Filomat 36:7 (2022), 2333–2344
https://doi.org/10.2298/FIL2207333H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Berezin Number Inequalities via Convex Functions

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Abstract. The Berezin symbol $\widetilde{A}$ of an operator A on the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ over some set Ω with the reproducing kernel k_ξ is defined by

$$\tilde{A}(\xi) = \left\langle A\frac{k_\xi}{\|k_\xi\|}, \frac{k_\xi}{\|k_\xi\|} \right\rangle, \ \xi \in \Omega.$$

The Berezin number of an operator A is defined by

$$\text{ber}(A) := \sup_{\xi\in\Omega} \left|\widetilde{A}(\xi)\right|.$$

We study some problems of operator theory by using this bounded function $\widetilde{A}$, including treatments of inner product inequalities via convex functions for the Berezin numbers of some operators. We also establish some inequalities involving of the Berezin inequalities.

1. Introduction

Let Ω be a subset of a topological space X such that the boundary $\partial\Omega$ is nonempty. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space complex-valued functions defined on Ω. We say that $\mathcal{H}$ is a reproducing kernel Hilbert space if the following two conditions are satisfied :

(i) for any $\xi \in \Omega$, the evaluation functionals $f \to f(\xi)$ are continuous on $\mathcal{H}$;

(ii) for any $\xi \in \Omega$, there exists $f_\xi \in \mathcal{H}$ such that $f_\xi(\xi) \neq 0$ (or equivalently, there is no $\xi_0 \in \Omega$ such that $f(\xi_0) = 0$ for every $f \in \mathcal{H}$).

According to the classical Riesz representation theorem, the assumption (i) implies that, for every $\xi \in \Omega$ there exists a unique function $k_\xi \in \mathcal{H}$ such that

$$f(\xi) = \langle f, k_\xi \rangle_{\mathcal{H}}, \ f \in \mathcal{H}.$$

The function $k_\xi(z)$ is called the reproducing kernel of $\mathcal{H}$ at point ξ. It is well known that every reproducing kernel Hilbert space is separable. So, if $\{e_n(z)\}_{n\geq 0}$ is any orthonormal basis of $\mathcal{H}$, then (see Aronzajn [3])

$$k_\xi(z) = \sum_{n=0}^{\infty} \overline{e_n(\xi)} e_n(z).$$

2020 *Mathematics Subject Classification*. Primary 47A30; Secondary 47A63
Keywords. Berezin symbol, Berezin number, reproducing kernel Hilbert space, mixed Schwarz inequality
Received: 18 June 2021; Revised: 12 October 2021; Accepted: 04 November 2021
Communicated by Fuad Kittaneh
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By virtue of assumption (ii), we surely have $k_\xi \neq 0$ and we denote by $\widehat{k_\xi}$ the normalized reproducing kernel, that is $\widehat{k_\xi} := \frac{k_\xi}{\|k_\xi\|_{\mathcal{H}}}$. Recall that if $\mathcal{B}(\mathcal{H})$ is the Banach algebra of all bounded linear operator on $\mathcal{H}$, then the Berezin symbol $\widetilde{A}$ of any operator $A \in \mathcal{B}(\mathcal{H})$ is the complex-valued function defined on the Ω by the formula (see, Berezin [8, 9])

$$\widetilde{A}(\xi) := \left\langle A\widehat{k_\xi}, \widehat{k_\xi} \right\rangle_{\mathcal{H}}, \ \xi \in \Omega.$$

The Berezin set of operator A is defined by

$$\operatorname{Ber}(A) = \left\{ \left\langle A\widehat{k_\xi}, \widehat{k_\xi} \right\rangle : \xi \in \Omega \right\} = \operatorname{Range}\left(\widetilde{A}\right),$$

and Berezin number $\operatorname{ber}(A)$ of operator A is the following number (see [25, 26])

$$\operatorname{ber}(A) := \sup_{\xi \in \Omega} \left| \widetilde{A}(\xi) \right|.$$

Since, $\left|\widetilde{A}(\xi)\right| \leq \|A\|$, Berezin symbol is a bounded function on Ω. Also, it is trivial by Cauchy-Schwarz inequality that $\operatorname{ber}(A) \leq \|A\|$. If $A = cI$ with $c \neq 0$, then obviously $\operatorname{ber}(A) = |c| > \frac{|c|}{2} = \frac{\|A\|}{2}$. But Karaev in [26] showed that in general

$$\frac{1}{2}\|A\| \leq \operatorname{ber}(A)$$

is not satisfied for every $A \in \mathcal{B}(\mathcal{H})$.

Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [25]. For the basic properties and facts on these new concepts, see [5–7, 26, 32, 34].

