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The Topological Riesz Algebras

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Abstract. The multiplicative order convergence was studied and investigated on Riesz algebras. This paper deals with Riesz algebras and different topologies on them. In this paper, we investigate Riesz algebras on which we define various kinds of continuities. We give the relation between them under certain specific conditions. We show some relations among locally full, locally convex and locally solid Riesz algebras. Also, we introduce the notions of order and topological continuity of algebraic multiplications on topological Riesz algebras. Also, we extend the multiplication to quotient spaces of Riesz algebras.

1. Introduction

It is known that the theory of lattice-ordered group is an important class of partially ordered algebraic systems. The concept of lattice-ordered groups was started to study in [13, 18], and it was followed in [22, 27, 38]. A lattice-ordered group with a topology satisfying the continuity of group and lattice operations is called a topological lattice-ordered group or topological *l*-group. That is a generalization of topologies on Riesz spaces. Topological lattice-ordered groups were firstly studied on Riesz spaces by Šmarda [36, 37]. We also refer to the reader for linear topologies to [11, 26, 29, 32, 34].

Another important class of Riesz space is Riesz algebra. A Riesz algebra is an associative algebra that is at the same time a Riesz space such that the partial ordering and the multiplication are compatible. It was introduced by Birkhoff and Pierce [14]. After then, some important works have been done on this concept [12, 15, 16, 24, 25, 33]. In recent years, contributions to the theory of Riesz algebras have been made in [6, 8, 10]. As far as we know, there seems to be no work about the concept of topology on Riesz algebras except Aydın that just constructed the multiplicative norm topology using the norm convergence in [7]. In the present paper, we introduce the concept of topological Riesz algebras with different types of linear topologies. Thus, we hope that this paper will fill this gap.

The structure of the paper is as follows. In Section 2, we recall some notations and terminologies of Riesz algebras and topological Riesz spaces are used in this paper. Section 3 contains the definition of topological Riesz algebras and examples. Theorem 3.3 and Theorem 3.6 give some relations between locally

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full and locally solid Riesz algebras. Also, an extension of the algebraic multiplication to topological and Dedekind completion is shown in Theorem 3.9. In Section 4, we present the notions of topological and order continuity of algebraic multiplication in topological Riesz algebras, and some examples. Moreover, we give some relations between them. The last section is devoted to the multiplicative continuity of Riesz algebras.

2. Notation and preliminaries

Let E be a non-empty set with an order relation (i.e. it is an antisymmetric, reflexive and transitive relation). A set E with an order relation is said to be a *lattice* whenever the supremum $x \vee y = \sup\{x, y\}$ and the infimum $x \wedge y = \inf\{x, y\}$ both exist for each pair of vectors $x, y \in E$.

An ordered set E is called an *ordered group* if $(E, *)$ is a group and $x \leq y$ implies $x * z \leq y * z$ and $z * x \leq z * y$ for all $x, y, z \in E$. A lattice E is called a *lattice ordered group* (an *l-group*, for short) whenever it is at the same time an ordered group.

A topology τ on a group $(E, *)$ is called a *group topology* if the mapping $f : G \times G \to G$ defined by $f(x, y) = x * y^{-1}$ is continuous. Also, an l-group E is said to be a *topological lattice group* (or, topological l-group) if the group and lattice operations are all continuous, i.e., E is a topological l-group whenever it satisfies the following properties:

(1) $(E, *)$ is a topological group;

(2) the mapping $(x, y) \to x \vee y$ is continuous;

(3) the mapping $(x, y) \to x \wedge y$ is continuous.

A real vector space E with an order relation is called an *ordered vector space* if the order relation is compatible with the algebraic structure of E which means that $y \leq x$ implies $y + z \leq x + z$ for all $z \in E$ and $\lambda y \leq \lambda x$ for each $0 \leq \lambda \in \mathbb{R}$. An ordered vector space E is called a *Riesz space* (or, *vector lattice*) whenever it is at the same time a lattice. An element x in a Riesz space E is said to be *positive* whenever $0 \leq x$, and also, the set of all positive elements in E is denoted by E_+. For an element x in a Riesz space E, $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := x \vee (-x)$ are called the *positive part*, *negative part*, and *module* of x, respectively. A subset A of a Riesz space E is called a *solid* if, for each $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. A Riesz space E is said to be *Archimedean* whenever $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E_+$. In this article, unless otherwise stated, all Riesz spaces are assumed to be Archimedean.

