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Zero-Dimensionality and Hausdorffness in Quantale-Valued Preordered Spaces

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Abstract. In this paper, we study the category of quantale-valued preordered spaces. We show that it is a normalized topological category and give characterization of zero-dimensionality and D-connectedness in the category of quantale-valued preordered spaces. Moreover, we characterize explicitly each of $\overline{T_0}$, T_0, T_1, pre-$\overline{T_2}$, $\overline{T_2}$ and NT_2 quantale-valued preordered spaces. Finally, we examine how these characterization are related to each other and show that the full subcategory $\mathbf{T_i}$(**pre-T$_2$**($\mathcal{L}$**-Prord**)) ($i = 0, 1, 2$) of **pre-T$_2$**($\mathcal{L}$**-Prord**), and the full subcategory $\mathbf{T_i}$ ($\mathcal{L}$**-Prord**) ($i = 1, 2$) of $\mathcal{L}$**-Prord** are isomorphic.

1. Introduction

Order theory is a branch of mathematics that deals with many kinds of binary relations. These binary relations apprehend the intuitive concept of mathematical ordering which covers the field of mathematics and its related areas like computer science (see, [15, 34–36]). Domain theory is an interface between mathematics and computer science and it is a fast-growing branch and deals with special kinds of partially ordered sets that are commonly known as domains. Thus, domain theory can be considered a branch of order theory. These domains were firstly studied by Dana Scott in the 1960s. The primary motivation for the study of domains was the search of denotational semantics of lambda calculus, especially for functional programming languages in computer science (see, [34–36]).

In 1921, zero-dimensional spaces were defined by Sierpinski. A topological space (X, τ) is called zero-dimensional provided that X has a basis consisting of clopen sets [18] and it has been used to construct many useful classes of topological spaces (see [14, 23]). This notion has been extended to an arbitrary topological category by Stine [37, 38].

Classical separation axioms of topology have been extended to topological category by several authors [12, 29]. In 1991, Baran [3] introduced these axioms in a set-based topological category in terms of initial, final structures and discreteness. Also, he introduced pre-Hausdorff objects in an arbitrary topological category [3, 10] which are reduced to pre-Hausdorff topological space (X, τ), where (X, τ) is called pre-Hausdorff provided that for any two distinct points, if there is a neighborhood of one missing the other, then the two points have disjoint neighborhoods. One of the important use of pre-Hausdorff objects is to define several different forms of Hausdorff [6], regular, normal and completely regular objects [8] in

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arbitrary topological categories. In 1994, M. V. Mielke [30] proved that the pre-Hausdorff objects play an important role in the theory of geometric realization.

With the advancement of fuzzy theory, distinct mathematical frameworks have been acquainted with fuzzy structures including fuzzy topology [28], quantale-valued approach space [20, 21], quantale-valued metric space [22], fuzzy convergence space [19, 31] and fuzzy closure space [27]. Considering the fuzzy counterparts of ordered structures, it has been generalized by introducing some suitable quantales on ordered structures [13, 16, 28, 39] and several interesting results have been found. This motivates us to consider zero-dimensionality and separation axioms of topological category in quantale-valued preordered spaces.

The aims of this paper are stated as under:

(i) to give the characterization of zero-dimensional and D-connected quantale-valued preordered spaces,

(ii) to characterize each of $\overline{T_0}$, T_0, T_1, pre-$\overline{T_2}$, $\overline{T_2}$ and NT_2 objects in the category of quantale-valued preordered spaces,

(iii) to examine how these characterization are related,

(iv) to show that the full subcategory $\mathbf{T_i}$(**pre-T$_2$**($\mathcal{L}$**-Prord**)) ($i = 0, 1, 2$) of **pre-T$_2$**($\mathcal{L}$**-Prord**), and the full subcategory $\mathbf{T_i}$ ($\mathcal{L}$**-Prord**) ($i = 1, 2$) of $\mathcal{L}$**-Prord** are isomorphic.

2. Preliminaries

Recall, [21] that a partially ordered set or poset $(L, \leq)$ is called a complete lattice if all subsets of L have both supremum ($\bigvee$) and infimum ($\bigwedge$). For any complete lattice, the top element and bottom element is denoted by $\top$ and $\bot$, respectively.

In any complete lattice $(L, \leq)$, we define the *well-below relation*, $\alpha \lhd \beta$ if for all subsets $A \subseteq L$ such that $\beta \leq \bigvee A$ there is $\delta \in A$ such that $\alpha \leq \delta$. Similarly, we define the *well-above relation*, $\alpha \prec \beta$ if for all subsets $A \subseteq L$ such that $\bigwedge A \leq \alpha$ there exists $\delta \in A$ such that $\delta \leq \beta$. Furthermore, a complete lattice $(L, \leq)$ is called a *completely distributive lattice* provided that we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$.

The triple $(L, \leq, *)$ is called a *quantale* if $(L, *)$ is a semi group, and the operation $*$ satisfies the following properties: for all $\alpha_i, \beta \in L$, $(\bigvee_{i\in I} \alpha_i) * \beta = \bigvee_{i\in I}(\alpha_i * \beta)$ and $\beta * (\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I}(\beta * \alpha_i)$ and $(L, \leq)$ is a complete lattice.

A quantale $(L, \leq, *)$ is called *commutative* if $(L, *)$ is a commutative semi group and it is called *integral* if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$.

Note that we denote a quantale by $\mathcal{L} = (L, \leq, *)$ if it is commutative and integral where $(L, \leq)$ is completely distributive.

A quantale $\mathcal{L} = (L, \leq, *)$ is called a *value quantale* if $(L, \leq)$ is completely distributive lattice such that $\forall \alpha, \beta \lhd \top$, $\alpha \vee \beta \lhd \top$ [16].

Definition 2.1. ([22, 39]) Let X be a nonempty set. A map $\mathcal{R} : X \times X \longrightarrow \mathcal{L} = (L, \leq, *)$ is called an $\mathcal{L}$-preordered relation on X if it satisfies (i) for all $x \in X$, $\mathcal{R}(x, x) = \top$ (Reflexivity), and (ii) for all $x, y, z \in X$, $\mathcal{R}(x, y) * \mathcal{R}(y, z) \leq \mathcal{R}(x, z)$ (transitivity). The pair $(X, \mathcal{R})$ is called an $\mathcal{L}$-preordered space.

Note that an $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is called an $\mathcal{L}$-equivalence space $(X, \mathcal{R})$ if for all $x, y \in X$, $\mathcal{R}(x, y) = \mathcal{R}(y, x)$ (symmetric property). Also, $(X, \mathcal{R})$ is called separated $\mathcal{L}$-preordered space if $x = y$ whenever $\mathcal{R}(x, y) = \top$.

A map $f : (X, \mathcal{R}_X) \longrightarrow (Y, \mathcal{R}_Y)$ is called an $\mathcal{L}$-order preserving map if $\mathcal{R}_X(x_1, x_2) \leq \mathcal{R}_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

Let $\mathcal{L}$**-Prord** denotes the category whose objects are $\mathcal{L}$-preordered spaces and morphisms are $\mathcal{L}$-order preserving mappings.

Example 2.2. (i) For $\mathcal{L} = 2 = (\{0, 1\}, \leq, \wedge)$, **2-Prord** $\cong$ **Prord**, where **Prord** is the category of preordered sets and monotone maps.

(ii) For $\mathcal{L} = ([0,\infty], \geq, +)$ (Lawvere's quantale), $[0,\infty]$**-Prord** $\cong \infty$**qMet**, where ∞**qMet** is the category of extended quasi metric spaces and non-expansive maps.

