Filomat 36:7 (2022), 2269–2280
https://doi.org/10.2298/FIL2207269E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An Extension of Apostol Type of Hermite-Genocchi Polynomials and their Probabilistic Representation

Beih S. El-Desouky[a], Rabab S. Gomaa[a], Alia M. Magar[a]

[a]*Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt*

Abstract. The main purpose of this paper is to introduce and investigate the various properties of a new generalization of Apostol Hermite-Genocchi polynomials. We derive many useful results involving new generalized Apostol Hermite-Genocchi polynomials. We also consider some statistical applications of the new family in probability distribution theory and reliability.

1. Introduction

The generalized Apostol-Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order α are defined by (see, for example [15], [16] and [35])

$$\left(\frac{2t}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},$$

$$(|\, t \,|< \pi, \ \text{ when } \ \lambda = 1; \ \ |\, t \,|<|\log(-\lambda)\,|, \ \text{ when } \ \lambda \neq 1, \ \ 1^{\alpha} := 1). \tag{1}$$

Remark 1.1. *When $\lambda \neq -1$ in* (1), *the order α of generalized Apostol-Genocchi polynomials should tacitly be restricted to non negative integer values.*

Some interesting generalizations of Apostol type polynomials have been investigated in the literature (see, for example [11], [12], [14], [17], [18], [19], [29], [31], [32], [33] and [34])
Srivastava [27] defined a class of generalized Hermite polynomials by the generating function

$$\sum_{n=0}^{\infty} \gamma_n^m(x)\frac{t^n}{n!} = e^{mxt-t^m}, \tag{2}$$

for more details, see ([4], [6], [5] and [30]).
Also, the Hermite based Appel polynomials have been introduced and investigated in the literature (see,

2020 *Mathematics Subject Classification*. 11B68, 33C45, 33C99, 05A10.
Keywords. Hermite distribution, Generalized Hermite distribution, Hermite polynomials, Genocchi polynomials, Hermite-Genocchi polynomials, Apostol-Genocchi polynomials, Discrete distribution, Reliability.
Received: 09 April 2019; Revised: 31 December 2019; Accepted: 10 January 2021
Communicated by Hari M. Srivastava
Email addresses: b_desouky@yahoo.com (Beih S. El-Desouky), dr.rsg12@yahoo.com (Rabab S. Gomaa), alia.ma16@yahoo.com (Alia M. Magar)

for example [2],[3], [22], [23], [33] and [34]).
Araci et al. [1] introduced a new concept of the Apostol Hermite-Genocchi polynomials by

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x,y;a,b,c;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} c^{xt+h(t,y)}, \qquad \left(|\, t \,|<|\, \frac{\log(-\lambda)}{\log(\frac{b}{a})}\,|;\ a\in\mathbb{C}\backslash\{0\},\ b,\ c\in\mathbb{R}^+;\ 1^{\alpha}:=1\right). \tag{3}$$

Recently, by using the exponential as well as trigonometric generating functions, Srivastava et al. [34] defined two parametric kinds of each of generated Apostol-Bernoulli $B_n^{(c,\alpha)}(p,q;\lambda)$, Apostol-Euler $E_n^{(c,\alpha)}(p,q;\lambda)$ and Apostol-Genocchi polynomials $G_n^{(c,\alpha)}(p,q;\lambda)$ of order α, as follows

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{pt}\cos(qt) = \sum_{n=0}^{\infty} B_n^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{n!}, \tag{4}$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{pt}\cos(qt) = \sum_{n=0}^{\infty} E_n^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{n!}, \tag{5}$$

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{pt}\cos(qt) = \sum_{n=0}^{\infty} G_n^{(c,\alpha)}(p,q;\lambda)\frac{t^n}{n!}. \tag{6}$$

In this article as a motivation of these works, we introduced and investigated the new generalization of the Apostol Hermite-Genocchi polynomials and some basic properties are derived.
On the other side, discrete data or count data comprises of observations are common medical sciences and epidemiology. Nakagawa and Osaki [21] were the first to study a discrete life time distribution. Salvia and Bollinger [26] introduced basic results about discrete reliability and illustrated them with the simple discrete life distributions. The characterization of discrete distributions has been studied by Roy et al. [25]. Gupta et al. [9] introduced classes of discrete distributions with increasing failure rate. Nair and Asha [20] derived some classes of multivariate life distributions in discrete time.
According to that, we introduced new multivariate distribution which combined the features of the class of generalized power series distribution and new generalized polynomials.
The rest of this paper is organized as follows. In Section 2, we introduce a new generalization of the Apostol Hermite-Genocchi polynomials and their properties. In Section 3, we give implicit summation formulas for this generalization. In Section 4, we consider some statistical applications of the new family in probability distribution theory and reliability.

2. A new generalization of the Apostol Hermite-Genocchi polynomials

Definition 2.1.
Let a, b and c be positive integers with the condition $a \neq b$. A new generalization of the Apostol Hermite-Genocchi polynomials ${}_H\boldsymbol{M}_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r)$ for nonnegative integer n is defined by means of the generating function

$$\sum_{n=0}^{\infty} {}_H\boldsymbol{M}_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!} = \frac{(-1)^r t^{rk} 2^{r(1-k)}}{\prod\limits_{i=0}^{r-1}(\alpha_i b^t - a^t)}\, c^{xt+h(t,y)},$$

$$\left(|\, t \,|<|\, \frac{\log(\alpha_i)}{\log(\frac{b}{a})}\,|;\ a,b,c\in\mathbb{R}^+;\ \alpha_i \neq 1;\ \forall\ i=0,1,\cdots,r-1\right), \tag{7}$$

where $k \in \mathbf{N}_0$; $r \in \mathbb{C}$; $\bar{\alpha}_r = (\alpha_0, \alpha_1, \cdots, \alpha_{r-1})$ is a sequence of complex numbers.