Let I be a partially ordered set. Then it is called a *directed set* if, for each $\alpha_1, \alpha_2 \in I$, there is another $\alpha \in I$ such that $\alpha \geq \alpha_1$ and $\alpha \geq \alpha_2$ (or, $\alpha \leq \alpha_1$ and $\alpha \leq \alpha_2$). A function from a directed set I into a set E is called a *net* in E. Let $(x_\alpha)_{\alpha \in A}$ be a net in a Riesz space E. Then it is called *order convergent* (or, shortly, *o-convergent*) to $x \in E$ whenever there is another net $(y_\beta)_{\beta \in B}$ satisfying $y_\beta \downarrow 0$ and, for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. Thus, we write $x_\alpha \xrightarrow{o} x$. We refer the reader for some different types of the order convergence to [2].

A Riesz space E with an associative algebra multiplication (i.e., the multiplicative operation "$\cdot$" from $E \times E$ to E satisfies the properties: $u \cdot (v + w) = u \cdot v + u \cdot w$, $(v + w) \cdot u = v \cdot u + w \cdot u$, $v(\alpha u) = \alpha v \cdot u$, and $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ for all $u, v, w \in E$) is called a *Riesz algebra* (or, for short, an *l-algebra*) if the positive cone E_+ is closed under the algebra multiplication, i.e., $x \cdot y \in E_+$ whenever $x, y \in E_+$. Moreover, if $x \cdot y = y \cdot x$ holds for all $x, y \in E$ then l-algebra E is called *commutative*. In this paper, in order to simplify the presentation, we always suppose l-algebras under consideration to be commutative. An algebra ideal in a Riesz algebra E (i.e., a linear subspace of E which is a two-sided ring ideal) is called an r-ideal. Moreover, an r-ideal which is also an order ideal is said to be an l-ideal (cf. [25, 33]).

A linear topology τ on a vector space E is a topology on E that makes the addition and the scalar multiplication continuous, i.e., the topology τ makes the mappings $+ : E \times E \to E$ defined by $(x, y) \to x + y$ and $\cdot : \mathbb{R} \times E \to E$ defined by $(\lambda, x) \to \lambda x$ continuous. It is well known that the topological convergence of nets on topological vector spaces is linear. Also, for each topological vector space, there is a base $\mathfrak{N}$ consisting of zero neighborhoods and which satisfies the following properties: for each $U \in \mathfrak{N}$, $\lambda U \in \mathfrak{N}$ for all $|\lambda| \leq 1$; for any $U_1, U_2 \in \mathfrak{N}$, there is another zero neighborhood $U \in \mathfrak{N}$ such that $U \subseteq U_1 \cap U_2$; for every $U \in \mathfrak{N}$, there is a zero neighborhood $V \in \mathfrak{N}$ with $V + V \subseteq U$; for each zero neighborhood U in $\mathfrak{N}$ and scalar $\lambda > 0$, we have $\lambda U \in \mathfrak{N}$ (cf. [4]). A topological Riesz space (E, τ) has the Lebesgue property whenever $x_\alpha \xrightarrow{o} x$ implies $x_\alpha \xrightarrow{\tau} x$.

We remind that a subset A in an ordered vector space is said to be a *full set* whenever $[x, y] \subseteq A$ for every $x, y \in A$ with $x \leq y$. A linear topology τ on an ordered vector space is called *locally full* whenever it has a τ-neighborhood base at zero consisting of full neighborhoods (cf. [3, Exer.1, p.72]). A linear topology τ on a Riesz space E is called a *solid topology* if τ has a base that consists of solid sets, and so, (E, τ) is called a *locally solid Riesz space*. It can be seen that topological Riesz spaces may not be locally solid (cf. [28, Exam.2.3]). Similarly, a locally convex topology on a vector space is a linear topology that has a base at zero consisting of convex sets. We refer the reader for unexplained notions and terminology to [3–5, 9, 19, 21, 30, 31, 39, 40].