It is well-known that

$$\operatorname{ber}(A) \leq w(A) \leq \|A\| \tag{1}$$

and

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\| \tag{2}$$

for any $A \in \mathcal{B}(\mathcal{H})$. The inequalities in (2) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A normal. For basic properties of the numerical radius, we refer to [20] and [21]. The inequalities in (2) have been improved considerably by the second author in [29] and [31]. It has been shown in [29] and [31], respectively, that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{1/2}\right), \tag{3}$$

where $|A| = (A^*A)^{1/2}$ is the absolute value of A, and

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|.$$

The inequalities in (3), which refine the second inequality in (2), have been utilized in [29] to derive an estimate for the numerical radius of the Frobenius companion matrix (also see [1, 2, 12, 23]).

The purpose of this paper is to establish some inequalities involving of the Berezin number inequalities of operators by using convex function $\tilde{A}$. Usual operator norm inequalities and a related Berezin number inequality of operators are also presented. Related results are contained in [15–19, 22, 35–37].

2. Berezin Number Inequalities

2.1. Lemmas

In order to prove our results, we need the following sequence of lemmas.

Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. In this case we will write $A \geq 0$. The classical operator Jensen inequality for the positive operators $A \in \mathcal{B}(\mathcal{H})$ is

$$\langle Ax, x\rangle^r \leq (\geq) \langle A^r x, x\rangle, \ r \geq 1 \ (0 \leq r \leq 1) \tag{4}$$

for any unit vector $x \in \mathcal{H}$.

Lemma 2.1. *We have the Power-Mean inequality, that reads*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b \leq (\lambda a^p + (1-\lambda) b^p)^{\frac{1}{p}}, \tag{5}$$

for $a, b \geq 0$, $0 \leq \lambda \leq 1$, *and* $p \geq 1$.

The following inequality is the spectral theorem for positive operators and Jensen inequality (see [14]) which states that if f is a convex function on an interval containing the spectrum of A, then

$$f(\langle Ax, x\rangle) \leq \langle f(A) x, x\rangle \tag{6}$$

which A is a positive operators in $\mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$ is an unit vector. If f is concave, then (6) holds in the reverse direction.

The mixed Schwarz inequality was introduced in [21], as follows:

Lemma 2.2. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and let* $x \in H$ *be a unit vector. Then*

$$|\langle Ax, x\rangle|^2 \leq \langle |A| x, x\rangle \langle |A^*| x, x\rangle. \tag{7}$$

Another inequality, which can be found by Aujla and Silva [4], gives a norm inequality involving convex function of positive operator which assert

$$\left\| f\left(\frac{A+B}{2}\right)\right\| \leq \left\|\frac{f(A)+f(B)}{2}\right\| \tag{8}$$

which f be a non-negative nondecreasing convex function on $[0, \infty)$ and $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators.

In 2004, Kittaneh [30] has shown follow inequality and this follow inequality is considered as a refined triangle inequality for positive operators.

Lemma 2.3. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$\left\| |A|^2 + |A^*|^2 \right\| \leq \left\| A^2 \right\| + \|A\|^2. \tag{9}$$

The following lemma contains a special case of a more general norm inequality that is equivalent to some Löwner–Heinz type inequalities (see [13, 27]).

Lemma 2.4. *If* $A, B \in \mathcal{B}(\mathcal{H})$ *are positive operators, then*

$$\left\| A^{1/2} B^{1/2} \right\| \leq \|AB\|^{1/2}.$$

The last lemma contains a recent norm inequality for sums of positive operators that is sharper than the triangle inequality (see [28]).

Lemma 2.5. *If* $A, B \in \mathcal{B}(\mathcal{H})$ *are positive operators, then*

$$\|A+B\| \leq \frac{1}{2}(\|A\| + \|B\|) + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{1/2}B^{1/2}\right\|^2}.$$

2.2. Main Results

Now, we are ready to state the main results of this section. Our first main result can be stated as follows.

Theorem 2.6. *If $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ and $f : [0,\infty) \to \mathbb{R}$ is an increasing convex function, then*

$$f\left(\left|\widetilde{A}(\xi)\widetilde{B}(\xi)\right|^2\right) \le \frac{1}{2} f\left(\left|\widetilde{BA}(\xi)\right|^2\right) + \frac{1}{2}\left\langle\left(\lambda f\left(|A|^{\frac{2}{\lambda}}\right) + (1-\lambda) f\left(|B^*|^{\frac{2}{1-\lambda}}\right)\right)\widehat{k_\xi}, \widehat{k_\xi}\right\rangle \tag{10}$$

for $0 \le \lambda \le 1$. *Further,*

$$f\left(\left|\widetilde{A}(\xi)\widetilde{B}(\xi)\right|\right) \le \frac{1}{2} f\left(\left|\widetilde{BA}(\xi)\right|\right) + \frac{1}{4}\left\langle\left(f\left(|A|^2\right) + f\left(|B^*|^2\right)\right)\widehat{k_\xi}, \widehat{k_\xi}\right\rangle. \tag{11}$$