(iii) For $\mathcal{L} = (\triangle^+, \leq, *)$ (distance distribution functions quantale defined in [21]), then $\triangle^+$**-Prord** $\cong$ **ProbqMet**, where **ProbqMet** is the category of probabilistic quasi metric spaces and non-expansive maps [16].

Note that in some literature, $\mathcal{L}$-preordered space is often called a continuity space if $\mathcal{L}$ is a value quantale (see [16]), an $\mathcal{L}$-metric space (see [22]) and an $\mathcal{L}$-category (see [17]).

Recall, [1] a functor $\mathcal{U} : \mathcal{C} \to$ **Set** (the category of sets and functions) is called topological if (*i*) $\mathcal{U}$ is concrete (i.e., faithful and amnestic) (*ii*) $\mathcal{U}$ consists of small fibers and (*iii*) every $\mathcal{U}$-source has a unique initial lift, i.e., if for every source $(f_i : X \to (X_i, \zeta_i))_{i\in I}$ there exists a unique structure ζ on X such that $g : (Y, \eta) \to (X, \zeta)$ is a morphism iff for each $i \in I$, $f_i \circ g : (Y, \eta) \to (X_i, \zeta_i)$ is a morphism. Moreover, a topological functor is called a discrete (resp. indiscrete) if it has a left (resp. right) adjoint. In addition, a functor is called a normalized topological functor if constant objects, i.e., subterminals, have a unique structure.

3. $\mathcal{L}$-Prord as a Normalized Topological Category

Note that the forgetful functor $\mathcal{U} : \mathcal{L}$**-Prord** $\to$ **Set** is a topological (see [17]), that are defined as follows.

Lemma 3.1. ([17]) *Let $(X_i, \mathcal{R}_i)$ be a collection of $\mathcal{L}$-preordered spaces. A source $(f_i : (X, \mathcal{R}) \to (X_i, \mathcal{R}_i))_{i\in I}$ is initial in $\mathcal{L}$-**Prord** iff for all $x, y \in X$,*

$$\mathcal{R}(x, y) = \bigwedge_{i\in I} \mathcal{R}_i(f_i(x), f_i(y)).$$

Lemma 3.2. ([17]) *Let X be a non-empty set and $(X, \mathcal{R})$ be an $\mathcal{L}$-preordered space. For all $x, y \in X$,*

(i) The discrete $\mathcal{L}$-preordered structure on X is given by

$$\mathcal{R}_{dis}(x, y) = \begin{cases} \top, & x = y, \\ \bot, & x \neq y. \end{cases}$$

(ii) The indiscrete $\mathcal{L}$-preordered structure on X is given by

$$\mathcal{R}_{ind}(x, y) = \top.$$

Remark 3.3. The topological functor $\mathcal{U} : \mathcal{L}$**-Prord** $\to$ **Set** is a normalized since a unique $\mathcal{L}$-preordered structure exists on $X = \emptyset$ or $X = \{x\}$, where $X \in Obj(\mathcal{L}$**-Prord**$)$.

4. Zero-Dimensional and D-Connected $\mathcal{L}$-Preordered Spaces

Recall that a topological space (X, τ) is called zero-dimensional provided that X has a basis consisting of clopen sets. In 1997, Stine [37] examined that a topological space (X, τ) is zero-dimensional provided that for all $i \in I$, there exists a family of functions $f_i : (X, \tau) \to (X_i, \tau_{i_{dis}})$ such that τ is the topology induced (i.e., initial topology) by $(X_i, \tau_{i_{dis}})$ via f_i, where $(X_i, \tau_{i_{dis}})$ is the family of discrete topological spaces. Considering the categorical counterparts, we have the following definition given in [38].

Definition 4.1. (cf. [38]) Let $\mathcal{U} : \mathcal{C} \to \mathcal{E}$ be a topological and $\mathcal{D} : \mathcal{E} \to \mathcal{C}$ be a discrete functor. Any object $X \in Obj(\mathcal{C})$ is called a zero-dimensional object provided that for all $i \in I$, there exists $A_i \in Obj(\mathcal{E})$ and morphisms $f_i : \mathcal{U}(X) \to A_i$ such that $(\overline{f}_i : X \to \mathcal{D}(A_i))_{i\in I}$ is the initial lift of $(f_i : \mathcal{U}(X) \to \mathcal{U}(\mathcal{D}(A_i)) = A_i)_{i\in I}$.

Remark 4.2. (i) For $\mathcal{C}$ = **Top** (category of topological spaces and continuous maps) and $\mathcal{E}$ = **Set**, By Theorem 4.3.1 of [37], Definition 4.1 reduces to usual zero-dimensional topological space.

(ii) For $\mathcal{C}$ = **Prord** and $\mathcal{E}$ = **Set**, by Theorem 5.3 of [11], (X, R) is zero-dimensional iff R is an equivalence relation on X.

(iii) If $\mathcal{U} : \mathcal{C} \to \mathcal{E}$ is a normalized topological functor, by Theorems 4.3.4 and 5.3.1 of [37], then every indiscrete object in $\mathcal{C}$ is a zero-dimensional object.

Theorem 4.3. *Every $\mathcal{L}$-preordered space $(X, \mathcal{R})$ with $card(X) = 1$ is zero-dimensional.*

Proof. Let $(X, \mathcal{R})$ be an $\mathcal{L}$-preordered space and $X = \{x\}$. Then, $\mathcal{R} = \mathcal{R}_{dis} = \mathcal{R}_{ind}$. By Remarks 3.3 and 4.2 (iii), $(X, \mathcal{R})$ is zero-dimensional. □

Theorem 4.4. *Let $(X, \mathcal{R})$ be an $\mathcal{L}$-preordered space with $card(X) \geq 2$ and $(X_i, \mathcal{R}_{i_{dis}})$ be discrete $\mathcal{L}$-preordered space for $i \in I$. $(X, \mathcal{R})$ is zero-dimensional if and only if there exists $f_i \colon (X, \mathcal{R}) \to (X_i, \mathcal{R}_{i_{dis}})$ such that for all $x, y \in X$,*

$$\mathcal{R}(x, y) = \begin{cases} \top, & f_i(x) = f_i(y), \forall i \in I \\ \bot, & f_i(x) \neq f_i(y), \exists i \in I. \end{cases}$$

Proof. Suppose $(X, \mathcal{R})$ is a zero-dimensional $\mathcal{L}$-preordered space. By Definition 4.1, there exist non-empty discrete $\mathcal{L}$-preordered spaces $(X_i, \mathcal{R}_{i_{dis}})$, $i \in I$ and a family of functions $f_i \colon X \to X_i$ such that $f_i \colon (X, \mathcal{R}) \to (X_i, \mathcal{R}_{i_{dis}})$ is the initial lift of $f_i \colon X \to X_i$. Note that for all $x, y \in X$,

$$\begin{aligned} \mathcal{R}(x, y) &= \bigwedge_{i \in I} \{\mathcal{R}_{i_{dis}}(f_i(x), f_i(y))\} \\ &= \bigwedge_{i \in I} \begin{cases} \top, & f_i(x) = f_i(y) \\ \bot, & f_i(x) \neq f_i(y). \end{cases} \end{aligned}$$

Case I: If $f_i(x) = f_i(y)$, $\forall i \in I$, by definition of initial structure, $\mathcal{R}(x, y) = \top$.

Case II: Similarly, if $f_i(x) \neq f_i(y)$, $\exists i \in I$, then $\mathcal{R}(x, y) = \bot$.