3. Topological *l*-algebras

The notion of topological *l*-group with a locally solid topology has been introduced on a lattice ordered group (cf. [27, 29]) recently. We prefer an approach based on the following definition which reflects our understanding of this notion.

Definition 3.1. *An l-algebra E with a linear topology τ is called a* topological Riesz algebra, *or* topological *l*-algebra, *for short.*

If the topology is, in particular, a kind of an arbitrary linear topology τ on E then we shall speak that it is a τ topological *l*-algebra.

Example 3.2. *It is well known that every Riesz seminorm is a Riesz pseudonorm (cf. [40, Exer.100.19]) and a family of Riesz pseudonorms generates a locally solid topology (cf. [3, Thm.2.28]). Now, let $\mathcal{N}$ be a nonempty family of Riesz seminorms on a Riesz algebra E. Thus, the topology generated by the family of seminorms $\rho_j(x) := j(|x|)$ for all $j \in \mathcal{N}$ and $x \in E$ is the absolute weak topology $|\sigma|(E, \mathcal{N})$ on E relative to $\mathcal{N}$. Therefore, the topology $\sigma(E, \mathcal{G})$ is locally convex, solid, and full (cf. [3, 4]), where $\mathcal{G} = \{\rho_j : j \in \mathcal{N}\}$. Thus, E with the absolute weak topology is a locally convex, solid, and full l-algebra.*

Theorem 3.3. *Every locally solid l-algebra is a locally full l-algebra.*

Proof. Suppose that (E, τ) is a locally solid *l*-algebra. It is enough to show that τ is a locally full topology. Consider the solid base at zero $\mathcal{N}$ consisting of solid neighborhoods. Then, for a fixed $U \in \mathcal{N}$, there exists another zero neighborhood solid set $V \in \mathcal{N}$ such that $V + V \subseteq U$. Take the full hull $W := \cup\{[x, y] : x, y \in V\}$ of V. It is clear that $V \subseteq W$ and W is a zero neighborhood set. Now, for an arbitrary $w \in W$, there are some elements $x, y \in V$ such that $x \leq w \leq y$, and so, we have $|w| \leq |x| + |y| \in V + V \subseteq U$. Hence, we obtain that $V \subset W \subseteq U$. Therefore, τ is a locally full topology. □

Remark 3.4. *(i) It follows from Theorem 3.3 and [3, Thm.2.25] that if the topology τ of a topological l-algebra (E, τ) is generated by a family of Riesz seminorms then E is a locally convex, solid, and full l-algebra.*

(ii) It follows from [10, Lem.2.1] that a topological l-algebra (E, τ) is a locally full l-algebra if and only if $x_\alpha \xrightarrow{\tau} 0$ and $z_\alpha \xrightarrow{\tau} 0$ imply $y_\alpha \xrightarrow{\tau} 0$ for all nets $(x_\alpha)_{\alpha \in A}$, $(y_\alpha)_{\alpha \in A}$ and $(z_\alpha)_{\alpha \in A}$ in E with $x_\alpha \leq y_\alpha \leq z_\alpha$ for all $\alpha \in A$.

The converse of Theorem 3.3 does not hold in general. It means that a locally full *l*-algebra need not to be a locally solid *l*-algebra. To see this, we consider [10, Exam.3].

Example 3.5. *Take the Riesz algebra $E := L_\infty[0, 1]$ with the pointwise multiplication. Then it follows from Theorem 3.6 that the weak topology ω on E is a locally full, but not locally solid because $x_n \xrightarrow{\omega} 0$ yet $|x_n| \xrightarrow{\omega} 1$ for a sequence $x_n \in X$ of Rademacher's functions on $[0, 1]$.*

By considering [32, Thm.8.1], we show the following result.