Proof. Let $\widehat{k_\xi}$ be normalized reproducing kernel. The following refinement of the Cauchy-Schwarz inequality proved by Buzano [11]:

$$\|x\|\|y\| \ge \left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right| + \left|\langle x,e\rangle\langle e,y\rangle\right| \ge \left|\langle x,y\rangle\right|, \tag{12}$$

for all $x, y, e \in \mathcal{H}$ and $\|e\| = 1$. From inequality (12), we conclude that

$$\frac{1}{2}\left(\|x\|\|y\| + \left|\langle x,y\rangle\right|\right) \ge \left|\langle x,e\rangle\langle e,y\rangle\right|.$$

Putting $e = \widehat{k_\xi}$, $x = A\widehat{k_\xi}$ and $y = B^*\widehat{k_\xi}$ in the above, we have

$$\frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right| + \left\|A\widehat{k_\xi}\right\|\left\|B^*\widehat{k_\xi}\right\|\right) \ge \left|\left\langle A\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\left\langle B\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|. \tag{13}$$

Hence, by the function $t \to t^2$ is convex,

$$\begin{aligned}
\left|\left\langle A\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\left\langle B\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 &\le \left(\frac{\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right| + \left\|A\widehat{k_\xi}\right\|\left\|B^*\widehat{k_\xi}\right\|}{2}\right)^2 \\
&\le \frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 + \left\|A\widehat{k_\xi}\right\|^2\left\|B^*\widehat{k_\xi}\right\|^2\right) \\
&= \frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 + \left\langle A\widehat{k_\xi}, A\widehat{k_\xi}\right\rangle\left\langle B^*\widehat{k_\xi}, B^*\widehat{k_\xi}\right\rangle\right) \\
&= \frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 + \left\langle |A|^2\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\left\langle |B^*|^2\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right) \\
&= \frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 + \left\langle\left(|A|^{\frac{2}{\lambda}}\right)^{\lambda}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\left\langle\left(|B^*|^{\frac{2}{1-\lambda}}\right)^{1-\lambda}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right) \\
&\le \frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 + \left\langle |A|^{\frac{2}{\lambda}}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle^{\lambda}\left\langle |B^*|^{\frac{2}{1-\lambda}}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle^{1-\lambda}\right) \\
&\quad \text{(by the inequality (4))} \\
&\le \frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 + \lambda\left\langle |A|^{\frac{2}{\lambda}}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle + (1-\lambda)\left\langle |B^*|^{\frac{2}{1-\lambda}}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right) \\
&\quad \text{(by the inequality (5))}.
\end{aligned}$$

Hence,

$$\left|\left\langle A\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\left\langle B\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 \le \frac{1}{2}\left(\left|\left\langle BA\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right|^2 + \lambda\left\langle |A|^{\frac{2}{\lambda}}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle + (1-\lambda)\left\langle |B^*|^{\frac{2}{1-\lambda}}\widehat{k_\xi}, \widehat{k_\xi}\right\rangle\right) \tag{14}$$

Now since f is increasing and convex, (14) implies

$$
\begin{aligned}
& f\left(\left|\left\langle A\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\left\langle B\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2\right)\\
& \leq f\left(\frac{1}{2}\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2+\lambda\left\langle |A|^{\frac{2}{\lambda}}\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+(1-\lambda)\left\langle |B^*|^{\frac{2}{1-\lambda}}\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)\right)\\
& \leq \frac{f\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2\right)+f\left(\lambda\left\langle |A|^{\frac{2}{\lambda}}\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+(1-\lambda)\left\langle |B^*|^{\frac{2}{1-\lambda}}\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)}{2}\\
& \leq \frac{f\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2\right)+\lambda f\left(\left\langle |A|^{\frac{2}{\lambda}}\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)+(1-\lambda) f\left(\left\langle |B^*|^{\frac{2}{1-\lambda}}\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right)}{2}\\
& \leq \frac{f\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2\right)+\lambda \left\langle f\left(|A|^{\frac{2}{\lambda}}\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+(1-\lambda) \left\langle f\left(|B^*|^{\frac{2}{1-\lambda}}\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{2}\\
& \text{(by inequality (6))}\\
& \leq \frac{f\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|^2\right)+\left\langle\left(\lambda f\left(|A|^{\frac{2}{\lambda}}\right)+(1-\lambda) f\left(|B^*|^{\frac{2}{1-\lambda}}\right)\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{2}
\end{aligned}
$$

which is equivalent to

$$
f\left(\left|\widetilde{A}(\xi)\,\widetilde{B}(\xi)\right|^2\right)\leq \frac{1}{2}f\left(\left|\widetilde{BA}(\xi)\right|^2\right)+\frac{1}{2}\left\langle\left(\lambda f\left(|A|^{\frac{2}{\lambda}}\right)+\frac{1}{2}(1-\lambda) f\left(|B^*|^{\frac{2}{1-\lambda}}\right)\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle.
$$