Conversely, suppose the condition holds. We show that $(X, \mathcal{R})$ is zero-dimensional, i.e., by Definition 4.1, $f_i \colon (X, \mathcal{R}) \to (X_i, \mathcal{R}_{i_{dis}})$ is the initial lift of $f_i \colon X \to X_i$. It is obvious that for each $i \in I$, $f_i \colon (X, \mathcal{R}) \to (X_i, \mathcal{R}_{i_{dis}})$ is an $\mathcal{L}$-order preserving map. Suppose $g \colon (Y, \mathcal{R}_Y) \to (X, \mathcal{R})$ is a mapping. We prove that g is an $\mathcal{L}$-order preserving if and only if $f_i \circ g$ is an $\mathcal{L}$-order preserving for all $i \in I$. The necessity is obvious. Let $f_i \circ g$ is an $\mathcal{L}$-order preserving map for each $i \in I$. It follows that for $x, y \in Y$,

$$\mathcal{R}_Y(x, y) \leq \bigwedge_{i \in I} \{\mathcal{R}_{i_{dis}}(f_i(g(x)), f_i(g(y)))\}$$

and

$$\mathcal{R}(g(x), g(y)) = \begin{cases} \top, & f_i(g(x)) = f_i(g(y)), \forall i \in I \\ \bot, & f_i(g(x)) \neq f_i(g(y)), \exists i \in I. \end{cases}$$

Case I: If $f_i(g(x)) = f_i(g(y))$, $\forall i \in I$, then $\mathcal{R}_Y(x, y) \leq \mathcal{R}(g(x), g(y)) = \top$.

Case II: Let $f_i(g(x)) \neq f_i(g(y))$, $\exists i \in I$. It follows that $\mathcal{R}(g(x), g(y)) = \bot$, and $\mathcal{R}_Y(x, y) = \bot$ since $\mathcal{R}_Y(x, y) \leq \bigwedge_{i \in I} \{\mathcal{R}_{i_{dis}}(f_i(g(x)), f_i(g(y)))\} = \bot$. Consequently, $\mathcal{R}_Y(x, y) \leq \mathcal{R}(g(x), g(y))$.

Hence, $g \colon (Y, \mathcal{R}_Y) \to (X, \mathcal{R})$ is an $\mathcal{L}$-order preserving and consequently, $(X, \mathcal{R})$ is zero-dimensional. □

Corollary 4.5. *Every discrete $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is zero-dimensional.*

Example 4.6. Let $\mathcal{L} = ([0, 1], \leq, *)$ be a triangular norm with a binary operation $*$ defined as $\forall \alpha, \beta \in [0, 1]$, $\alpha * \beta = (\alpha - 1 + \beta) \vee 0$ (Lukasiewicz t-norm) [24], where the bottom and top elements are $\bot = 0$ and $\top = 1$. Let $X = \{x, y, z\}$, $Y_i = \{a_i, b_i\}$ for $i = 1, 2, 3$ and $\mathcal{R}_{i_{dis}}$ be the discrete $\mathcal{L}$-preordered relation on Y_i for $i = 1, 2, 3$ with the Lukasiewicz t-norm $\mathcal{L} = ([0, 1], \leq, *)$. The map $f_i \colon (X, \mathcal{R}) \longrightarrow (Y_i, \mathcal{R}_{i_{dis}})$, $i = 1, 2, 3$, is defined as

$$f_i(t) = \begin{cases} a_i, & t = x \\ b_i, & t = y, z. \end{cases}$$

Define an $\mathcal{L}$-preordered relation $\mathcal{R} : X \times X \longrightarrow \mathcal{L}$ by

$$\mathcal{R}(u,v) = \mathcal{R}(v,u) = \begin{cases} \top, & u = v \text{ or } u = y \text{ and } v = z \\ \bot, & u = x \text{ and } v = y, z. \end{cases}$$

It follows that $(X, \mathcal{R})$ is zero-dimensional.

Now, we give the characterization of D-connected objects in **$\mathcal{L}$-Prord**.

Definition 4.7. ([9, 32]) Let $\mathcal{U} : C \to$ **Set** be a topological functor and $X \in Obj(C)$. X is D-connected provided that any morphism from X to any discrete object is constant.

Theorem 4.8. *An $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is D-connected if and only if $\mathcal{R}(x,y) > \bot$ or $\mathcal{R}(y,x) > \bot$ for some $x, y \in X$ with $x \neq y$.*

Proof. Suppose $(X, \mathcal{R})$ is D-connected and $\mathcal{R}(x,y) = \bot = \mathcal{R}(y,x)$ for all $x, y \in X$ with $x \neq y$. Let $(Y, \mathcal{R}_{dis})$ be a discrete $\mathcal{L}$-preordered space with $card(Y) > 1$ and $f : (X, \mathcal{R}) \to (Y, \mathcal{R}_{dis})$ be an $\mathcal{L}$-order preserving map. Then,

$$\bot = \mathcal{R}(x,y) \leq \mathcal{R}_{dis}(f(x), f(y))$$

and

$$\bot = \mathcal{R}(y,x) \leq \mathcal{R}_{dis}(f(y), f(x)).$$

It follows that f does not have to be a constant map, a contradiction.

Conversely, suppose that the condition holds. We show that $(X, \mathcal{R})$ is D-connected. Let $(Y, \mathcal{R}_{dis})$ be a discrete $\mathcal{L}$-preordered space and $f : (X, \mathcal{R}) \to (Y, \mathcal{R}_{dis})$ be an $\mathcal{L}$-order preserving map. If $card(Y) = 1$, then $(X, \mathcal{R})$ is D-connected since f is a constant map. Suppose $card(Y) > 1$ and f is not a constant map. Then, there exist distinct points x and y in X such that $f(x) \neq f(y)$ and consequently,

$$\mathcal{R}(x,y) \leq \mathcal{R}_{dis}(f(x), f(y)) = \bot$$

and

$$\mathcal{R}(y,x) \leq \mathcal{R}_{dis}(f(y), f(x)) = \bot.$$

It follows that $\mathcal{R}(x,y) = \bot = \mathcal{R}(y,x)$, a contradiction since $\mathcal{R}(x,y) > \bot$ or $\mathcal{R}(y,x) > \bot$ for some distinct points $x, y \in X$. Hence, f is a constant map and by Definition 4.7, $(X, \mathcal{R})$ is D-connected. □

Theorem 4.9. *Let $(X, \mathcal{R}_X)$ and $(Y, \mathcal{R}_Y)$ be $\mathcal{L}$-preordered spaces and $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be an $\mathcal{L}$-order preserving map. If $(X, \mathcal{R}_X)$ is D-connected and f is surjective, then $(Y, \mathcal{R}_Y)$ is D-connected.*

Proof. Let $f(x), f(y) \in f(X)$ with $f(x) \neq f(y)$. Since f is an $\mathcal{L}$-order preserving map, it follows that $\mathcal{R}_X(x,y) \leq \mathcal{R}_Y(f(x), f(y))$. By the assumption that $(X, \mathcal{R}_X)$ is D-connected, $\bot < \mathcal{R}_X(x,y) \leq \mathcal{R}_Y(f(x), f(y))$ which implies $\bot < \mathcal{R}_Y(f(x), f(y))$. Similarly, we have $\bot < \mathcal{R}_Y(f(y), f(x))$. Therefore, $f(X)$ is D-connected. Since f is surjective, it follows that $f(X) = Y$ is D-connected. □

Theorem 4.10. *Every D-disconnected (not D-connected) $\mathcal{L}$-preordered space is zero-dimensional.*

Proof. Let $(X, \mathcal{R})$ be an $\mathcal{L}$-preordered space. If $(X, \mathcal{R})$ is D-disconnected, then, by Theorem 4.8, $\mathcal{R}(x,y) = \bot = \mathcal{R}(y,x)$ for all $x, y \in X$ with $x \neq y$. It follows that $(X, \mathcal{R})$ is discrete and by Corollary 4.5, $(X, \mathcal{R})$ is zero-dimensional. □

5. Hausdorff Objects in $\mathcal{L}$-Prord

Let X be a non-empty set and the wedge $X^2 \vee_{\triangle} X^2$ be the pushout of the diagonal $\triangle : X \to X^2$ along itself. More precisely, if i_1 and $i_2 : X^2 \to X^2 \vee_{\triangle} X^2$ denote the inclusion of X^2 as the first and second factor, respectively, then $i_1\triangle = i_2\triangle$ is the pushout diagram.