Theorem 3.6. *A locally full l-algebra is a locally solid l-algebra if and only if the lattice operations are topological continuous.*

Proof. Assume that the topology τ in a locally full l-algebra is locally solid. Then we show that $x \to |x|$ is continuous. All other continuities of lattice operations are analogous. Take an element $x \in E$ and a neighborhood U such that $|x| \in U$. Then there is a solid neighborhood of zero set V such that $|x| + V \subset U$. Choose $W := x + V$ as a neighborhood of x. Note that for every $w \in W$, $w - x \in V$ and $||w| - |x|| \leq |w - x|$. Since V is a solid set, $|w| - |x| \in V$. Thus, we obtain the desired result from $|w| \in |x| + V \subseteq U$.

For the converse, suppose that (E, τ) is a locally full l-algebra and the lattice operations are τ-continuous in E. Take a full neighborhood of zero set V. Then there exists another full neighborhood of zero U such that $\{x^+ : x \in U\} \subseteq V$ since the operation $x \to x^+$ is continuous. It is clear that U is a subset of its solid hull $Sol(U) = \{x \in E : \exists u \in U, |x| \leq |u|\}$, and also, $Sol(U)$ is a solid neighborhood of zero. On the other hand, for each $u \in U$, we have $u^+ \in V$, and so, one can see that $x \in V$ for all $x \in E$ with $|x| \leq u^+$ because V is a full set. Therefore, we obtain that $Sol(U) \subseteq V$. Thus, it follows that τ is also a locally solid topology. □

For a relation between solid and full topologies on l-algebras, we observe the following example.

Example 3.7. *Let E be an l-algebra with a locally full topology τ. Since the full topological convergence is linear, the multiplicative τ-convergence on E (i.e., $x_\alpha \xrightarrow{m\tau} x$ whenever $u \cdot |x_\alpha - x| \xrightarrow{\tau} 0$ for all $u \in E_+$) is topological with respect to the locally solid topology τ_m on E; see [10, Thm.17]. Then (E, τ_m) is a locally solid l-algebra.*

We give the following example to see that a topological l-algebra does not need to be locally full.

Example 3.8. *Consider the Riesz space E consisting of all eventually zero real sequences with coordinatewise ordering and algebra multiplication. Then E is a Riesz algebra. Take a norm q on E such that $q(x)$ is the sum of all $\frac{1}{n}|x_n - x_{n+1}|$ for all n. Then E with the topology generated by the norm q is not a locally full l-algebra because q is not a monotone norm.*

Now, we turn our attention to the completion of locally solid l-algebras. It is known that every Hausdorff topological vector space (E, τ) has a unique, up to a topological and algebraic isomorphism, Hausdorff topological completion $(\hat{E}, \hat{\tau})$ (cf. [1, p.1]).

Theorem 3.9. *Let (E, τ) be a locally solid l-algebra. Then we have:*

(i) If every order bounded increasing sequence in E is a τ-Cauchy sequence and τ is a Hausdorff topology then the topological completion $(\hat{E}, \hat{\tau})$ of (E, τ) is also a Hausdorff locally solid l-algebra.

(ii) If E is a Dedekind complete locally solid l-algebra and E is an order ideal in the topological completion $(\hat{E}, \hat{\tau})$ of (E, τ) then $(\hat{E}, \hat{\tau})$ is a locally solid l-algebra.

Proof. (*i*) It follows from [3, Thm.2.40] that the topological completion $(\hat{E}, \hat{\tau})$ of (E, τ) exists, and also, it is a Hausdorff locally solid Riesz space. By applying [1, Thm.3.1(ii)], one can obtain that the topological completion $\hat{E}$ is a Dedekind complete Riesz space. Now, by considering the same as above item in [33], we show that $\hat{E}$ is the l-algebra with an extension of the algebra multiplication "$*$" in E to the multiplication "$*$" in $\hat{E}$. Indeed, take $\hat{x}, \hat{y} \in \hat{E}$. Then there exist $x, y \in E_+$ such that $0 \leq \hat{x} \leq x$ and $0 \leq \hat{y} \leq y$ because E is a majorizing set in its Dedekind completion $\hat{E}$. Moreover, define $\hat{x} = \sup\{w \in E_+ : 0 \leq w \leq \hat{x}\} \leq x$ and $\hat{y} = \sup\{z \in E_+ : 0 \leq z \leq \hat{y}\} \leq y$. From [33, Prop3.2(ii)], we have $0 \leq w * z \leq x * y$ for all $0 \leq w, z \in E$ with $0 \leq w \leq \hat{x}$ and $0 \leq z \leq \hat{y}$. Then there exists the element