Setting $h(t,y) = yt^2$ in (7), we get the following definition.

Definition 2.2.
Let a, b and c be positive integers with the condition $a \neq b$. A new generalization of the Apostol Hermite-Genocchi polynomials ${}_HM_n^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r)$ for nonnegative integer n is defined by means of the generating function

$$\sum_{n=0}^{\infty} {}_HM_n^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r)\frac{t^n}{n!} = \frac{(-1)^r t^{rk} 2^{r(1-k)}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)}\ c^{xt+yt^2},$$

$$\left(|\, t \,|<|\, \frac{\log(\alpha_i)}{\log(\frac{b}{a})}\,|;\ a, b, c \in \mathbb{R}^+;\ \alpha_i \neq 1;\ \forall\ i = 0, 1, \cdots, r-1\right), \tag{8}$$

where $k \in \mathbf{N}_0$; $r \in \mathbb{C}$; $\bar{\alpha}_r = (\alpha_0, \alpha_1, \cdots, \alpha_{r-1})$ is a sequence of complex numbers.

Remark 2.3. *When $\alpha_i \neq 1,\ i = 0, 1, ..., r-1$ in* (7) *and* (8)*, the generalized Apostol-Genocchi polynomials should tacitly be restricted to non negative integer values.*

Remark 2.4.
If we set $x = y = 0$ in (8)*, then we obtain the new unified generalized Apostol Hermite-Genocchi numbers, as*

$${}_HM_n^{(r)}(0, 0; a, b, 1; \bar{\alpha}_r) = {}_HM_n^{(r)}(a, b; \bar{\alpha}_r).$$

Remark 2.5. *By comparing the generating function in* (7) *and* (4)*,* (5) *and* (6)*, we obtain the following relationships*

1. ${}_H\boldsymbol{M}_n^{(r)}(x, y; 1, 1, e, e, \lambda) = (-1)^r B_n^{(c,r)}(x, y; \lambda)$.
2. ${}_H\boldsymbol{M}_n^{(r)}(x, y; 0, 1, e, e, -\lambda) = E_n^{(c,r)}(x, y; \lambda)$.
3. ${}_H\boldsymbol{M}_n^{(r)}(x, y; 1, 1, e, e, -\lambda) = (2)^{-r} G_n^{(c,r)}(x, y; \lambda)$.

Moreover, the family of polynomials ${}_H\mathbf{M}_n^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r)$ and ${}_HM_n^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r)$ includes well known polynomials, some of which we list below:

1. ${}_H\mathbf{M}_n^{(r)}(x, y; 1, a, b, c; -\lambda) = 2^{-r}\ G_n^{(r)}(x, y; a, b, c; \lambda)$.
   (Generalized Apostol-Genocchi polynomials, see [1])
2. ${}_HM_n^{(r)}(x, y; 1, a, b, c; -\lambda) = 2^{-r}\ {}_HG_n^{(r)}(x, y; a, b, c; \lambda)$.
   (Generalized Apostol Hermite-Genocchi polynomials of order r, see [8])
3. ${}_HM_n^{(r)}(x, 0; k, 1, e, e; \bar{\alpha}_r) = (-1)^{-r} M_n^{(r)}(x; k, \bar{\alpha}_r)$.
   (Unified family of generalized Apostol-Bernoulli, Euler and Genocchi polynomials, see [7]).

For the other known polynomials which are related with the family ${}_HM_n^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r)$, we refer to [8], [13], [28], [33] and [34].
The family of polynomials ${}_HM_n^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r)$ possess the following interesting properties. These are stated as Theorems 2.6 and 2.7 below.

Theorem 2.6.
Let $a,\ b$ and c be positive integers with $a \neq b$. For $x \in \mathbf{R}$ and $n \geq 0$. Then we have

$${}_HM_n^{(r+\beta)}(x + y, z + u; k, a, b, c; \bar{\alpha}_r) = \sum_{k=0}^{\infty}\binom{n}{k} {}_HM_n^{(r)}(y, z; k, a, b, c; \bar{\alpha}_r) {}_HM_n^{(\beta)}(x, u; k, a, b, c; \bar{\alpha}_r). \tag{9}$$

Proof. From (8), we get

$$\sum_{n=0}^{\infty} {}_HM_n^{(r+\beta)}(x + y, z + u; k, a, b, c; \bar{\alpha}_r) = \frac{(-1)^{r+\beta} 2^{(r+\beta)(1-k)} t^{(r+\beta)k}}{\prod_{i=0}^{(r+\beta)-1}(\alpha_i b^t - a^t)} c^{(x+y)t+(z+u)t^2}$$

$$= \left(\sum_{n=0}^{\infty} {}_{H}M_{n}^{(r)}(y, z; k, a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} {}_{H}M_{n}^{(\beta)}(x, u; k, a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} \right).$$

By using Cauchy product and comparing the coefficients of t^n on both sides, yields (9). □

Theorem 2.7.
Let a, b and c be positive integers with the rule $a \neq b$ for $x \in \mathbf{R}$ and $n \geq 0$. Then we have

$$ {}_{H}M_{n}^{(r)}(x + z, y; k, a, b, c; \bar{\alpha}_r) = \sum_{m=0}^{n} \binom{n}{m} {}_{H}M_{n-m}^{(r)}(z; k, a, b, c; \bar{\alpha}_r) H_m(x, y, c). \tag{10}$$