A point (x, y) in $X^2 \vee_{\triangle} X^2$ is denoted as $(x, y)_1$ (resp. $(x, y)_2$) if it lies in the first (resp. second) component. Note that $(x, y)_1 = (x, y)_2$ if and only if $x = y$.

Definition 5.1. (cf. [3]) A map $A : X^2 \vee_{\triangle} X^2 \longrightarrow X^3$ is called a *principal axis map* provided that

$$A(x, y)_i = \begin{cases} (x, y, x), & i = 1 \\ (x, x, y), & i = 2. \end{cases}$$

Definition 5.2. (cf. [3]) A map $S : X^2 \vee_{\triangle} X^2 \longrightarrow X^3$ is called a *skewed axis map* provided that

$$S(x, y)_i = \begin{cases} (x, y, y), & i = 1 \\ (x, x, y), & i = 2. \end{cases}$$

Definition 5.3. (cf. [3]) A map $\nabla : X^2 \vee_{\triangle} X^2 \longrightarrow X^2$ is called a *folding map* provided that $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$.

Definition 5.4. Let $\mathcal{U} : \mathcal{C} \longrightarrow$ **Set** be a topological functor and $X \in Obj(\mathcal{C})$ with $\mathcal{U}(X) = B$.

- (i) X is called $\overline{T_0}$ provided that the initial lift of the $\mathcal{U}$-source $\{A : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor [3].
- (ii) X is called T_0 provided that X doesn't contain an indiscrete subspace with at least two points [29].
- (iii) X is called T_1 provided that the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete [3].
- (iv) X is called pre-$\overline{T_2}$ provided that the initial lift of the $\mathcal{U}$-source $\{A : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ and $S : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3\}$ agree [3, 10].
- (v) X is called $\overline{T_2}$ provided that X is $\overline{T_0}$ and pre-$\overline{T_2}$ [6].
- (vi) X is called NT_2 provided that X is T_0 and pre-$\overline{T_2}$ [6].

In **Top**, by Theorem 2.1 of [7] and [29], $\overline{T_0}$ and T_0 (resp. T_1) reduce to the usual T_0 (resp. T_1) separation property, and $\overline{T_2}$ and NT_2 reduce to usual Hausdorff separation property.

Theorem 5.5. *An $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is $\overline{T_0}$ if and only if $\mathcal{R}(x, y) \wedge \mathcal{R}(y, x) = \perp$ for all $x, y \in X$ with $x \neq y$.*

Proof. Suppose $(X, \mathcal{R})$ is $\overline{T_0}$ and $x, y \in X$ with $x \neq y$. Let $\mathcal{R}_{dis}$ be the discrete $\mathcal{L}$-preordered structure on X^2 and $pr_k : X^3 \to X$ be the projection map for $k = 1, 2, 3$. For $u = (x, y)_1, v = (x, y)_2 \in X^2 \vee_{\triangle} X^2$ with $u \neq v$. Note that

$$\begin{aligned} \mathcal{R}_{dis}(\nabla u, \nabla v) &= \mathcal{R}_{dis}((x, y), (x, y)) = \top, \\ \mathcal{R}(pr_1 Au, pr_1 Av) &= \mathcal{R}(pr_1(x, y, x), pr_1(x, x, y)) = \mathcal{R}(x, x) = \top, \\ \mathcal{R}(pr_2 Au, pr_2 Av) &= \mathcal{R}(pr_2(x, y, x), pr_2(x, x, y)) = \mathcal{R}(y, x), \\ \mathcal{R}(pr_3 Au, pr_3 Av) &= \mathcal{R}(pr_3(x, y, x), pr_3(x, x, y)) = \mathcal{R}(x, y). \end{aligned}$$

Since $u \neq v$ and $(X, \mathcal{R})$ is $\overline{T_0}$, by Lemma 3.1 and Definition 5.4,

$$\begin{aligned} \perp &= \bigwedge \{\mathcal{R}_{dis}(\nabla u, \nabla v), \mathcal{R}(pr_k Au, pr_k Av) : k = 1, 2, 3\} \\ &= \bigwedge \{\top, \mathcal{R}(x, y), \mathcal{R}(y, x)\} \\ &= \mathcal{R}(x, y) \wedge \mathcal{R}(y, x). \end{aligned}$$

Conversely, let $\mathcal{R}_1$ be the initial $\mathcal{L}$-preordered structure on $X^2 \vee_\triangle X^2$ induced by $A : X^2 \vee_\triangle X^2 \to U(X^3, \mathcal{R}^3) = X^3$ and $\nabla : X^2 \vee_\triangle X^2 \to U(X^2, \mathcal{R}_{dis}) = X^2$, where $\mathcal{R}_{dis}$ is the discrete $\mathcal{L}$-preordered structure on X^2 and $\mathcal{R}^3$ is the product structure on X^3 induced by the projection map $pr_k : X^3 \to X$ for $k = 1, 2, 3$.

Suppose $\mathcal{R}(x, y) \wedge \mathcal{R}(y, x) = \bot$ for all $x, y \in X$ with $x \neq y$ and $u, v \in X^2 \vee_\triangle X^2$.

Case I: If $u = v$, then $\mathcal{R}_1(u, v) = \mathcal{R}_1(u, u) = \top$

Case II: If $u \neq v$ and $\nabla u \neq \nabla v$, then $\mathcal{R}_{dis}(\nabla u, \nabla v) = \bot$. By Lemma 3.1,

$$\begin{aligned}\mathcal{R}_1(u, v) &= \bigwedge \{\mathcal{R}_{dis}(\nabla u, \nabla v), \mathcal{R}(pr_k Au, pr_k Av) : k = 1, 2, 3\} \\ &= \bigwedge \{\bot, \mathcal{R}(pr_k Au, pr_k Av) : k = 1, 2, 3\} \\ &= \bot.\end{aligned}$$

Case III: Suppose $u \neq v$ and $\nabla u = \nabla v$. If $\nabla u = (x, y) = \nabla v$ for some $x, y \in X$ with $x \neq y$, then $u = (x, y)_1$ and $v = (x, y)_2$ or $u = (x, y)_2$ and $v = (x, y)_1$ since $u \neq v$.

If $u = (x, y)_1$ and $v = (x, y)_2$, then

$$\mathcal{R}_{dis}(\nabla u, \nabla v) = \mathcal{R}_{dis}((x, y), (x, y)) = \top,$$

$$\begin{aligned}\mathcal{R}(pr_1 Au, pr_1 Av) &= \mathcal{R}(pr_1 A(x, y)_1, pr_1 A(x, y)_2) \\ &= \mathcal{R}(x, x) = \top,\end{aligned}$$

$$\begin{aligned}\mathcal{R}(pr_2 Au, pr_2 Av) &= \mathcal{R}(pr_2 A(x, y)_1, pr_2 A(x, y)_2) \\ &= \mathcal{R}(y, x)\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}(pr_3 Au, pr_3 Av) &= \mathcal{R}(pr_3 A(x, y)_1, pr_3 A(x, y)_2) \\ &= \mathcal{R}(x, y).\end{aligned}$$

It follows that for $k = 1, 2, 3$,

$$\begin{aligned}\mathcal{R}_1(u, v) &= \bigwedge \{\mathcal{R}_{dis}(\nabla(x, y)_1, \nabla(x, y)_2), \mathcal{R}(pr_k A(x, y)_1, pr_k A(x, y)_2)\} \\ &= \bigwedge \{\top, \mathcal{R}(x, y), \mathcal{R}(y, x)\} \\ &= \mathcal{R}(x, y) \wedge \mathcal{R}(y, x).\end{aligned}$$

By the assumption that $\mathcal{R}(x, y) \wedge \mathcal{R}(y, x) = \bot$ and we have $\mathcal{R}_1(u, v) = \bot$.