$$\hat{u} := \sup\{w * z : w, z \in E_+, w \leq \hat{x}, z \leq \hat{y}\}.$$

Thus, we define $\hat{u} = \hat{x} * \hat{v}$ in $\hat{E}$. Moreover, it can be seen that the multiplication makes $\hat{E}$ an l-algebra and extends the original multiplication in E to $\hat{E}$. Also, one can show that the extension "$*$"to $\hat{E}$ is unique.

(*ii*) From [1, Thm.2.6(ii)], the topological completion $(\hat{E}, \hat{\tau})$ of (E, τ) is Dedekind complete because E is Dedekind complete. By using the fact that E is an order ideal in $\hat{E}$ and the same argument in the proof of (*i*), we get the desired result. □

Problem 3.10. *Is it possible to extend the algebraic multiplication to the topological completion without condition of Dedekind completeness?*

We now turn our attention to quotient algebras. Let I be an order ideal in a Riesz space E. Then the set $E/I = \{x+I : x \in E\}$ is a Riesz space with respect to the order relation $[x] \leq [y]$ whenever there exist elements $u_x \in [x]$ and $v_y \in [y]$ satisfying $u_x \leq v_y$ (cf.[5, Thm.2.22]), where we denote $x+I$ with $[x]$. Then the Riesz space E/I is called the quotient space of E with respect to the ideal I. On the other hand, we remind that an r-ideal which is also an order ideal is said to be an l-ideal.

Theorem 3.11. *Let E be a locally solid Riesz algebra and I be an l-ideal in E. Then E/I is a locally solid Riesz algebra.*

Proof. Suppose that $(E, \tau, *)$ is a locally solid l-algebra. We define a mapping "$\circ$" on E/I denoted by $[x] \circ [y] := [x * y]$. Then it is a binary operation on E/I. Indeed, assume that $([x], [y]) = ([a], [b])$ holds in $E/I \times E/I$. Then we have $x - a \in I$ and $y - b \in I$ (cf. [30, Thm.27.2]). Then it follows that

$$\begin{aligned} [x] \circ [y] &= [x * y] = x * y + I = (x - a + a) * (y - b + b) + I \\ &= (x - a) * (y - b) + (x - a) * b + a * (y - b) + a * b + I \\ &= a * b + I \\ &= [a] \circ [b]. \end{aligned}$$

So the mapping "$\circ$" is well defined, and so, it is a function. Next, we show that $(E/I, \circ)$ is an algebra. For arbitrary $x, y, z \in E$ and $\lambda \in \mathbb{R}$, we observe the following facts:

(1) $[x] \circ ([y] + [z]) = [x] \circ ([y + z]) = [x * (y + z)] = [x * y + x * z] = [x * y] + [x * z] = [x] \circ [y] + [x] \circ [z]$;
(2) $[x] \circ (\lambda[z]) = [x] \circ [\lambda z] = [x * (\lambda z)] = \lambda[x * y] = \lambda([x] \circ [z])$;
(3) $[x] \circ ([y] \circ [z]) = [x] \circ ([y * z]) = [x * (y * z)] = [(x * y) * z] = ([x] \circ [y]) \circ [z])$.