Proof. From (8), we get

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{H}M_{n}^{(r)}(x + z, y; k, a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} &= \frac{(-1)^r 2^{r(1-k)} t^{rk}}{\prod\limits_{i=0}^{r-1} (\alpha_i b^t - a^t)} c^{(x+z)t + yt^2} \\ &= \sum_{n=0}^{\infty} {}_{H}M_{n}^{(r)}(z; k, a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x, y, c) \frac{t^m}{m!}, \end{aligned}$$

using Cauchy product and equating the coefficients of t^n on both sides of the last equation, yields (10). □

3. Implicit summation formulas on the generalized Apostol Hermite-Genocchi polynomials

Theorem 3.1.
Let a, b and c be positive integers with $a \neq b$ for $x,\ y \in \mathbf{R}$ and $n \geq 0$. Then we have

$$ {}_{H}M_{n}^{(r)}(x + z, y; k, a, b, c; \bar{\alpha}_r) = \sum_{l=0}^{n} \binom{n}{l} z^{n-l} (\ln c)^{n-l} {}_{H}M_{n}^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r). \tag{11}$$

Proof. From Eq. (8), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_{H}M_{n}^{(r)}(x + z, y; k, a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} &= \frac{(-1)^r 2^{r(1-k)} t^{rk}}{\prod\limits_{i=0}^{r-1} (\alpha_i b^t - a^t)} c^{(x+z)t + yt^2} \\ &= \sum_{i=0}^{\infty} {}_{H}M_{i}^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r) \frac{t^i}{i!} \sum_{l=0}^{\infty} \frac{(z\ t\ \log c)^l}{l!}, \end{aligned}$$

using Cauchy product and equating the coefficients of t^n on both sides of the last equation, yields (11). □

Theorem 3.2.
Let a, b and c positive integers, by $a \neq b$ then, for $x,\ y \in \mathbf{R}$ and $n,\ m \geq 0$, we have

$$ {}_{H}M_{n+m}^{(r)}(z, y; k, a, b, c; \bar{\alpha}_r) = \sum_{s=0}^{m} \sum_{\ell=0}^{n} \binom{m}{s} \binom{n}{\ell} (\log c)^{s+\ell} (z - x)^{s+\ell}\ {}_{H}M_{n+m-s-\ell}^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r). \tag{12}$$

Proof. Replacing t by $t + u$ and rewrite the generating function (8) as the following

$$\frac{(-1)^r (t + u)^{rk} 2^{r(1-k)}}{\prod\limits_{i=0}^{r-1} (\alpha_i b^{t+u} - a^{t+u})} c^{x(t+u) + y(t+u)^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_{H}M_{n+m}^{(r)}(x, y; k, a, b, c; \bar{\alpha}_r) \frac{t^n}{n!} \frac{u^m}{m!}$$

$$\frac{(-1)^r(t+u)^{rk}2^{r(1-k)}}{\prod_{i=0}^{r-1}(\alpha_i b^{t+u}-a^{t+u})}\,c^{y(t+u)^2}=c^{-x(t+u)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}. \tag{13}$$

Replacing x by z in Eq. (13), we have

$$c^{-x(t+u)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}=c^{-z(t+u)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(z,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$c^{(z-x)(t+u)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(z,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}. \tag{14}$$

By applying [see Pathan and khan [24], p. 52]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}$$

to $c^{(z-x)(t+u)}$ in Eq. (14), we get

$$\sum_{N=0}^{\infty}(\log c)^N\frac{[(z-x)(t+u)]^N}{N!}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(z,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}.$$

$$\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\log c)^{s+\ell}(z-x)^{s+\ell}}{\ell!s!}\,t^\ell\,u^s\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(z,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}. \tag{15}$$

Replacing n by $n-\ell$ and m by $m-s$ in Eq. (15), we have

$$\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\log c)^{s+\ell}(z-x)^{s+\ell}}{\ell!s!}\,t^\ell\,u^s\sum_{n=\ell}^{\infty}\sum_{m=s}^{\infty}{}_HM^{(r)}_{n+m-\ell-s}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^{n-\ell}}{(n-\ell)!}\frac{u^{m-s}}{(m-s)!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(z,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}.$$

By using the lemma in [Srivastava [28], p. 100], we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\log c)^{s+\ell}(z-x)^{s+\ell}}{\ell!s!}\;{}_HM^{(r)}_{n+m-\ell-s}(x,y;k,a,b,c;\bar{\alpha}_r)\right)\frac{t^n}{(n-\ell)!}\frac{u^m}{(m-s)!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HM^{(r)}_{n+m}(z,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}\frac{u^m}{m!}.$$

Comparing the coefficients of $t^n\ u^m$, yields (12). □

Theorem 3.3.
Let a, b and c be positive integers with the rule $a \neq b$ for $x,\ y \in \mathbf{R}$ and $n \geq 0$, we have

$${}_HM^{(r)}_n(x,y;k,a,b,c;\bar{\alpha}_r)=\sum_{m=0}^{n}\binom{n}{m}{}_HM^{(r)}_{n-m}(a,b;\bar{\alpha}_r)H_m(x,y,c). \tag{16}$$

Proof. From Eq. (8), we have

$$\sum_{n=0}^{\infty} {}_H M_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!} = \frac{(-1)^r 2^{r(1-k)} t^{rk}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{(x)t+yt^2}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} {}_H M_{n-m}^{(r)}(a,b;\bar{\alpha}_r) H_m(x,y,c)\right)\frac{t^n}{n!}.$$

Thus, equating the coefficients of t^n on both sides of last equation, yields (16). □

Theorem 3.4.
For arbitrary real or complex parameter (r), the following implicit summation formula holds true

$$ {}_H M_n^{(r)}(x+1,y;k,a,b,c;\bar{\alpha}_r) = \sum_{k=0}^{n}\binom{n}{k}(\log c)^{n-k}\ {}_H M_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r). \tag{17}$$