Similarly, if $u = (x, y)_2$ and $v = (x, y)_1$, then $\mathcal{R}_1(u, v) = \bot$.

Therefore, for all $u, v \in X^2 \vee_\triangle X^2$, we have

$$\mathcal{R}_1(u, v) = \begin{cases} \top, & u = v \\ \bot, & u \neq v. \end{cases}$$

and by Lemma 3.2 (i), $\mathcal{R}_1$ is the discrete $\mathcal{L}$-preordered structure on $X^2 \vee_\triangle X^2$. Hence, by Definition 5.4, $(X, \mathcal{R})$ is $\overline{T_0}$. □

In a quantale $(L, \leq, *)$, if $\alpha \in L$ and $\alpha \neq \top$, then α is called *prime element* provided that $\zeta \wedge \eta \leq \alpha$ implies $\zeta \leq \alpha$ or $\eta \leq \alpha$ for all $\zeta, \eta \in L$.

Corollary 5.6. *Let $(X, \mathcal{R})$ be an $\mathcal{L}$-preordered space, where $\mathcal{L}$ has a prime bottom element. Then, $(X, \mathcal{R})$ is $\overline{T_0}$ if and only if $\mathcal{R}(x, y) = \bot$ or $\mathcal{R}(y, x) = \bot$ for all $x, y \in X$ with $x \neq y$.*

Proof. It follows from Theorem 5.5 and the definition of prime bottom element. □

Theorem 5.7. *An $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is T_0 if and only if $(X, \mathcal{R})$ is separated.*

Proof. Let $(X,\mathcal{R})$ be T_0, for distinct $x,y \in X$, $A = \{x,y\} \subset X$ and $\mathcal{R}_A$ be the initial $\mathcal{L}$-preordered structure induced by the inclusion map $i : A \to (X,\mathcal{R})$. For all $x,y \in X$ with $x \neq y$, $\mathcal{R}_A(x,y) \leq \mathcal{R}(i(x),i(y)) = \mathcal{R}(x,y)$ or $\mathcal{R}_A(y,x) \leq \mathcal{R}(i(y),i(x)) = \mathcal{R}(y,x)$. It follows that $\mathcal{R}(x,y) < \top$ or $\mathcal{R}(y,x) < \top$ otherwise $\mathcal{R}(x,y) = \top = \mathcal{R}(y,x)$ and X contains an indiscrete subspace with at least two element.

Conversely, suppose $(X,\mathcal{R})$ is separated. Let A be an indiscrete subspace of X with at least two elements $x,y \in A$ with $x \neq y$. Let $\mathcal{R}_A$ be the initial $\mathcal{L}$-preordered structure induced by the inclusion map $i : A \to (X,\mathcal{R})$. It follows that $\top = \mathcal{R}_A(x,y) \leq \mathcal{R}(i(x),i(y)) = \mathcal{R}(x,y)$ and $\top = \mathcal{R}_A(y,x) \leq \mathcal{R}(i(y),i(x)) = \mathcal{R}(y,x)$ and consequently, $\mathcal{R}(x,y) = \top = \mathcal{R}(y,x)$, a contradiction. Therefore, X does not contain an indiscrete subspace with at least two elements. Hence, by Definition 5.4, $(X,\mathcal{R})$ is T_0. □

Theorem 5.8. *An $\mathcal{L}$-preordered space $(X,\mathcal{R})$ is T_1 if and only if $\mathcal{R}(x,y) = \bot = \mathcal{R}(y,x)$ for all $x,y \in X$ with $x \neq y$.*

Proof. Suppose that $(X,\mathcal{R})$ is T_1 and $x,y \in X$ with $x \neq y$. Let $u = (x,y)_1$, $v = (x,y)_2 \in X^2 \vee_\triangle X^2$. Note that

$$\begin{aligned}
\mathcal{R}_{dis}(\nabla u, \nabla v) &= \mathcal{R}_{dis}((x,y),(x,y)) = \top, \\
\mathcal{R}(pr_1 Su, pr_1 Sv) &= \mathcal{R}(pr_1(x,y,y), pr_1(x,x,y)) = \mathcal{R}(x,x) = \top, \\
\mathcal{R}(pr_2 Su, pr_2 Sv) &= \mathcal{R}(pr_2(x,y,y), pr_2(x,x,y)) = \mathcal{R}(y,x), \\
\mathcal{R}(pr_3 Su, pr_3 Sv) &= \mathcal{R}(pr_3(x,y,y), pr_3(x,x,y)) = \mathcal{R}(y,y) = \top,
\end{aligned}$$

where $\mathcal{R}_{dis}$ is the discrete $\mathcal{L}$-preordered structure on $X^2 \vee_\triangle X^2$ and $pr_k : X^3 \to X$ are the projection maps for $k = 1,2,3$. Since $u \neq v$ and $(X,\mathcal{R})$ is T_1, by Lemma 3.1 and Definition 5.4,

$$\begin{aligned}
\bot &= \bigwedge \{\mathcal{R}_{dis}(\nabla u, \nabla v), \mathcal{R}(pr_k Su, pr_k Sv) : k = 1,2,3\} \\
&= \bigwedge \{\top, \mathcal{R}(y,x)\} = \mathcal{R}(y,x).
\end{aligned}$$

Similarly, if $u = (x,y)_2$, $v = (x,y)_1 \in X^2 \vee_\triangle X^2$, then

$$\begin{aligned}
\bot &= \bigwedge \{\mathcal{R}_{dis}(\nabla u, \nabla v), \mathcal{R}(pr_k Su, pr_k Sv) : k = 1,2,3\} \\
&= \bigwedge \{\top, \mathcal{R}(x,y)\} = \mathcal{R}(x,y).
\end{aligned}$$

Conversely, let $\mathcal{R}_1$ be the initial $\mathcal{L}$-preordered structure on $X^2 \vee_\triangle X^2$ induced by $S : X^2 \vee_\triangle X^2 \to U(X^3,\mathcal{R}^3) = X^3$ and $\nabla : X^2 \vee_\triangle X^2 \to U(X^2,\mathcal{R}_{dis}) = X^2$, where $\mathcal{R}_{dis}$ is the discrete $\mathcal{L}$-preordered structure on X^2 and $\mathcal{R}^3$ is the product structure on X^3 induced by the projection map $pr_k : X^3 \to X$ for $k = 1,2,3$.

Suppose $\mathcal{R}(x,y) = \bot = \mathcal{R}(y,x)$ for all $x,y \in X$ with $x \neq y$ and $u,v \in X^2 \vee_\triangle X^2$.

Case I: If $u = v$, then $\mathcal{R}_1(u,v) = \mathcal{R}_1(u,u) = \top$.