Therefore, $(E/I, \circ)$ is an associative algebra. Now, take a pair of positive elements $0 \leq [x], [y] \in E/I$. Then there exist $a_x \in [x], b_y \in [y]$ and $u_x, v_y \in [0]$ such that $u_x \leq a_x$ and $v_y \leq b_y$. Hence, it follows from $0 \leq a_x - u_x$ and $0 \leq b_y - v_y$ that

$$(a_x - u_x) * (b_y - v_y) = a_x * b_y - a_x * v_y - u_x * b_y + u_x * v_y \geq 0$$

because "$*$" is an algebraic Riesz multiplication. Then, $[0] \leq [a_x * b_y - a_x * v_y - u_x * b_y + u_x * v_y] = [a_x * b_y]$ because I is an l-ideal. Thus, following from the order relation on E/I, we obtain $[x] \circ [y] = [x * y] \geq [0]$. As a result, E/I is an l-algebra. It follows from [3, Thm.2.24] that E/I is a locally solid l-algebra. □

4. The continuity of multiplications

Definition 4.1. *Let (E, τ) be a topological l-algebra with the algebraic multiplication "$*$". Then "$*$"is called*

(1) a topological continuous multiplication *if the mapping $x \to x * y$ is topological continuous on E for all $x, y \in E$;*
(2) an order continuous multiplication *if the mapping $x \to x * y$ is order continuous on E for all $x, y \in E$.*

Both continuities are not well-matched in topological l-algebras. To see this, we consider [10, Exam.6].

Example 4.2. *Let U be an ultra filter on $\mathbb{N}$. Define the multiplication "$*$" on ℓ_∞ by $x * y := (\lim_{\mathcal{U}} x_n) \cdot (\lim_{\mathcal{U}} y_n) \cdot \mathbb{1}$, where $\mathbb{1}$ is a sequence of reals identically equal to 1, and $\lim_{\mathcal{U}}$ is the limit of real sequences with respect to the convergence along $\mathcal{U}$. Thus, Banach lattice ℓ_∞ is a topological l-algebra, and the algebra multiplication "$*$" in ℓ_∞ is topological continuous, but not order continuous.*

All multiplications in topological l-algebras do not need to be topological continuous.

Example 4.3. *Take the l-algebra $X = (\ell_\infty, *)$ from Example 4.2 with the locally solid topology inherited from the Tychonoff topology τ on $X^u = s = \mathbb{R}^{\mathbb{N}}$. Then the multiplication "$*$" in the locally solid l-algebra $(\ell_\infty, *, \tau)$ is not topological continuous.*

Remark 4.4.

(i) *An order bounded operation π on a Riesz space is called an* orthomorphism *whenever it follows from $x \perp y$ that $\pi(x) \perp y$. Let E be a Banach lattice. Thus, it follows from Remark 3.4(i), [5, Thm.4.77] and [40, Thm.140.9] that $Orth(E)$ is a locally solid, convex Riesz algebra with an order continuous multiplication.*

(ii) *An l-algebra E is called an f-*algebra *if $x \wedge y = 0$ implies $(u \cdot x) \wedge y = (x \cdot u) \wedge y = 0$ for all $u \in E_+$. Every algebraic multiplication in any topological f-algebra is order continuous (cf. [24, p.57]).*

Recall that the topological convergence on metric spaces and the relatively uniform convergence on Riesz spaces are sequential. But, not all topological spaces are sequential. It is well know that a topology τ in any topological space X is sequential if and only if, for every $Y \subseteq X$ and for every $x \in cl_\tau(Y)$, there exists a sequence $x_n \xrightarrow{\tau} x$ in Y. We remind the following classical example of not sequential topological spaces.

Example 4.5. *Take the order topology on the ordinal $w_1 + 1 = [0, w_1]$. Then every sequence of countable ordinals has a countable supremum because w_1 has cofinality w_1. So, w_1 is not open as w_1 is a limit ordinal. So the order topology on $[0, w_1]$ is not sequential.*

A standard theorem in metric spaces states that the sequential continuity is equivalent to the topological continuity. This is not true in arbitrary topological spaces. However, we have the following standard fact from the point-set topology.