Proof. From (8), we have

$$\sum_{n=0}^{\infty} {}_H M_n^{(r)}(x+1,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!} = \frac{(-1)^r 2^{r(1-k)} t^{rk}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{(x+1)t+yt^2}$$
$$= \left(\sum_{k=0}^{\infty} {}_H M_k^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}(\log c)^n\frac{t^n}{n!}\right).$$

By using Cauchy product and equating the coefficients of t^n on both sides of last equation, yields (17). □

Theorem 3.5.
For arbitrary real or complex parameter (r), the following implicit summation formula holds true

$$\sum_{k=0}^{n}\binom{n}{k}(\log ab)^k r^k {}_H M_n^{(r)}(-x,y;k,a,b,c;\bar{\alpha}_r) = (-1)^{n-rk} {}_H M_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r). \tag{18}$$

Proof. From (8), we have

$$\sum_{n=0}^{\infty}[1-(-1)^n] {}_H M_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!} = \frac{(-1)^r 2^{r(1-k)t^{rk}}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{xt+yt^2} - \frac{(-1)^r 2^{r(1-k)(-t)^{rk}}}{\prod_{i=0}^{r-1}(\alpha_i b^{-t} - a^{-t})} c^{-xt+yt^2}$$

$$= c^{yt^2}\left[\frac{(-1)^r 2^{r(1-k)t^{rk}}}{\prod_{i=0}^{r-1}(\alpha_i b^t - a^t)} c^{xt} - (-1)^{rk}(ab)^{rt}\frac{(-1)^r 2^{r(1-k)(t)^{rk}}}{\prod_{i=0}^{r-1}(\alpha_i a^t - b^t)} c^{-xt}\right]$$
$$= \sum_{n=0}^{\infty} {}_H M_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!} - (-1)^{rk}\left(\sum_{k=0}^{\infty}(\log a\ b)^k\frac{rt^k}{k!}\right)\sum_{n=0}^{\infty} {}_H M_n^{(r)}(-x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} {}_H M_n^{(r)}(x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!} - (-1)^{rk}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(\log a\ b)^k r^k\ {}_H M_{n-k}^{(r)}(-x,y;k,a,b,c;\bar{\alpha}_r)\frac{t^n}{n!}.$$

Equating the coefficients of t^n on both sides, yields (18). □

Theorem 3.6.
Let a, b and c be positive integers by $a \neq b$. For $x, y \in \mathbb{R}$ and $n \geq 0$. Then we have

$$ {}_HM_n^{(r)}(x+r, y; k, a, b, c; \bar{\alpha}_r) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} y^k (\log c)^k {}_HM_{n-2k}^{(r)}(x; k, \frac{a}{c}, \frac{b}{c}, c; \bar{\alpha}_r), \tag{19} $$

where [.] is Gauss notation, and represents the maximum integer which does not exceed a number in the square brackets.

Proof. From (8), we have

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_HM_n^{(r)}(x+r, y; k, a, b, c; \bar{\alpha}_r)\frac{t^n}{n!} &= \frac{(-1)^r 2^{r(1-k)} t^{rk}}{\prod\limits_{i=0}^{r-1}(\alpha_i(\frac{b}{c})^t - (\frac{a}{c})^t)} c^{xt} c^{yt^2} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} (\log c)^k y^k M_{n-2k}^{(r)}(x; k, \frac{a}{c}, \frac{b}{c}, c; \bar{\alpha}_r)\frac{t^n}{n!}. \end{aligned} $$

Equating the coefficients of t^n, yields (19). □

Theorem 3.7.
Explicit formula of the generalized Apostol Hermite-Genocchi polynomials is given by

$$ {}_HM_n^{(r)}(\alpha, \beta; \bar{\alpha}_r) = 2^r \sum_{x_1,\ldots,x_n=0} \prod_{i=1}^{r} (\alpha_{_{i-1}})^{x_i} \sum_{k=0}^{[\frac{n}{m}]} \frac{\beta^k n!}{k!(n-mk)!}(\alpha + X)^{n-mk}, \tag{20} $$

where $X = x_1 + x_2 + \ldots + x_r$.

Proof. Put $x = \alpha,\ \ y = \beta,\ \ b = c = e,\ \ a = 1,\ \ k = 0\ \ and\ \ h(t, \beta) = \beta t^m\ \ in$ (7) we have

$$ \sum_{n=0}^{\infty} {}_HM_n^{(r)}(\alpha, \beta; \bar{\alpha}_r)\frac{t^n}{n!} = 2^r \sum_{x_1,\ldots,x_r=0} \prod_{i=1}^{r} (\alpha_{_{i-1}})^{x_i}\ e^{(x_1+\ldots+x_r+\alpha)t+\beta t^m}. $$

Let $x_1 + x_2 + ... + x_r = X$,we have:

$$ \sum_{n=0}^{\infty} {}_HM_n(r)(\alpha, \beta; \bar{\alpha}_r)\frac{t^n}{n!} = 2^r \sum_{x_1,\ldots,x_r=0} \prod_{i=1}^{r} (\alpha_{_{i-1}})^{x_i}\ e^{(X+\alpha)t+\beta t^m}, $$

but the generating function of the generalized Hermite polynomial is given by

$$ e^{\alpha t+\beta t^m} = \sum_{n=0}^{\infty} H_{n,m}(\alpha, \beta)\frac{t^n}{n!}, $$

then

$$ \sum_{n=0}^{\infty} {}_HM_n^{(r)}(\alpha, \beta; \bar{\alpha}_r)\frac{t^n}{n!} = 2^r \sum_{n=0}^{\infty}[\sum_{x_1,\ldots,x_r=0} \prod_{i=1}^{r} (\alpha_{_{i-1}})^{x_i}\ H_{n,m}(\alpha + X, \beta)]\frac{t^n}{n!}. $$