Case II: If $u \neq v$ and $\nabla u \neq \nabla v$, then $\mathcal{R}_{dis}(\nabla u, \nabla v) = \bot$ since $\mathcal{R}_{dis}$ is the discrete structure on X^2. By Lemma 3.1,

$$\begin{aligned}
\mathcal{R}_1(u,v) &= \bigwedge \{\mathcal{R}_{dis}(\nabla u, \nabla v), \mathcal{R}(pr_k Su, pr_k Sv) : k = 1,2,3\} \\
&= \bigwedge \{\bot, \mathcal{R}(pr_1 Su, pr_1 Sv), \mathcal{R}(pr_2 Su, pr_2 Sv), \mathcal{R}(pr_3 Su, pr_3 Sv)\} \\
&= \bot.
\end{aligned}$$

Case III: Suppose $u \neq v$ and $\nabla u = \nabla v$. If $\nabla u = (x,y) = \nabla v$ for some $x,y \in X$ with $x \neq y$, then $u = (x,y)_1$ and $v = (x,y)_2$ or $u = (x,y)_2$ and $v = (x,y)_1$ since $u \neq v$.

If $u = (x,y)_1$ and $v = (x,y)_2$, then by Lemma 3.1,

$$\begin{aligned}
\mathcal{R}_1(u,v) &= \bigwedge \{\mathcal{R}_{dis}(\nabla u, \nabla v), \mathcal{R}(pr_k Su, pr_k Sv) : k = 1,2,3\} \\
&= \bigwedge \{\top, \mathcal{R}(y,x)\} \\
&= \mathcal{R}(y,x) = \bot
\end{aligned}$$

since $x \neq y$ and $\mathcal{R}(y,x) = \bot$.

Similarly, if $u = (x,y)_2$ and $v = (x,y)_1$, then $\mathcal{R}_1(u,v) = \bot$.

Hence, for all $u, v \in X^2 \vee_\triangle X^2$, we get

$$\mathcal{R}_1(u,v) = \begin{cases} \top, & u = v \\ \bot, & u \neq v \end{cases}$$

and it follows that $\mathcal{R}_1$ is the discrete $\mathcal{L}$-preordered structure on $X^2 \vee_\triangle X^2$. By Definition 5.4, $(X, \mathcal{R})$ is T_1. □

Theorem 5.9. *An $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is pre-$\overline{T_2}$ if and only if the following conditions are satisfied.*

(I) For all $x, y \in X$ with $x \neq y$, $\mathcal{R}(x,y) \wedge \mathcal{R}(y,x) = \mathcal{R}(x,y) = \mathcal{R}(y,x)$.
(II) For any three distinct points $x, y, z \in X$, $\mathcal{R}(y,x) \wedge \mathcal{R}(z,x) \wedge \mathcal{R}(y,z) = \mathcal{R}(y,x) \wedge \mathcal{R}(z,x) = \mathcal{R}(x,y) \wedge \mathcal{R}(z,y) = \mathcal{R}(z,x) \wedge \mathcal{R}(y,z)$.
(III) For any four distinct points $x, y, z, w \in X$, $\mathcal{R}(x,z) \wedge \mathcal{R}(y,z) \wedge \mathcal{R}(y,w) = \mathcal{R}(x,z) \wedge \mathcal{R}(y,z) \wedge \mathcal{R}(x,w) = \mathcal{R}(x,w) \wedge \mathcal{R}(y,z) \wedge \mathcal{R}(y,w) = \mathcal{R}(x,z) \wedge \mathcal{R}(y,w) \wedge \mathcal{R}(x,w)$.

Proof. Suppose that $(X, \mathcal{R})$ is pre-$\overline{T_2}$ and $x, y \in X$ with $x \neq y$. Let $pr_k : X^3 \to X$ be the projection map for $k = 1,2,3$ and $u = (x,y)_1, v = (x,y)_2 \in X^2 \vee_\triangle X^2$. By Definition 5.4 (iv), we have

$$\begin{aligned} \bigwedge\{\mathcal{R}(pr_k Au, pr_k Av) : k = 1,2,3\} &= \bigwedge\{\mathcal{R}(pr_k Su, pr_k Sv) : k = 1,2,3\} \\ \bigwedge\{\top, \mathcal{R}(x,y), \mathcal{R}(y,x)\} &= \bigwedge\{\top, \mathcal{R}(y,x)\} \\ \mathcal{R}(x,y) \wedge \mathcal{R}(y,x) &= \mathcal{R}(y,x). \end{aligned}$$

Similarly, if $u = (x,y)_2, v = (x,y)_1$, then we have $\mathcal{R}(x,y) \wedge \mathcal{R}(y,x) = \mathcal{R}(x,y)$. Hence, $\mathcal{R}(x,y) \wedge \mathcal{R}(y,x) = \mathcal{R}(x,y) = \mathcal{R}(y,x)$.

Let x, y, z be any three distinct points of X. Since $(X, \mathcal{R})$ is pre-$\overline{T_2}$ and by Definition 5.4 (iv), for $k = 1,2,3$, we have

$$\begin{aligned} \bigwedge\{\mathcal{R}(pr_k A(y,z)_1, pr_k A(x,z)_2)\} &= \bigwedge\{\mathcal{R}(pr_k S(y,z)_1, pr_k S(x,z)_2)\} \\ \bigwedge\{\mathcal{R}(y,x), \mathcal{R}(z,x), \mathcal{R}(y,z)\} &= \bigwedge\{\top, \mathcal{R}(y,x), \mathcal{R}(z,x)\}, \end{aligned}$$

$$\begin{aligned} \bigwedge\{\mathcal{R}(pr_k A(x,z)_1, pr_k A(y,z)_2)\} &= \bigwedge\{\mathcal{R}(pr_k S(x,z)_1, pr_k S(y,z)_2)\} \\ \bigwedge\{\mathcal{R}(x,y), \mathcal{R}(z,y), \mathcal{R}(x,z)\} &= \bigwedge\{\top, \mathcal{R}(x,y), \mathcal{R}(z,y)\}, \end{aligned}$$

$$\begin{aligned} \bigwedge\{\mathcal{R}(pr_k A(x,y)_1, pr_k A(z,y)_2)\} &= \bigwedge\{\mathcal{R}(pr_k S(x,y)_1, pr_k S(z,y)_2)\} \\ \bigwedge\{\mathcal{R}(x,z), \mathcal{R}(y,z), \mathcal{R}(x,y)\} &= \bigwedge\{\top, \mathcal{R}(x,z), \mathcal{R}(y,z)\}. \end{aligned}$$

By the condition (I), it follows that $\mathcal{R}(y,x) \wedge \mathcal{R}(z,x) \wedge \mathcal{R}(y,z) = \mathcal{R}(y,x) \wedge \mathcal{R}(z,x) = \mathcal{R}(x,y) \wedge \mathcal{R}(z,y) = \mathcal{R}(z,x) \wedge \mathcal{R}(y,z)$.

Let x, y, z, w be any four distinct points of X. Since $(X, \mathcal{R})$ is pre-$\overline{T_2}$ and by Definition 5.4 (iv), for $k = 1,2,3$, we have

$$\begin{aligned} \bigwedge\{\mathcal{R}(pr_k A(x,y)_1, pr_k A(z,w)_2)\} &= \bigwedge\{\mathcal{R}(pr_k S(x,y)_1, pr_k S(z,w)_2)\} \\ \bigwedge\{\mathcal{R}(x,z), \mathcal{R}(y,z), \mathcal{R}(x,w)\} &= \bigwedge\{\mathcal{R}(x,z), \mathcal{R}(y,z), \mathcal{R}(y,w)\}, \end{aligned}$$

$$\begin{aligned} \bigwedge\{\mathcal{R}(pr_k A(x,y)_1, pr_k A(w,z)_2)\} &= \bigwedge\{\mathcal{R}(pr_k S(x,y)_1, pr_k S(w,z)_2)\} \\ \bigwedge\{\mathcal{R}(x,w), \mathcal{R}(y,w), \mathcal{R}(x,z)\} &= \bigwedge\{\mathcal{R}(x,w), \mathcal{R}(y,w), \mathcal{R}(y,z)\}, \end{aligned}$$

$$\bigwedge\{\mathcal{R}(pr_kA(w,z)_1, pr_kA(y,x)_2)\} = \bigwedge\{\mathcal{R}(pr_kS(w,z)_1, pr_kS(y,x)_2)\}$$
$$\bigwedge\{\mathcal{R}(w,y),\mathcal{R}(z,y),\mathcal{R}(w,x)\} = \bigwedge\{\mathcal{R}(w,y),\mathcal{R}(z,y),\mathcal{R}(z,x)\}.$$

By the condition (I), it follows that $\mathcal{R}(x,z)\wedge\mathcal{R}(y,z)\wedge\mathcal{R}(y,w) = \mathcal{R}(x,z)\wedge\mathcal{R}(y,z)\wedge\mathcal{R}(x,w) = \mathcal{R}(x,w)\wedge\mathcal{R}(y,z)\wedge\mathcal{R}(y,w) = \mathcal{R}(x,z)\wedge\mathcal{R}(y,w)\wedge\mathcal{R}(x,w)$.