On the other hand, from (13), we get

$$
\begin{aligned}
\left|\left\langle A\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\left\langle B\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right| & \leq \frac{1}{2}\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|+\left\langle A\widehat{k}_\xi,A\widehat{k}_\xi\right\rangle^{\frac{1}{2}}\left\langle B^*\widehat{k}_\xi,B^*\widehat{k}_\xi\right\rangle^{\frac{1}{2}}\right)\\
& = \frac{1}{2}\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|+\left\langle |A|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\frac{1}{2}}\left\langle |B^*|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle^{\frac{1}{2}}\right)\\
& \leq \frac{1}{2}\left(\left|\widetilde{BA}(\xi)\right|+\frac{\left\langle |A|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle |B^*|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{2}\right)\\
& \qquad \text{(by the AM-GM inequality).}
\end{aligned}
$$

Again, since f is increasing and convex, we obtain

$$
\begin{aligned}
& f\left(\left|\left\langle A\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\left\langle B\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|\right)\\
& \leq f\left(\frac{1}{2}\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|+\frac{\left\langle |A|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle |B^*|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{2}\right)\right)\\
& \leq \frac{1}{2}\left(f\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|\right)+f\left(\frac{\left\langle |A|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle |B^*|^2\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{2}\right)\right)\\
& \leq \frac{1}{2}\left(f\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|\right)+\frac{\left\langle f\left(|A|^2\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle+\left\langle f\left(|B^*|^2\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle}{2}\right)\\
& = \frac{1}{2}f\left(\left|\left\langle BA\widehat{k}_\xi,\widehat{k}_\xi\right\rangle\right|\right)+\frac{1}{4}\left\langle\left(f\left(|A|^2\right)+f\left(|B^*|^2\right)\right)\widehat{k}_\xi,\widehat{k}_\xi\right\rangle
\end{aligned}
$$

and

$$f\left(\left|\widetilde{A}(\xi)\widetilde{B}(\xi)\right|\right) \leq \frac{1}{2} f\left(\left|\widetilde{BA}(\xi)\right|\right) + \frac{1}{4}\left\langle\left(f\left(|A|^2\right) + f\left(|B^*|^2\right)\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle.$$

We obtain the desired inequality. □

Noting that the function $f(t) = t^p$, $p \geq 1$ satisfes the conditions in Theorem 2.6, we obtain the following particular.

Corollary 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$. Then for any $p \geq 1$ and $0 \leq \lambda \leq 1$,*

$$\left|\widetilde{A}(\xi)\widetilde{B}(\xi)\right|^{2p} \leq \frac{1}{2}\left|\widetilde{BA}(\xi)\right|^{2p} + \frac{1}{2}\left\langle\left(\lambda|A|^{\frac{2p}{\lambda}} + (1-\lambda)|B^*|^{\frac{2p}{1-\lambda}}\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle,$$

and

$$\left|\widetilde{A}(\xi)\widetilde{B}(\xi)\right|^{p} \leq \frac{1}{2}\left|\widetilde{BA}(\xi)\right|^{p} + \frac{1}{4}\left\langle\left(|A|^{2p} + |B^*|^{2p}\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle.$$

The first application of Theorem 2.6 and Corollary 2.7 is the following ber-norm and Berezin number inequality for the product of two operators.

Corollary 2.8. *If $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ and $f : [0,\infty) \to \mathbb{R}$ is an increasing convex function, then*

$$f\left(\mathrm{ber}^2(B^*A)\right) \leq \frac{1}{2} f\left(\mathrm{ber}\left(|B|^2|A|^2\right)\right) + \frac{1}{4}\left\|f\left(|A|^4\right) + f\left(|B|^4\right)\right\|_{\mathrm{ber}}.$$

In particular, if $p \geq 1$, then

$$\mathrm{ber}^{2p}(B^*A) \leq \frac{1}{2}\mathrm{ber}^p\left(|B|^2|A|^2\right) + \frac{1}{4}\left\||A|^{4p} + |B|^{4p}\right\|_{\mathrm{ber}}. \tag{15}$$

Proof. Replacing A and B by $|A|^2$ and $|B|^2$ respectively Theorem 2.6, then the inequality (11) reduces to

$$f\left(\left|\left\langle|A|^2\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\left\langle|B|^2\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|\right) \leq \frac{1}{2} f\left(\left|\left\langle|B|^2|A|^2\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|\right) + \frac{1}{4}\left\langle\left(f\left(|A|^4\right) + f\left(|B^*|^4\right)\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle. \tag{16}$$

On the other hand,

$$\begin{aligned}\left|\left\langle B^*A\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^2 &= \left|\left\langle A\widehat{k}_\xi, B\widehat{k}_\xi\right\rangle\right|^2 \\ &\leq \left\|A\widehat{k}_\xi\right\|^2\left\|B\widehat{k}_\xi\right\|^2 \\ &\quad \text{(by the Cauchy-Schwarz inequality)} \\ &= \left\langle|A|^2\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\left\langle|B|^2\widehat{k}_\xi, \widehat{k}_\xi\right\rangle.\end{aligned}$$