Equating the coefficients t^n on both sides and form the definition of generalized Hermite polynomials, we have

$$ H_{n,m}(\alpha + X, \beta) = \sum_{k=0}^{[\frac{n}{m}]} \frac{\beta^k n!}{k!(n-mk)!}(\alpha + X)^{n-mk}, $$

this yields (20). □

4. Application involving probability distribution and some concepts of reliability

Definition 4.1.
Let $X_1, X_2, ..., X_r$ be nonnegative random variable. Then $\underline{\mathbf{X}} = (X_1, X_2, ..., X_r)$ is said to be generalized Hermite-Genocchi distribution, if its probability mass function is

$$P(\underline{\mathbf{X}}) = \mathbf{B} \prod_{i=1}^{r} (\alpha_{i-1})^{x_i} \; H_{n,m}\left(\sum_{i=1}^{r} x_i + \gamma, \beta\right), \qquad \beta, \gamma \geq 0;\; r \geq 1,\; m \in N \tag{21}$$

where normalizing constant is given by

$$\frac{1}{\mathbf{B}} = {}_H M_n^{(r)}(\gamma, \beta; \bar{\alpha}_r) = \sum_{\ell_1, \ell_2, ..., \ell_r = 0}^{\infty} \prod_{i=1}^{r} (\alpha_{i-1})^{\ell_i} H_{n,m}\left(\sum_{i=1}^{r} \ell_i + \gamma, \beta\right),$$

and

$$H_{n,m}(\sum_{i=1}^{r} x_i + \gamma, \beta) = \sum_{k=0}^{[\frac{n}{m}]} \frac{\beta^k}{k!} \frac{n!}{(n-mk)!} \left(\sum_{i=1}^{r} x_i + \gamma\right)^{n-mk}.$$

${}_H M_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)$ is convergent and positive for $\bar{\alpha}_r = (\alpha_0, \alpha_1, ..., \alpha_{r-1})$, $0 < \alpha_r < 1$ and the distribution is denoted by $GHG[m; \bar{\alpha}_r; \gamma, \beta]$.

Definition 4.2.
Let $m = 2$ in (21), then $\underline{X}$ is said to have Hermite-Genocchi distribution, if its probability mass function is

$$P(\underline{\mathbf{X}}) = \mathbf{B} \prod_{i=1}^{r} (\alpha_{i-1})^{X_i} \; H_{n,2}\left(\sum_{i=1}^{r} x_i + \gamma, \beta\right), \tag{22}$$

where

$$H_{n,2}(\sum_{i=1}^{r} x_i + \gamma, \beta) = \sum_{k=0}^{[\frac{n}{2}]} \frac{\beta^k}{k!} \frac{n!}{(n-2k)!} \left(\sum_{i=1}^{r} x_i + \gamma\right)^{n-2k}.$$

The distribution is denoted by $HG[\bar{\alpha}_r; \gamma, \beta]$.

Result 4.1: Let $\underline{X} = (X_1, ..., X_r)$, following the generalized Hermite-Genocchi distribution, then the probability generating function of $\underline{X}$ is

$$G_{\underline{X}}(t) = \mathbf{B}\; {}_H M_n^{(r)}(\gamma, \beta; \mathbf{t}\bar{\alpha}_r) \tag{23}$$

Proof. Form the definition of the probability generating function, we have

$$G_{\underline{X}}(t) = E(\mathbf{t}^{\underline{X}}) = \sum_{x_1, x_2, ..., x_r = 0} P(x_1, x_2, ..., x_r)\; \mathbf{t}^{\underline{X}},$$

where $\mathbf{t} = (t_1, t_2, ..., t_r)$, this yield (23). □

Result 4.2: The moment generating function of the generalized Apostol Hermite-Genocchi distribution is

$$M_{\underline{X}}(t) = \mathbf{B}\; {}_H M_n^{(r)}(\gamma, \beta; e^{\mathbf{t}}\bar{\alpha}_r) \tag{24}$$

Result 4.3: The $\ell - th$ factorial moments $\mu_{[\ell]}$ with moment generating function $M_{\underline{X}}(t)$ of the generalized Hermite-Genocchi distribution is

$$\mu_{[\ell]} = \mathbf{B} \sum_{x_1, x_2, ..., x_r = 0}^{\infty} (x_i)_{\underline{\ell}} \prod_{i=1}^{r} (\alpha_{i-1})^{x_i} \; H_{n,m}\left(\sum_{i=1}^{r} x_i + \gamma, \beta\right). \tag{25}$$

Result 4.4: The $\ell-th$ moments $\acute{\mu}_\ell$ with moment generating function $M_{\underline{X}}(t)$ of the generalized Hermite-Genocchi distribution is

$$\acute{\mu}_\ell = \mathbf{B} \sum_{x_1,x_2,\ldots,x_r=0}^{\infty} x_i^\ell \prod_{i=1}^{r}(\alpha_{i-1})^{x_i} \ H_{n,m}\left(\sum_{i=1}^{r} x_i+\gamma,\beta\right). \tag{26}$$