Assertion 4.6. *A mapping $T : X \to Y$ is continuous between topological spaces if and only if $x_\alpha \xrightarrow{\tau} x$ implies $T(x_\alpha) \xrightarrow{\tau} T(x)$ for every net $(x_\alpha)_{\alpha \in A}$ in X.*

It can be seen from Assertion 4.6 that the algebra multiplication in a locally solid l-algebra E is topological continuous if and only if $x_\alpha \xrightarrow{\tau} x$ implies $x_\alpha \cdot y \xrightarrow{\tau} x \cdot y$ for each net $(x_\alpha)_{\alpha \in A}$ and every x, y in E. One can give the same argument for the order continuity of the algebraic multiplication. It follows from [10, Exam.11] that the algebra multiplication in any universally complete l-algebra does not need to be topological continuous. Also, we can observe the following example.

Example 4.7. *Consider a Riesz space E consisting of all eventually zero real sequences with the coordinatewise ordering and algebra multiplication. Then E is a Riesz algebra. Since E is a subset of ℓ_∞, one can take the topology τ on E as the supremum norm topology. Then it follows from [3, Thm.2.17] that the lattice operations are continuous because τ is a solid topology. Then the algebra multiplication on E is topological continuous.*

It is well known that the order convergence could be easily not topological (cf. [19, 23]). Thus, the topological convergence of nets does not agree with the order convergence. However, it was proved that the order convergence of nets in a Riesz space is topological if and only if the Riesz space is finite dimensional in [20].

Definition 4.8. *An algebraic multiplication "$*$" in any topological l-algebra (E, τ) is said to have the* multiplicative Lebesgue property *whenever it follows $x_\alpha \downarrow x$ that $x_\alpha * y \xrightarrow{\tau} x * y$ for all $y \in E$.*

Theorem 4.9. *Let E be a topological l-algebra with a Hausdorff locally solid topology. Then the Lebesgue property of multiplication implies the order continuity of it in E.*

Proof. Assume that the multiplication in a Hausdorff locally solid l-algebra (E, τ) satisfies the multiplicative Lebesgue property. Then take a net $x_\alpha \downarrow x$ and an arbitrary element y in E. Thus, we have $x_\alpha * y \xrightarrow{\tau} x * y$ for all $y \in E$ because of the multiplicative Lebesgue property. Without loss of generality, we assume that $y \in E_+$. Since the net $(x_\alpha)_{\alpha \in A}$ is decreasing, we have $(x_\alpha * y)_{\alpha \in A} \downarrow$. Thus, for an arbitrary index α_0, we have $0 \leq x_{\alpha_0} * y - x_\alpha * y$ for all $\alpha \geq \alpha_0$. Then $x_{\alpha_0} * y - x_\alpha * y \xrightarrow{\tau} x_{\alpha_0} * y - x * y \geq 0$ because τ is a Hausdorff solid topology. Hence $x * y$ is a lower bound of $(x_\alpha * y)_{\alpha \in A}$. Take another lower bound w of $(x_\alpha * y)_{\alpha \in A}$, and so, we have $x_\alpha * y - w \xrightarrow{\tau} x * y - w \geq 0$. Therefore, we obtain $w \leq x * y$. This shows that $x_\alpha * y \downarrow x * y$, and so, the multiplication is order continuous. □

We observe the following straightforward fact.

Proposition 4.10. *Let (E, τ) be a topological l-algebra with the Lebesgue property. Then the order continuity of the algebraic multiplication implies the topological continuity.*

Recall that an element x in an l-algebra is called *nilpotent* if there exists a natural number $n \in \mathbb{N}$ such that $x^n = 0$ (cf. [33, 40]). We denote the set of all nilpotent elements in E by $N(E)$. Thus, we complete the section with the following result.

Proposition 4.11. *Let E be a locally solid f-algebra. Then the algebra multiplication is topological continuous on the set $N(E)$.*

Proof. Let $(x_\alpha)_{\alpha\in A}$ be a net in $N(E)$ such that $x_\alpha \xrightarrow{\tau} x \in E$. Fix an element $y \in E$. Then it follows from [33, Thm.10.2(iii)] that

$$y \cdot (x_\alpha - x) = y \cdot x_\alpha - y \cdot x = y \cdot x \xrightarrow{\tau} 0.$$

Therefore, we get the desired result. □

5. The multiplicative continuous Riesz algebras

Definition 5.1. *Let (E, τ) be a locally solid l-algebra. Then E is said to be a* multiplicative continuous Riesz algebra*, or an mc-algebra, for short, if the following conditions hold:*

(1) the mapping $(x, y) \to x * y$ from $E \times E$ to E is continuous;
(2) the lattice operations are continuous.