Since f is increasing, we get

$$f\left(\left|\left\langle B^*A\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right|^2\right) \leq f\left(\left\langle|A|^2\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\left\langle|B|^2\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right)$$

and this together with (16) imply

$$f\left(\left|\widetilde{B^*A}(\xi)\right|^2\right) \leq \frac{1}{2} f\left(\left|\widetilde{|B|^2|A|^2}(\xi)\right|\right) + \frac{1}{4}\left\langle\left(f\left(|A|^4\right) + f\left(|B^*|^4\right)\right)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle.$$

By taking supremum over $\xi \in \Omega$, we have

$$f\left(\mathrm{ber}^2(B^*A)\right) \leq \frac{1}{2} f\left(\mathrm{ber}\left(|B|^2|A|^2\right)\right) + \frac{1}{4}\left\|f\left(|A|^4\right) + f\left(|B|^4\right)\right\|_{\mathrm{ber}}.$$

Consider the function $f(t) = t^p$, $p \geq 1$, then we get the second inequality. This completes the proof. □

Remark 2.9. *Since for $p = 1$ and $A = B$, we get on both sides of (15) the same quantity $\|A\|_{\text{ber}}^4$.*

Corollary 2.10. *If $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ then*

$$\text{ber}(B^*A) \leq \frac{1}{2}\left\||A|^2 + |B|^2\right\|$$

and

$$\text{ber}^{2p}(B^*A) \leq \frac{1}{2}\left\||A|^{4p} + |B|^{4p}\right\|, \; p \geq 1. \tag{17}$$

Proof. We recall the following arithmetic-geometric mean inequality obtained in [10]

$$\|B^*A\| \leq \frac{1}{4}\left\|(|A| + |B|)^2\right\|. \tag{18}$$

Hence, by the inequality (1),

$$\begin{aligned}\text{ber}(B^*A) \leq \|B^*A\| &\leq \frac{1}{4}\||A| + |B|\|^2 \;\; \text{(by (18))} \\ &= \left\|\left(\frac{|A| + |B|}{2}\right)^2\right\| \\ &\leq \frac{1}{2}\left\||A|^2 + |B|^2\right\| \;\; \text{(by the inequality (8))}.\end{aligned}$$

Notice that

$$\text{ber}^p\left(|B|^2|A|^2\right) \leq \frac{1}{2}\left\||A|^{4p} + |B|^{4p}\right\|.$$

Also Corollary 2.8 implies that

$$\begin{aligned}\text{ber}^{2p}(B^*A) &\leq \frac{1}{2}\text{ber}^p\left(|A|^2|B|^2\right) + \frac{1}{4}\left\||A|^{4p} + |B|^{4p}\right\| \\ &\leq \frac{1}{2}\left\||A|^{4p} + |B|^{4p}\right\|,\end{aligned}$$

explaining why Corollary 2.8 provide a refinement of the inequality (17). Further, the first inequality in Corollary 2.8 provides a generalization of (17). □

Now Theorem 2.6 is utilized to obtain the following one-operator Berezin number inequality.

Corollary 2.11. *If $E \in \mathcal{B}(\mathcal{H}(\Omega))$ and $f : [0, \infty) \to \mathbb{R}$ is an increasing convex function, then for $0 \leq \lambda \leq 1$,*

$$f\left(\text{ber}^4(E)\right) \leq \frac{1}{2}f\left(\text{ber}^2(|E||E^*|)\right) + \frac{1}{2}\left\|(1-\lambda)f\left(|E|^{\frac{2}{1-\lambda}}\right) + \lambda f\left(|E^*|^{\frac{2}{\lambda}}\right)\right\|_{\text{ber}},$$

and

$$f\left(\text{ber}^2(E)\right) \leq \frac{1}{2}\left(f(\text{ber}(|E||E^*|)) + \frac{1}{2}\left\|f\left(|E|^2\right) + f\left(|E^*|^2\right)\right\|_{\text{ber}}\right).$$

In particular, if $p \geq 1$, then

$$\text{ber}^{4p}(E) \leq \frac{1}{2}\text{ber}^{2p}(|E||E^*|) + \frac{1}{2}\left\|(1-\lambda)|E|^{\frac{2p}{1-\lambda}} + \lambda|E^*|^{\frac{2p}{\lambda}}\right\|_{\text{ber}},$$

and

$$\text{ber}^{2p}(E) \leq \frac{1}{2}\text{ber}^p(|E||E^*|) + \frac{1}{4}\left\||E|^{2p} + |E^*|^{2p}\right\|_{\text{ber}}. \tag{19}$$