Conversely, suppose that the conditions hold. We show that $(X,\mathcal{R})$ is pre-$\overline{T_2}$. Let $\mathcal{R}_A$ and $\mathcal{R}_S$ be initial structures on $X^2\vee_\triangle X^2$ induced by $A : X^2\vee_\triangle X^2 \to U(X^3,\mathcal{R}^3) = X^3$ and $S : X^2\vee_\triangle X^2 \to U(X^3,\mathcal{R}^3) = X^3$ respectively, and $\mathcal{R}^3$ be the product structure on X^3 induced by the projection maps $pr_k : X^3 \to X$ for $k = 1,2,3$. We need to show that $\mathcal{R}_A = \mathcal{R}_S$.

First, note that $\mathcal{R}_A$ and $\mathcal{R}_S$ are symmetric by the assumption (I).

Suppose u and v are any two points in $X^2\vee_\triangle X^2$.

If $u = v$, then $\mathcal{R}_A(u,v) = \top = \mathcal{R}_S(u,v)$.

If $u \neq v$ and they are in the same component of the wedge $X^2\vee_\triangle X^2$, i.e., $u = (x,y)_i$ and $v = (z,w)_i$ for $i = 1,2$, then

$$\begin{aligned}\mathcal{R}_A(u,v) &= \bigwedge\{\mathcal{R}(pr_kAu, pr_kAv) : k = 1,2,3\}\\ &= \bigwedge\{\mathcal{R}(x,z),\mathcal{R}(y,w)\}\\ &= \bigwedge\{\mathcal{R}(pr_kSu, pr_kSv) : k = 1,2,3\}\\ &= \mathcal{R}_S(u,v).\end{aligned}$$

Suppose $u \neq v$ and they are in the different component of the wedge $X^2\vee_\triangle X^2$. We have the following cases for u and v:

Case I: $u = (x,y)_1$ or $(y,x)_1$ and $v = (x,y)_2$ or $(y,x)_2$ for $x \neq y$.

If $u = (x,y)_1$ and $v = (x,y)_2$ (resp. $v = (y,x)_2$), then for $k = 1,2,3$

$$\mathcal{R}_A(u,v) = \bigwedge\{\mathcal{R}(pr_kAu, pr_kAv)\} = \mathcal{R}(x,y)\wedge\mathcal{R}(y,x) \;\; (resp.\ \mathcal{R}(x,y))$$

and

$$\mathcal{R}_S(u,v) = \bigwedge\{\mathcal{R}(pr_kSu, pr_kSv)\} = \mathcal{R}(y,x) \;\; (resp.\ \mathcal{R}(x,y)\wedge\mathcal{R}(y,x)).$$

By the assumption (I), it follows that $\mathcal{R}_A(u,v) = \mathcal{R}_S(u,v)$.

Similarly, if $u = (y,x)_1$ and $v = (x,y)_2$ (resp. $v = (y,x)_2$), then we have $\mathcal{R}_A(u,v) = \mathcal{R}_S(u,v)$.

Case II: $u = (x,y)_1, (x,z)_1, (y,z)_1, (y,x)_1, (z,x)_1$ or $(z,y)_1$ and $v = (x,y)_2, (x,z)_2, (y,z)_2, (y,x)_2, (z,x)_2$ or $(z,y)_2$ for three distinct points x,y,z of X.

If $u = (x,y)_1$ or $(y,x)_1$ and $v = (x,y)_2$ or $(y,x)_2$, $u = (x,z)_1$ or $(z,x)_1$ and $v = (x,z)_2$ or $(z,x)_2$, $u = (y,z)_1$ or $(z,y)_1$ and $v = (y,z)_2$ or $(z,y)_2$, then by the case I, we have $\mathcal{R}_A(u,v) = \mathcal{R}_S(u,v)$.

If $u = (x,y)_1$ and $v = (x,z)_2$ or $(y,z)_2$ (resp. $u = (y,x)_1$ and $v = (x,z)_2$ or $(y,z)_2$), then by the assumption (I),

$$\begin{aligned}\mathcal{R}_A(u,v) &= \bigwedge\{\mathcal{R}(pr_kAu, pr_kAv) : k = 1,2,3\}\\ &= \mathcal{R}(y,x)\wedge\mathcal{R}(x,z) \;\; (resp.\ \mathcal{R}(y,x)\wedge\mathcal{R}(y,z))\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_S(u,v) &= \bigwedge\{\mathcal{R}(pr_kSu, pr_kSv) : k = 1,2,3\}\\ &= \mathcal{R}(y,x)\wedge\mathcal{R}(y,z) \;\; (resp.\ \mathcal{R}(y,x)\wedge\mathcal{R}(x,z)).\end{aligned}$$

By the assumption (II), we have $\mathcal{R}_A(u,v) = \mathcal{R}_S(u,v)$.

Similarly, if $u = (x,y)_1$ or $(y,x)_1$ and $v = (z,x)_2$ or $(z,y)_2$, $u = (x,z)_1$ or $(z,x)_1$ and $v = (x,y)_2$ or $(y,z)_2$ or $(y,x)_2$ or $(z,y)_2$, $u = (y,z)_1$ or $(z,y)_1$ and $v = (x,y)_2$ or $(x,z)_2$ or $(y,x)_2$ or $(z,x)_2$, then by the assumption (II), we have $\mathcal{R}_A(u,v) = \mathcal{R}_S(u,v)$.

Case III: Let x, y, z, w be four distinct points of X.

If $u = (x, y)_1$ and $v = (z, w)_2$ (resp. $u = (z, w)_1$ and $v = (x, y)_2$), then by the assumption (I),

$$\begin{aligned} \mathcal{R}_A(u, v) &= \bigwedge\{\mathcal{R}(pr_k Au, pr_k Av) : k = 1, 2, 3\} \\ &= \mathcal{R}(x, z) \wedge \mathcal{R}(y, z) \wedge \mathcal{R}(x, w), \\ \mathcal{R}_S(u, v) &= \bigwedge\{\mathcal{R}(pr_k Su, pr_k Sv) : k = 1, 2, 3\} \\ &= \mathcal{R}(x, z) \wedge \mathcal{R}(y, z) \wedge \mathcal{R}(y, w) \\ &= (\textit{resp. } \mathcal{R}(x, z) \wedge \mathcal{R}(y, w) \wedge \mathcal{R}(x, w)) \end{aligned}$$

and by the assumption (III), we have $\mathcal{R}_A(u, v) = \mathcal{R}_S(u, v)$.