An *mc*-algebra is a pair (E, τ) of an l-algebra E and a locally solid topology τ on E. It is clear that the structure of a topological Riesz algebra is richer than a topological l-group. So, we expect to achieve stronger results.

Remark 5.2. *Every mc-algebra is also a topological lattice group because addition and scalar multiplication operations are continuous on Riesz algebras.*

Proposition 5.3. *Let (E, τ) be a locally full topological Riesz algebra. Then (E, τ) is an mc-algebra in each of the following cases for an arbitrary net $(x_\alpha)_{\alpha\in A}$ in E:*

(i) $x_\alpha \xrightarrow{\tau} x$ implies $|x_\alpha| \xrightarrow{\tau} |x|$;
(ii) $x_\alpha \xrightarrow{\tau} x$ implies $(x_\alpha)^+ \xrightarrow{\tau} x^+$;
(iii) $x_\alpha \xrightarrow{\tau} x$ implies $(x_\alpha)^- \xrightarrow{\tau} x^-$.

It follows from [32, Thm.8.1] that the properties $(i)-(iii)$ of Proposition 5.3 are equivalent. Also, it follows from [5, Thm.1.3] and the linearity of the solid topologies that the continuity of the infimum and supremum operations are equivalent. Hence, by using Theorem 3.6, one can omit the continuity requirements of the lattice operations in Definition 5.1. Recall that a net $(x_\alpha)_{\alpha\in A}$ is said to be *eventually order bounded* in a Riesz space E if we can find an index $\alpha_0 \in A$ such that the set $(x_\alpha)_{\alpha\geq\alpha_0}$ is order bounded in E. Chuchaev proved that the topological convergence agrees with the order convergence of eventually topologically bounded nets; see [17].

Theorem 5.4. *Let E be an l-algebra with the τ-continuous multiplication. If every net in E is eventually order bounded then E is an mc-algebra.*

Proof. Assume that $(x_\alpha)_{\alpha\in A} \xrightarrow{\tau} x$ and $(y_\beta)_{\beta\in B} \xrightarrow{\tau} y$ hold in E. Also, without loss of generality, suppose that there exist an index α_0 and an element $w \in E$ such that $|x_\alpha| \leq w$ for all $\alpha \geq \alpha_0$. Following from the inequality $|x_\alpha \cdot y_\beta - x \cdot y| \leq |x_\alpha| \cdot |y_\beta - y| + |y| \cdot |x_\alpha - x|$, we have

$$|x_\alpha \cdot y_\beta - x \cdot y| \leq w \cdot |y_\beta - y| + |y| \cdot |x_\alpha - x|$$

for all $\alpha \in A$, $\alpha \geq \alpha_0$, $\beta \in B$. Now, one can obtain $(w \cdot |y_\beta - y|)_\beta \xrightarrow{\tau} 0$ and $(|y| \cdot |x_\alpha - x|)_{\alpha\geq\alpha_0} \xrightarrow{\tau} 0$ by using Theorem 3.6 and the τ-continuity of the multiplication in E. Thus, we get $(x_\alpha \cdot y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{\tau} x \cdot y$. □

It follows from [3, Thm.2.19] that every order bounded subset in a locally solid Riesz space is topological bounded. But, the converse does not hold in general (cf. [28, Exam.2.2]). Every topological bounded subset is order bounded in a locally solid Riesz space whenever it has an order bounded zero neighborhood (cf. [28, Thm.2.2]). Thus, we obtain the following result.

Corollary 5.5. *s If every τ-convergent net is eventually τ-bounded in an l-algebra E with the τ-continuous multiplication and an order bounded zero neighborhood then it is an mc-algebra.*

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