha_{21}\exp\alpha_{22}x \text{ and } \rho = 0.5$$
$$g_3(x,\alpha) = 0.48 + 0.1\sin x, \text{ by assuming } \zeta_3(x,\alpha) = \alpha_{31} + \alpha_{32}\sin x \text{ and } \rho = 0.5$$

For estimation of the model parameters, the value of bandwidths must be obtained. The bandwidth depends on the sample size N for consistency of the kernel function then we must have $b_n \to 0$ and $Nb_n \to \infty$ for $N \to \infty$. But for practical implementation, this condition is not very helpful. So, the bandwidth b_n was chosen by an opening the window technique, i.e., by trying several bandwidths [11].

Table 4 summarizes the parameters estimation and MSE criteria for TSP according to model (1) using these nonlinear AR drift terms with different bandwidths. The independent sample paths of the model using these drift terms were simulated. Figures 4-6 shows the curves of the observations and its semiparametric estimator with selected bandwidth $b_n = 0.04$ and drift terms g_1, g_2, and g_3 respectively. The blue and red lines are the curves of the observations and the semiparametric predictor and the green line show the path of drift term. Considering the measures of MSE for three models, we get that the exponential model g_1 is more efficient than the other models.

Table 3: Descriptive statistics for Tabarrok data

	Mean	Median	LCL Mean	UCL Mean
	2180.3604	2187.5000	2161.8968	2198.8240
	Variance	Std.deviation	Skewness	Kurtosis
Tabarrok data	7416.1861	86.1172	-0.0707	-0.5670
		W (Shapiro-Wilk)	P-Value	
		0.97967	0.1939	



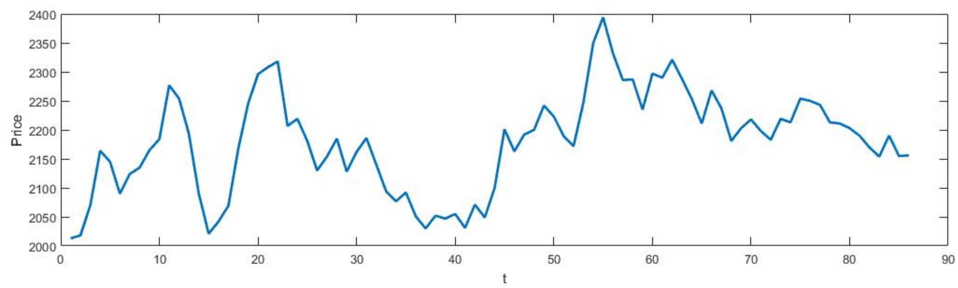


Figure 3: Plot of S_t against S_{t-1} (Top) and time series plot of TSP (bottom).

Table 4: Parameters estimation and MSE criteria for different $g(x,\theta)$

Autoregressive Function	$b_n = 0.01$	$b_n = 0.04$	$b_n = 0.08$
$g_1(x,\theta) = 0.5\exp(-x^2) + 0.3\cos x$	$\hat{\beta} = 0.1192$ $\hat{\sigma} = 0.2318$ $MSE = 0.2625$	$\hat{\beta} = 0.1199$ $\hat{\sigma} = 0.1719$ $MSE = 0.2469$	$\hat{\beta} = 0.1199$ $\hat{\sigma} = 0.1839$ $MSE = 0.2501$
$g_2(x,\theta) = \exp(-3x) + 0.1\sin x$	$\hat{\beta} = 0.1189$ $\hat{\sigma} = 0.0468$ $MSE = 0.2705$	$\hat{\beta} = 0.1190$ $\hat{\sigma} = 0.0472$ $MSE = 0.2666$	$\hat{\beta} = 0.1199$ $\hat{\sigma} = 0.0467$ $MSE = 0.2748$
$g_3(x,\theta) = 0.48 + 0.1\sin x$	$\hat{\beta} = 0.1190$ $\hat{\sigma} = 0.0140$ $MSE = 0.2706$	$\hat{\beta} = 0.1195$ $\hat{\sigma} = 0.0186$ $MSE = 0.2720$	$\hat{\beta} = 0.1185$ $\hat{\sigma} = 0.0164$ $MSE = 0.2727$