Proof. Replacing $A = |E^*|$ and $B = |E|$ in the inequality (10), we get

$$f\left(\left|\left\langle |E|\widehat{k_\xi},\widehat{k_\xi}\right\rangle\left\langle |E^*|\widehat{k_\xi},\widehat{k_\xi}\right\rangle\right|^2\right) \le \frac{f\left(\left|\left\langle |E|\,|E^*|\widehat{k_\xi},\widehat{k_\xi}\right\rangle\right|^2\right)+\left\langle\left\{(1-\lambda)\,f\left(|E|^{\frac{2}{1-\lambda}}\right)+\lambda f\left(|E^*|^{\frac{2}{\lambda}}\right)\right\}\widehat{k_\xi},\widehat{k_\xi}\right\rangle}{2}.$$

Since f is increasing, it follows from inequality (7) that

$$f\left(\left|\left\langle E\widehat{k_\xi},\widehat{k_\xi}\right\rangle\right|^4\right) \le \frac{f\left(\left|\left\langle |E|\,|E^*|\widehat{k_\xi},\widehat{k_\xi}\right\rangle\right|^2\right)+\left\langle\left\{(1-\lambda)\,f\left(|E|^{\frac{2}{1-\lambda}}\right)+\lambda f\left(|E^*|^{\frac{2}{\lambda}}\right)\right\}\widehat{k_\xi},\widehat{k_\xi}\right\rangle}{2}.$$

and

$$\begin{aligned}\sup_{\xi\in\Omega} f\left(\left|\widetilde{E}\,(\xi)\right|^4\right) &\le \frac{1}{2}\sup_{\xi\in\Omega} f\left(\left|\widetilde{|E|\,|E^*|}\,(\xi)\right|^2\right)\\ &+\frac{1}{2}\sup_{\xi\in\Omega}\left\langle\left\{(1-\lambda)\,f\left(|E|^{\frac{2}{1-\lambda}}\right)+\lambda f\left(|E^*|^{\frac{2}{\lambda}}\right)\right\}\widehat{k_\xi},\widehat{k_\xi}\right\rangle.\end{aligned}$$

which is equivalent to

$$f\left(\mathrm{ber}^4\,(E)\right) \le \frac{1}{2} f\left(\mathrm{ber}^2\,(|E|\,|E^*|)\right) + \frac{1}{2}\left\|(1-\lambda)\,f\left(|E|^{\frac{2}{1-\lambda}}\right)+\lambda f\left(|E^*|^{\frac{2}{\lambda}}\right)\right\|_{\mathrm{ber}},$$

and completes the proof of the first inequality of the theorem. By using (11) inequality, the second inequality follows similarly way. The other two inequalities follow by letting $f\,(t) = t^p$, $p \ge 1$. $\square$

The following result will be needed for further investigation.

Proposition 2.12. *If $A \in \mathcal{B}(\mathcal{H}\,(\Omega))$, then for any $p \ge 1$ and $0 \le \lambda \le 1$,*

$$\mathrm{ber}^{2p}\,(|A|\,|A^*|) \le \left\|(1-\lambda)\,|A|^{\frac{2p}{1-\lambda}} + \lambda\,|A^*|^{\frac{2p}{\lambda}}\right\|_{\mathrm{ber}},$$

and

$$\mathrm{ber}^{p}\,(|A|\,|A^*|) \le \frac{1}{2}\left\||A|^{2p} + |A^*|^{2p}\right\|_{\mathrm{ber}}. \tag{20}$$

Proof. Let $\widehat{k_\xi} \in \mathcal{H}$ be a normalized reproducing kernel. We have

$$\begin{aligned}\left|\left\langle |A|\,|A^*|\widehat{k_\xi},\widehat{k_\xi}\right\rangle\right|^{2p} &= \left|\left\langle |A^*|\widehat{k_\xi},|A|\widehat{k_\xi}\right\rangle\right|^{2p}\\ &\le \left\||A^*|\widehat{k_\xi}\right\|^{2p}\left\||A|\widehat{k_\xi}\right\|^{2p}\\ &\quad\text{(by the Cauchy-Schwarz inequality)}\end{aligned}$$

$$\leq \left\langle |A|\widehat{k}_\xi, |A|\widehat{k}_\xi \right\rangle^p \left\langle |A^*|\widehat{k}_\xi, |A^*|\widehat{k}_\xi \right\rangle^p$$
$$\leq \left\langle |A|^2\widehat{k}_\xi, \widehat{k}_\xi \right\rangle^p \left\langle |A^*|^2\widehat{k}_\xi, \widehat{k}_\xi \right\rangle^p$$
$$\leq \left\langle |A|^{2p}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle \left\langle |A^*|^{2p}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle$$

(by the inequality (6))

$$\leq \left\langle \left(|A|^{\frac{2p}{1-\lambda}}\right)^{1-\lambda}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle \left\langle \left(|A^*|^{\frac{2p}{\lambda}}\right)^{\lambda}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle$$
$$\leq \left\langle \left(|A|^{\frac{2p}{1-\lambda}}\right)\widehat{k}_\xi, \widehat{k}_\xi \right\rangle^{1-\lambda} \left\langle \left(|A^*|^{\frac{2p}{\lambda}}\right)\widehat{k}_\xi, \widehat{k}_\xi \right\rangle^{\lambda}$$