Result 4.5: The mean and variance of the generalized Apostol Hermite-Genocchi distribution with $\ell-th$ moments $\acute{\mu}_\ell$ is

$$E(\underline{X}) = \mathbf{B} \sum_{x_1,x_2,\ldots,x_r=0}^{\infty} x_i \prod_{i=1}^{r}(\alpha_{_{i-1}})^{x_i} \ H_{n,m}\left(\sum_{i=1}^{r} x_i+\gamma,\beta\right). \tag{27}$$

$$\begin{aligned} Var(\underline{X}) &= \left[\mathbf{B} \sum_{x_1,x_2,\ldots,x_r=0}^{\infty} x_i^2 \prod_{i=1}^{r}(\alpha_{_{i-1}})^{x_i} \ H_{n,m}\left(\sum_{i=1}^{r} x_i+\gamma,\beta\right)\right] \\ &- \left[\mathbf{B} \sum_{x_1,x_2,\ldots,x_r=0}^{\infty} x_i \prod_{i=1}^{r-1}(\alpha_{_{i-1}})^{x_i} \ H_{n,m}\left(\sum_{i=1}^{r} x_i+\gamma,\beta\right)\right]^2. \end{aligned} \tag{28}$$

Lemma 4.3. *Let $X_i \sim GHG(m,\alpha_i,\gamma,\beta)$, $i=1,2,\ldots,r-1$ Then the marginal joint cumulative distribution function*

$$P(X_i \le x_i) = 1 - \mathbf{B}\ (\alpha_{i-1})^{x_i}\ {}_HM_n^{(r)}(\gamma+x_i+1,\beta;\bar{\alpha}_{_r}) \tag{29}$$

Proof. Since

$$\begin{aligned} P(X_i \le x_i) &= 1-P(X_i > x_i) \\ &= 1-\mathbf{B} \sum_{\ell_i=x_i+1}^{\infty} \sum_{\substack{\ell_1,\ldots,\ell_{i-1} \\ \ell_{i+1},\ldots,\ell_r=0}}^{\infty} (\alpha_0)^{\ell_1}(\alpha_1)^{\ell_2}\ldots(\alpha_{i-1})^{\ell_i}\ldots(\alpha_{r-1})^{\ell_r} H_{n,m}\left(\sum_{i=1}^{r}\ell_i+\gamma,\beta\right). \end{aligned}$$

If setting $\ell_i - x_i - 1 = \ell$, then

$$P(X_i \le x_i) = 1-\mathbf{B} \sum_{\substack{\ell,\ell_1,\ldots,\ell_{i-1} \\ \ell_{i+1},\ldots,\ell_r=0}}^{\infty} (\alpha_0)^{\ell_1}(\alpha_1)^{\ell_2}\ldots(\alpha_{i-1})^{x_i+\ell_i}+1\ldots(\alpha_{r-1})^{\ell_r} H_{n,m}\left(\sum_{i=1}^{r}\ell_i+\gamma+x_i+1,\beta\right).$$

From (21), we obtain (29). □

Theorem 4.4.
If $X_1,X_2,\ldots,X_r$ are mutually independent where $X_i \sim GHG(m,\alpha_i,\gamma,\beta), i=1,2,\ldots,r-1$. Then the multivariate cumulative distribution function is given by

$$P(X_1 \le x_1, X_2 \le x_2,\ldots,X_r \le x_r) = \prod_{i=1}^{r}\left(1-\mathbf{B}\ (\alpha_{i-1})^{x_i}\ {}_HM_n^{(r)}(\gamma+x_i+1,\beta;\bar{\alpha}_{_r})\right). \tag{30}$$

Proof. Since $X_1,X_2,\ldots,X_r$ are mutually independent, hence

$$P(X_1 \le x_1, X_2 \le x_2,\ldots,X_r \le x_r) = \prod_{i=1}^{r}\left(1-P(X_i > x_i)\right).$$

From (29), we obtain (30). □

4.1. Reliability concepts for GHG distributions

4.1.1. Multivariate reliability function

Theorem 4.5.
Let x $= (x_1, x_2, ..., x_r) \in \mathbf{R}_+^r$ *representing the lifetimes of* $r-$*component system with the multivariate reliability function, then*

$$R(\mathrm{x}) = \mathbf{B} \prod_{i=1}^{r} (\alpha_{i-1})^{x_i} \ {}_HM_n^{(r)}(\gamma + x_1 + x_2 + ... + x_r, \beta; \bar{\alpha}_r). \tag{31}$$

Proof. From the definition of the multivariate reliability function, see [20] and (21)

$$\begin{aligned} R(\mathrm{x}) &= P(X_1 \geq x_1, X_2 \geq x_2, ..., X_r \geq x_r) \\ &= \sum_{m_1 \geq x_1} \sum_{m_2 \geq x_2} ... \sum_{m_r \geq x_r} P(X_1 = m_1, X_2 = m_2, ..., X_r = m_r), \end{aligned}$$

hence we obtain (31). □

Theorem 4.6. $R(\mathrm{x})$ *is said to be*

i) *Multivariate new better than used (Multivariate new worse than used) MNBU (MNWU) if*

$${}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r) {}_HM_n^{(r)}(\mathbf{x_r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r) \leq (\geq) {}_HM_n^{(r)}(\mathbf{x_r} + \gamma, \beta; \bar{\alpha}_r) {}_HM_n^{(r)}(\acute{t} + \gamma, \beta; \bar{\alpha}_r) \tag{32}$$

ii) *Multivariate new better than used in expectation (Multivariate new worse than used in expectation) MNBUE (MNWUE) if*

$$\begin{aligned} &{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r) \sum_{t_1, t_2, ..., t_r = 0} \prod_{i=0}^{r-1} \alpha_i \ {}_HM_n^{(r)}(\mathbf{x_r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r) \leq (\geq) \\ &\qquad {}_HM_n^{(r)}(\mathbf{x_r} + \gamma, \beta; \bar{\alpha}_r) \sum_{t_1, t_2, ..., t_r = 0} \prod_{i=0}^{r-1} \alpha_i \ {}_HM_n^{(r)}(\acute{t} + \gamma, \beta; \bar{\alpha}_r), \end{aligned} \tag{33}$$

where $\mathbf{x_r} = x_1 + x_2 + ... + x_r$, $\acute{t} = t_1 + t_2 + ... + t_r$.