If $u = (x, y)_1$ and $v = (w, z)_2$ (resp. $u = (w, z)_1$ and $v = (x, y)_2$), then by the assumption (I),

$$\begin{aligned} \mathcal{R}_A(u, v) &= \mathcal{R}(x, z) \wedge \mathcal{R}(y, w) \wedge \mathcal{R}(x, w), \\ \mathcal{R}_S(u, v) &= \mathcal{R}(x, w) \wedge \mathcal{R}(y, z) \wedge \mathcal{R}(y, w) \\ &= (\textit{resp. } \mathcal{R}(x, z) \wedge \mathcal{R}(y, z) \wedge \mathcal{R}(x, w)) \end{aligned}$$

and by the assumption (III), we have $\mathcal{R}_A(u, v) = \mathcal{R}_S(u, v)$.

Similarly, if $u = (y, x)_1$ and $v = (z, w)_2$ or $(w, z)_2$, $u = (z, w)_1$ or $(w, z)_1$ and $v = (y, x)_2$, $u = (x, z)_1$ or $(z, x)_1$ and $v = (y, w)_2$ or $(w, y)_2$, $u = (y, w)_1$ or $(w, y)_1$ and $v = (x, z)_2$ or $(z, x)_2$, $u = (x, w)_1$ or $(w, x)_1$ and $v = (y, z)_2$ or $(z, y)_2$, $u = (y, z)_1$ or $(z, y)_1$ and $v = (x, w)_2$ or $(w, x)_2$, then by the assumption (III), we have $\mathcal{R}_A(u, v) = \mathcal{R}_S(u, v)$.

It is shown similarly that for all u and v in the cases above, we get $\mathcal{R}_A(v, u) = \mathcal{R}_S(v, u)$.

Hence, we have $\mathcal{R}_A(u, v) = \mathcal{R}_S(u, v)$ for all points $u, v \in X^2 \vee_\Delta X^2$, and by Lemma 3.1 and Definition 5.4 (iv), $(X, \mathcal{R})$ is pre-$\overline{T_2}$. □

Theorem 5.10. *An $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is $\overline{T_2}$ if and only if $\mathcal{R}(x, y) = \bot = \mathcal{R}(y, x)$ for all $x, y \in X$ with $x \neq y$.*

Proof. It follows from Definition 5.4 (v), Theorems 5.5 and 5.9. □

Theorem 5.11. *Let $(X, \mathcal{R})$ be an $\mathcal{L}$-preordered space. The followings are equivalent.*

(i) $(X, \mathcal{R})$ *is* T_1.
(ii) $(X, \mathcal{R})$ *is* $\overline{T_2}$.
(iii) $(X, \mathcal{R})$ *is discrete.*

Proof. It follows from Lemma 3.2 (i), and Theorems 5.8 and 5.10. □

Theorem 5.12. *An $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is NT_2 if and only if the following conditions hold.*

(I) *$(X, \mathcal{R})$ is a separated $\mathcal{L}$-equivalence space.*
(II) *For any three distinct points $x, y, z \in X$, $\mathcal{R}(y, x) \wedge \mathcal{R}(z, x) \wedge \mathcal{R}(y, z) = \mathcal{R}(y, x) \wedge \mathcal{R}(z, x) = \mathcal{R}(x, y) \wedge \mathcal{R}(z, y) = \mathcal{R}(z, x) \wedge \mathcal{R}(y, z)$.*
(III) *For any four distinct points $x, y, z, w \in X$, $\mathcal{R}(x, z) \wedge \mathcal{R}(y, z) \wedge \mathcal{R}(y, w) = \mathcal{R}(x, z) \wedge \mathcal{R}(y, z) \wedge \mathcal{R}(x, w) = \mathcal{R}(x, w) \wedge \mathcal{R}(y, z) \wedge \mathcal{R}(y, w) = \mathcal{R}(x, z) \wedge \mathcal{R}(y, w) \wedge \mathcal{R}(x, w)$.*

Proof. It follows from Definition 5.4 (vi), Theorems 5.7 and 5.9. □

Theorem 5.13. *Let $(X, \mathcal{R})$ be a pre-$\overline{T_2}$ $\mathcal{L}$-preordered space, then the following are equivalent.*

1. $(X, \mathcal{R})$ *is* $\overline{T_0}$
2. $(X, \mathcal{R})$ *is* T_1
3. $(X, \mathcal{R})$ *is* $\overline{T_2}$

Proof. It follows from Theorems 5.5, 5.8, 5.9, and 5.10. □

6. Conclusion

Let C be a topological category. By Theorem 3.4 of [10], the full subcategory **pre-T$_2$**(C) of C consisting of all pre-$\overline{T_2}$ objects in C is a topological category.

Let $\mathbf{T_i}(C)$ be the full subcategory of C consisting of all $\mathbf{T_i}$ objects, $i = 0, 1, 2$, where $\mathbf{T_0}$ is $\overline{T_0}$ or T_0 and $\mathbf{T_2}$ is $\overline{T_2}$ or NT_2, and C is **pre-T$_2$**($\mathcal{L}$**-Prord**) or $\mathcal{L}$**-Prord**.

Corollary 6.1. *The following categories are isomorphic.*

(a) T_1 ($\mathcal{L}$*-**Prord***)
(b) $\overline{T_2}$ ($\mathcal{L}$*-**Prord***)
(c) $\overline{T_0}$ (**pre**-$\mathbf{T_2}$($\mathcal{L}$*-**Prord***))
(d) T_1 (**pre**-$\mathbf{T_2}$($\mathcal{L}$*-**Prord***))
(e) $\overline{T_2}$ (**pre**-$\mathbf{T_2}$($\mathcal{L}$*-**Prord***))

Proof. It follows from Theorem 3.5 of [10] and Theorems 5.8, 5.10, and 5.13. □

Corollary 6.2. *Every T_1 $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is zero-dimensional but the converse is not true, in general.*

Corollary 6.3. *If an $\mathcal{L}$-preordered space $(X, \mathcal{R})$ is D-disconnected, then it is T_1.*

Remark 6.4. (I) For any arbitrary topological category, there is no relationship between $\overline{T_0}$ and T_0 [5]. Also , it is proved in [6], that the notions of $\overline{T_2}$ and NT_2 are independent of each other, in general.

(a) In **CP** (category of pairs and pair preserving maps), all objects are $\overline{T_0}$ and T_1 and $\overline{T_2}$ and pre-$\overline{T_2}$ [2].

(b) In **CHY** (category of Cauchy spaces and Cauchy continuous maps), $T_0 = \overline{T_0} = T_1 = \overline{T_2} \Rightarrow$ pre-$\overline{T_2}$ [25]. Similarly, in **Prox** (category of proximity spaces and proximity maps), if a proximity space (X, δ) is $\overline{T_0}$ or T_1 or $\overline{T_2}$, then (X, δ) is pre-$\overline{T_2}$ [26].

(c) In **ConFCO** (the category of constant filter convergence spaces and continuous maps), $\overline{T_2} = NT_2 \Rightarrow T_0 = \overline{T_0} = T_1$ but in **ConLFCO** (the category of constant local filter convergence spaces and continuous maps), $T_0 \Rightarrow \overline{T_0} = T_1$ and $T_0 = NT_2 \Rightarrow \overline{T_2}$ [4].

(d) In **V-Cls** (category of $\mathcal{V}$-valued closure spaces and contractive maps) [27], $\overline{T_2} = T_1 \Rightarrow \overline{T_0} \Rightarrow T_0$ [33].

(II) In $\mathcal{L}$**-Prord**, by Theorems 5.5, 5.7 and 5.11, $\overline{T_2} = T_1 \Rightarrow \overline{T_0} \Rightarrow T_0$. Moreover, by Theorems 5.10 and 5.12, if $(X, \mathcal{R})$ is $\overline{T_2}$, then it is NT_2.

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