(by the inequality (4))

$$\leq (1-\lambda)\left\langle |A|^{\frac{2p}{1-\lambda}}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle + \lambda \left\langle |A^*|^{\frac{2p}{\lambda}}\widehat{k}_\xi, \widehat{k}_\xi \right\rangle$$

(by the inequality (5))

$$\leq \left\langle \left((1-\lambda)|A|^{\frac{2p}{1-\lambda}} + \lambda |A^*|^{\frac{2p}{\lambda}}\right)\widehat{k}_\xi, \widehat{k}_\xi \right\rangle.$$

By taking the supremum over $\xi \in \Omega$ in the above inequality, we have

$$\sup_{\xi\in\Omega} \left|\widetilde{|A||A^*|}(\xi)\right|^{2p} \leq \sup_{\xi\in\Omega} \left\langle \left((1-\lambda)|A|^{\frac{2p}{1-\lambda}} + \lambda |A^*|^{\frac{2p}{\lambda}}\right)\widehat{k}_\xi, \widehat{k}_\xi \right\rangle$$

which clearly implies that

$$\mathrm{ber}^{2p}(|A||A^*|) \leq \left\|(1-\lambda)|A|^{\frac{2p}{1-\lambda}} + \lambda |A^*|^{\frac{2p}{\lambda}}\right\|_{\mathrm{ber}}. \tag{21}$$

Similar arguments implies

$$\left|\widetilde{|A||A^*|}(\xi)\right|^{p} \leq \frac{1}{2}\left\langle \left(|A|^{2p} + |A^*|^{2p}\right)\widehat{k}_\xi, \widehat{k}_\xi \right\rangle,$$

for any $\xi \in \Omega$. By taking supremum over $\lambda \in \Omega$, we have

$$\mathrm{ber}^{p}(|A||A^*|) \leq \frac{1}{2}\left\||A|^{2p} + |A^*|^{2p}\right\|_{\mathrm{ber}}.$$

Hence, we get the desired inequality (20). □

Remark 2.13. *By combining inequalities (19) and (20), we infer that*

$$\mathrm{ber}^{2p}(A) \leq \frac{1}{2}\mathrm{ber}^{p}(|A||A^*|) + \frac{1}{4}\left\||A|^{2p} + |A^*|^{2p}\right\|_{\mathrm{ber}} \leq \frac{1}{2}\left\||A|^{2p} + |A^*|^{2p}\right\|_{\mathrm{ber}}. \tag{22}$$

The inequalities (22) provide a refinement of the inequality (3) (also, [24, Theorem 1]).

Now we are in a position to present our refined Berezin number inequality.

Theorem 2.14. *If $A \in \mathcal{B}(\mathcal{H}(\Omega))$, then*

$$\mathrm{ber}(A) \leq \frac{1}{2}\left(\left\|A^2\right\|_{\mathrm{ber}}^{1/2} + \|A\|_{\mathrm{ber}}\right). \tag{23}$$

Proof. By the inequality (19) and by the AM-GM inequality, we have

$$\left|\left\langle A\widehat{k}_\xi, \widehat{k}_\xi \right\rangle\right| \leq \left\langle |A|\widehat{k}_\xi, \widehat{k}_\xi \right\rangle^{1/2} \left\langle |A^*|\widehat{k}_\xi, \widehat{k}_\xi \right\rangle^{1/2}$$
$$\leq \frac{1}{2}\left(\left\langle |A|\widehat{k}_\xi, \widehat{k}_\xi \right\rangle + \left\langle |A^*|\widehat{k}_\xi, \widehat{k}_\xi \right\rangle\right)$$
$$\leq \frac{1}{2}\left\langle (|A| + |A^*|)\widehat{k}_\xi, \widehat{k}_\xi \right\rangle$$

for every $\xi \in \Omega$. Thus

$$\begin{aligned}\operatorname{ber}(A) &= \sup_{\xi\in\Omega}\left|\widetilde{A}(\lambda)\right| = \sup_{\xi\in\Omega}\left|\left\langle A\widehat{k}_\xi, \widehat{k}_\xi\right\rangle\right| \\ &\leq \frac{1}{2}\sup_{\xi\in\Omega}\left\langle (|A| + |A^*|)\widehat{k}_\xi, \widehat{k}_\xi\right\rangle \\ &\leq \frac{1}{2}\,\||A| + |A^*|\|_{\mathrm{ber}}.\end{aligned} \tag{24}$$

Applying Lemmas 2.4 and 2.5 to the positive operators $|A|$ and $|A^*|$,and using the facts that $\||A|\| = \||A^*|\| = \|A\|$ and $\||A|\,|A^*|\| = \left\|A^2\right\|$, we have

$$\||A| + |A^*|\|_{\mathrm{ber}} \leq \left\|A^2\right\|^{1/2} + \|A\|_{\mathrm{ber}}. \tag{25}$$

The desired inequality (23) now follows from (24) and (25). □

The following result is a consequence of the inequality (23).