Proof. If

$$\begin{aligned} {}_HM_n^{(r)}(\mathbf{x_r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r) &\leq \frac{{}_HM_n^{(r)}(\mathbf{X_r} + \gamma, \beta; \bar{\alpha}_r) {}_HM_n^{(r)}(\acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)} \\ \frac{{}_HM_n^{(r)}(\mathbf{x_r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)} &\leq \frac{{}_HM_n^{(r)}(\mathbf{x_r} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)} \frac{{}_HM_n^{(r)}(\acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)} \\ \prod_{i=1}^{r-1} (\alpha_{i-1})^{x_i + t_i} \frac{{}_HM_n^{(r)}(\mathbf{x_r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)} &\leq \prod_{i=1}^{r-1} (\alpha_{i-1})^{x_i} \frac{{}_HM_n^{(r)}(\mathbf{x_r} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)} \prod_{i=1}^{r-1} (\alpha_{i-1})^{t_i} \frac{{}_HM_n^{(r)}(\acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM_n^{(r)}(\gamma, \beta; \bar{\alpha}_r)} \end{aligned}$$

hence, we get

$$R(x_1 + t_1, x_1 + t_1, ..., x_1 + t_1) \leq R(x_1, x_2, ..., x_r) R(t_1, t_2, ..., t_r)$$

From the definition of the Multivariate new better used,then $R(\mathrm{x})$ is MNBU.
Similarly, from the definition of MNBUE (multivariate new better used in expectation), then $R(\mathrm{x})$ is MN-BUE. □

4.1.2. Multivariate hazard rate function

The multivariate hazard rate function is defined as, see [20]

$$h(x) = (h_1(x), h_2(x), ..., h_r(x))$$

$$\begin{aligned} h_i &= P(X_i = x_i | X \geq x) \\ &= 1 - \frac{R(x_1, x_2, ..., x_{i-1}, x_i + 1, x_{i-1}, ..., x_r)}{R(x_1, x_2, ..., x_r)}, \end{aligned}$$

where $x = (x_1, x_2, ..., x_r) \in \mathbf{R}^r_+$, so we obtain the following theorems.

Theorem 4.7.
The multivariate hazard rate function of GHG distribution is given by

$$h_i(x) = 1 - \alpha_{i-1} \frac{{}_HM^{(r)}_n(x_1 + x_2 + ... + x_{i-1} + x_i + 1 + x_{i-1} + ... + x_r + \gamma, \beta; \bar{\alpha}_r)}{{}_HM^{(r)}_n(x_1 + x_2 + ... + x_r + \gamma, \beta; \bar{\alpha}_r)}. \tag{34}$$

Theorem 4.8.
Let $\grave{\mathbf{X}}_\mathbf{r} = (x_1 + x_2 + ... + x_{i-1} + x_i + 1 + x_{i+1} + ... + x_r)$, $\mathbf{x_r} = (x_1 + x_2 + ... + x_r)$ *and* $\acute{t} = t_1 + t_2 + ... + t_r$*, then the following statement about GHG distribution holds.*
The GHG distribution is multivariate increasing hazard rate (multivariate decreasing hazard rate) MIHR (MDHR) iff

$$\frac{{}_HM^{(r)}_n(\grave{\mathbf{X}}_\mathbf{r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM^{(r)}_n(\mathbf{x_r} + \mathbf{t} + \gamma, \beta; \bar{\alpha}_r)} \leq (\geq) \frac{{}_HM^{(r)}_n(\grave{\mathbf{X}}_\mathbf{r} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM^{(r)}_n(\mathbf{x_r} + \gamma, \beta; \bar{\alpha}_r)}. \tag{35}$$

Proof. If

$$\frac{{}_HM^{(r)}_n(\grave{\mathbf{X}}_\mathbf{r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM^{(r)}_n(\mathbf{x_r} + \mathbf{t} + \gamma, \beta; \bar{\alpha}_r)} \leq \frac{{}_HM^{(r)}_n(\grave{\mathbf{X}}_\mathbf{r} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM^{(r)}_n(\mathbf{x_r} + \gamma, \beta; \bar{\alpha}_r)},$$

then

$$1 - \alpha_{i-1} \frac{{}_HM^{(r)}_n(\grave{\mathbf{X}}_\mathbf{r} + \acute{t} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM^{(r)}_n(\mathbf{x_r} + \mathbf{t} + \gamma, \beta; \bar{\alpha}_r)} \geq 1 - \alpha_{i-1} \frac{{}_HM^{(r)}_n(\grave{\mathbf{X}}_\mathbf{r} + \gamma, \beta; \bar{\alpha}_r)}{{}_HM^{(r)}_n(\mathbf{x_r} + \gamma, \beta; \bar{\alpha}_r)},$$

hence

$$h_i(x_1 + t_1, ..., x_i + t_i, ..., x_r + t_r) \geq h_i(x_1, x_2, ..., x_r).$$

From the definition of the multivariate increasing Hazard rate, we have GHG distribution is MIHR (multivariate increasing Hazard rate). □

Acknowledgements

The authors would like to thank the referee(s) for their comments, which improved the previous version of this article.

References

[1] S. Araci, W. A. Khan, M. Acikgoz, C. Özel, P. Kumam, A new generalization of Apostol type Hermite-Genocchi polynomials and its applications, SpringerPlus 5(1) (2016) 1–17.
[2] E. T. Bell, Exponential polynomials, Annals of Mathematics 35 (1934) 258–277.
[3] G. Dattoli, S. Lorenzutta, C. Cesarano, Finite sums and generalized forms of Bernoulli polynomials, Rendicont di Mathematic 19 (1999) 385–391.