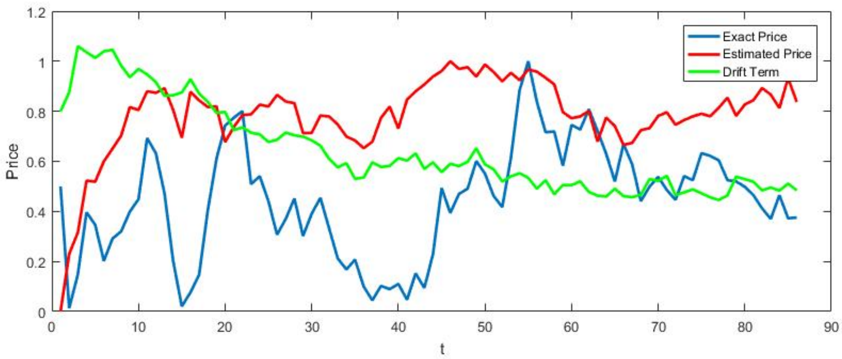


Figure 4: The exact and estimated price with AR function $g_1(x,\theta) = 0.5\exp(-x^2) + 0.3\cos x$.

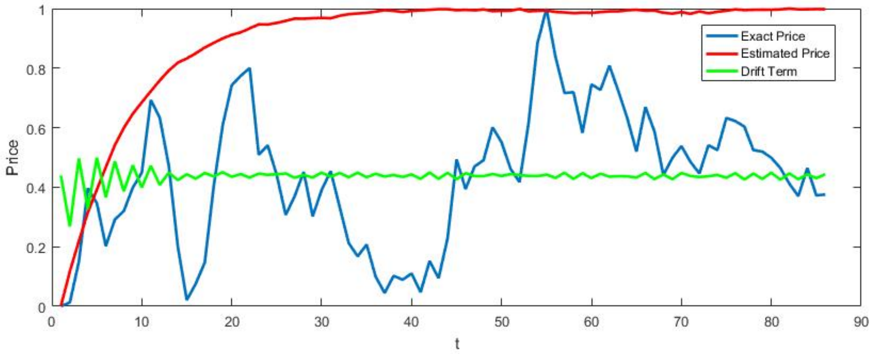


Figure 5: The exact and estimated price with AR function $g_2(x, \theta) = \exp(-3x) + 0.1 \sin x$.

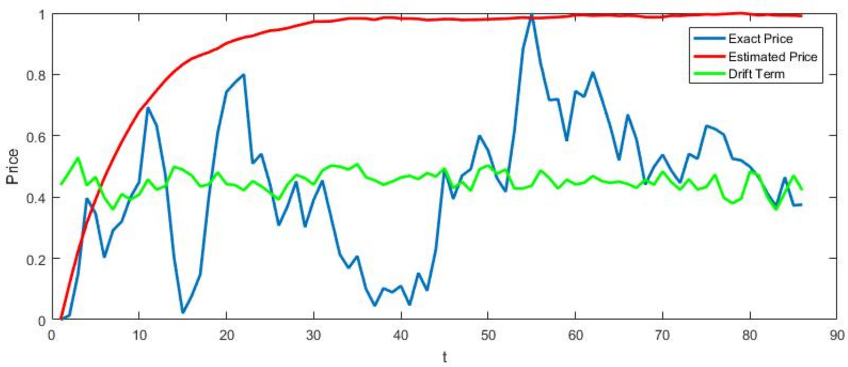


Figure 6: The exact and estimated price with AR function $g_3(x, \theta) = 0.48 + 0.1 \sin x$.

6. Conclusion

In this paper, a new three-factor mean-reverting OU process with nonlinear AR drift term with a dependent error has been presented. The stochastic drift term has many important characteristics, which can model the spikes of the prices in markets. For a stochastic system, the unique solvability of the presented model is demonstrated. The SP technique has been used to estimate the nonlinear AR function. Parameter estimation for the OU process has been carried out using the ML estimator. Finally, real-world data set and computer numerical simulations have been presented to support the accuracy of the findings.

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