Lemma 2.15. *If* $A \in \mathcal{B}(\mathcal{H}(\Omega))$ *is such that* $\operatorname{ber}(A) = \|A\|_{\mathrm{ber}}$, *then* $\left\|A^2\right\|_{\mathrm{ber}} = \|A\|_{\mathrm{ber}}^2$.

Proof. It follows from the inequality (23) that

$$2\operatorname{ber}(A) \leq \left\|A^2\right\|_{\mathrm{ber}}^{1/2} + \|A\|_{\mathrm{ber}}$$

for every $\xi \in \Omega$. Thus, if $\operatorname{ber}(A) = \|A\|_{\mathrm{ber}}$, then $\|A\|_{\mathrm{ber}} \leq \left\|A^2\right\|_{\mathrm{ber}}^{1/2}$, and hence $\|A\|_{\mathrm{ber}}^2 \leq \left\|A^2\right\|_{\mathrm{ber}}$. Also the reverse inequality is always true. Thus $\|A\|_{\mathrm{ber}}^2 = \left\|A^2\right\|_{\mathrm{ber}}$ as required. □

The following another result shows that the inequality (19) provides an improvoment of the inequality (23).

Corollary 2.16. *If* $A \in \mathcal{B}(\mathcal{H}(\Omega))$, *then*

$$\operatorname{ber}(A) \leq \frac{1}{2}\sqrt{2\operatorname{ber}(|A|\,|A^*|) + \left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}}} \leq \frac{1}{2}\left(\left\|A^2\right\|_{\mathrm{ber}}^{1/2} + \|A\|_{\mathrm{ber}}\right).$$

Proof. Let $\widehat{k}_\xi \in \mathcal{H}$ be a normalized reproducing kernel. We get

$$\begin{aligned}\operatorname{ber}(A) &\leq \frac{1}{2}\sqrt{2\operatorname{ber}(|A|\,|A^*|) + \left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}}} \\ &\quad \text{(by the inequality (19))} \\ &\leq \frac{1}{2}\sqrt{2\,\||A|\,|A^*|\| + \left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}}} \\ &\quad \text{(by the inequality in (1))} \\ &\leq \frac{1}{2}\sqrt{2\left\|A^2\right\| + \left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}}} \\ &\leq \frac{1}{2}\sqrt{2\left\|A^2\right\|_{\mathrm{ber}} + \left\|A^2\right\|_{\mathrm{ber}} + \|A\|_{\mathrm{ber}}^2} \\ &\quad \text{(by the inequality (9))} \\ &\leq \frac{1}{2}\sqrt{2\,\|A\|_{\mathrm{ber}}\left\|A^2\right\|_{\mathrm{ber}}^{1/2} + \left\|A^2\right\|_{\mathrm{ber}} + \|A\|_{\mathrm{ber}}^2} \\ &\leq \frac{1}{2}\sqrt{\left(\left\|A^2\right\|_{\mathrm{ber}}^{1/2} + \|A\|_{\mathrm{ber}}\right)^2} \\ &\leq \frac{1}{2}\left(\left\|A^2\right\|_{\mathrm{ber}}^{1/2} + \|A\|_{\mathrm{ber}}\right).\end{aligned}$$

This completes the proof. □

We give the following example which show that $\operatorname{ber}(A) = \max_{1\leq j\leq n} \left|a_{jj}\right|$ for any complex $n \times n$ matrix $A = \left(a_{jk}\right)_{j,k=1}^{n}$.

Example 2.17. *Let us consider the finite dimensional setting.* $A = \left(a_{jk}\right)_{j,k=1}^{n}$ *be a* $n\times n$ *matrix. Let* $v = (v_1, ..., v_n) \in \mathbb{C}^n$ *and* $X = \{1, ..., n\}$. *We can consider* $\mathbb{C}^n$ *as the set of all functions mapping* $X \to \mathbb{C}$ *by* $v(j) = v_j$. *Letting* e_j *be the jth standard basis vector for* $\mathbb{C}^n$ *under the standard inner product, we can view* $\mathbb{C}^n$ *as an RKHS with kernel*

$$k(i, j) = \left\langle e_j, e_i \right\rangle.$$

Note that $k_j = \widehat{k}_j$ *for each* $j = 1, ..., n$. *We have* $a_{jj} = \left\langle Ae_j, e_j\right\rangle$. *Thus, the Berezin set of A is simply*

$$\operatorname{Ber}(A) = \left\{a_{jj} : j = 1, ..., n\right\},$$

which is just the collection of diagonal elements of A. Therefore $\operatorname{ber}(A) = \max_{1\leq j\leq n} \left|a_{jj}\right|$.

Acknowledgement. The authors are very grateful to the referee for his/her useful remarks.

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