[4] G. B. Djordjević, Generalized Hermite polynomials, Publication de l'Instut Mathematique 53 (1993) 69–72.
[5] G. B. Djordjević, On some properties of generalized Hermite polynomials, Fibonacci Quart 34 (1996) 2–6.
[6] G. B. Djordjević, G. V. Milovanović, Polynomials related to the generalized Hermite polynomials, Facta Universitatis Series Mathematics and Informatics 8 (1993) 35–43.
[7] B. S. El-Desouky, R. S. Gomaa, A new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, Applied Mathematics and Computation 247 (2014) 695–702.
[8] S. Geboury, B. Kurt, Some relations involving Hermite-based Apostol-Genocchi polynomials, Applied Mathematics Sciences 6 (2012) 4091–4102.
[9] P. L. Gupta, R. C. Gupta, R. C. Tripathi, On the monotonic properties of the discrete failure rates, Journal of Statistical Planning and Inference 65 (1997) 255–268.
[10] D. D. Hanagal, Testing whether the survival function is multivariate new better than used, Statistical Papers 39 (1998) 203–211.
[11] Y. He, S. Araci, H. M. Srivastava, M. Acikgoz, Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials, Applied Mathematics and computation 262 (2015) 31–41.
[12] Y. He, S. Araci, H. M. Srivastava, M. Abdel-Aty, Higher-order convolutions for Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, Mathematics 6 (2018) 1–14.
[13] H. Jolany, H. Sharifi, R. E. Aliklaye, Some results for the Apostol-Genocchi polynomials of higher order, The Bulletin of the Malaysian Mathematical Society Series 2 (2013) 465–479.
[14] Q.-M. Luo, Apostol Euler polynomials of higher order and Gaussian hypergeometric functions, Taiwanese Journal of Mathematics 10 (2006) 917–925.
[15] Q.-M. Luo, Fourier expansions and integral representations for the Genocchi polynomials, Journal of Integer Sequences 12 (2009) 1–9.
[16] Q.-M. Luo, Extension of the Genocchi polynomials and their Fourier expansions and integral representations, Qsaka Journal of Mathematics 48 (2011) 291–309.
[17] Q.-M. Luo, H. M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, Journal of Mathematical Analysis and Applications 308(1) (2005) 290–302.
[18] Q.-M. Luo, H. M. Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, Computers and Mathematics with Applications 51(3) (2006) 631–642.
[19] Q.-M. Luo, H. M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, Applied Mathematics and Computation 217 (2011) 5702–5728.
[20] N. U. Nair, G. Asha, Some classes of multivariate life distribution in discrete time, Journal of Multivariate Analysis 62 (1997) 181–189.
[21] T. Nakagawa, S. Osaki, The discrete Weibull distribution, IEEE Transaction on Reliability 24 (1975) 300–301.
[22] M. A. Pathan, W. A. Khan, Some implicit summation formulas and symmetric identities for the generalizated Hermite-Bernoulli polynomials, Mediterranean Journal of Mathematics 12 (2015a) 679–695.
[23] M. A. Pathan, W. A. Khan, A new class of generalized polynomials associated with Hermite and Euler polynomials, Mediterranean Journal of Mathematics 13 (2015b) 913–928.
[24] M. A. Pathan, W. A. Khan, Some new classes of generalized Hermite-based Apostol-Euler and Apostol-Genocchi polynomials, Fasciculli Mathematici 55 (2015d) 153–170.
[25] D. Roy, R. P. Gupta, Characterizations and model selection through reliability measures in the discrete case, Statistics and Probability Letters 43 (1999) 197–206.
[26] A. A. Salvia, R. C. Bollinger, On discrete hazard functions, IEEE Transactions on Reliability 31 (1982) 458–459.
[27] H. M. Srivastava, A note on a generating function for the generalized Hermite polynomials, Indagationes Mathematicae (Proceedings) 79 (1976) 457–461.
[28] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, In Mathematical Proceedings of the Cambridge Philosophical Society 129 (2000) 77–84.
[29] H. M. Srivastava, Some generalizations and basic(or q-)extensions of the Bernoulli, Euler and Genocchi polynomials, Applied Mathematics and Information Sciences 5 (2011) 390–444.
[30] H. M. Srivastava, G. B. Djordjević, Some generating functions and other properties associated with the generalized Hermite and related polynomials, Integral Transforms and Special Functions 22 (2011) 895–906.
[31] H. M. Srivastava, M. Garg, S. Choudhary, Some new families of the generalized Euler and Genocchi polynomials, Taiwanese Journal of Mathematics 15 (2011) 283–305.
[32] H. M. Srivastava, B. Kurt, Y. Simsek, Some families of Genocchi type polynomials and their interpolation function, Integral Transforms Spec. Funct. 23 (12) (2012) 919–938 see also Corrigendum, Integral Transforms and Special Functions 23 (12) (2012) 939–940.
[33] H. M. Srivastava, M. Masjed-Jamei, M. R. Beyki, A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Gencchi polynomials, Applied Mathematics and Information Sciences 12 (2018) 907–916.
[34] H. M. Srivastava, M. Masjed-Jamei, M. R. Beyki, Some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Gencchi polynomials, Rocky Mountain Journal of Mathematics 49 (2019) 681–697.
[35] H. M. Srivastava, M. A. Özarslan, C. Kaanoğlu, Some generalized Lagrange-Based Apostol-Bernoulli, Apostol-Euler and Apostol-Gencchi polynomials Russian Journal of Mathematical Physics 20 (2013) 110–120.