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Iterative Algorithms for Determining Optimal Solution Set of Interval Linear Fractional Programming Problem

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Abstract. Determining the optimal solution (OS) set of interval linear fractional programming (ILFP) models is generally an NP-hard problem. Few methods have been proposed in this field which have only been able to obtain the optimal value of the objective function. Thus, there is a need for an appropriate method to determine the OS set of the ILFP model. In this paper, we introduce three algorithms to obtain the OS of ILFP. In the first and second algorithms, using the definition of strong and weak feasible solutions, the objective function of ILFP has been transformed to a linear objective function on the largest feasible region (LFR) and we obtain the OS of ILFP. These two algorithms, only introduce one point as the feasible OS. Since ILFP is an interval model, we seek an algorithm, where for the first time a solution set is obtained as the OS set by solving two sub-models. Hence, we transform the ILFP model into two pessimistic and optimistic sub-models, as one is in the smallest feasible region (SFR) and the other on the LFR. We add constraints to the optimistic model to ensure that the OS set is feasible. Then, we introduce pessimistic and modified optimistic model (PMOM) algorithm. In this algorithm, each PMOM is solved separately. The OSs obtained from these two models give the OS set so that this OS set is feasible. Note that the union of feasible OSs obtained from the proposed algorithms will be a more complete feasible OS set.

1. Introduction

Linear programming and its applications are touchable in various branches of human activities, especially economics. Since all real-world problems can not be incorporated into linear formats, the linear fractional programming problem in the 1960s and 1970s has received much attention from scholars and researchers. Also, academic studies of fractional programming have begun given the need to develop more efficient models for solving real-world problems, with a variety of methods proposed to solve such problems [9, 12, 16].

In the new branches of optimization and research in operations, investigating and solving various categories of fractional programming has become important. Increasing profitability has mean while been one of the major concerns of human activities, including humanities, economics, etc. Most of the time, as the mathematical model of profitability is taken into account, the result is the optimization of a fraction which is the ratio of the activity output to the input. For example, in many activities, optimization of the

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ratios of inventory / capital, real capital / required capital, foreign loans / total loans,... is a major goal in this regard.

Human beings routinely face problems where optimization of an objective over different constraints is required. For this purpose, various algorithms and highly efficient methods have been devised. The most important ways for solving such problems are have been tested by different researchers. In 1962, Charnes and Cooper [6] found that by changing a non-linear variable and adding a new constraint, a linear fractional programming problem can be reduced to a linear model. On the other hand, in 1967, Dinkelbach [7] transformed the linear fractional problem into a linear programming problem using a parametric method without changing the feasible region.

The inaccuracies that naturally appear in the input data of real-life problems can be manifested in a variety of ways. Thus, the main methods for the expression of inaccuracy and data ambiguity include usage of interval numbers or fuzzy numbers.

In the interval linear programming problem, many researchers have worked in this field and have proposed several methods, including Allahdadi et al. presented various methods for solving these problems such that the obtained OS was feasible [2–4, 13].

Given the importance of linear fractional programming problems with data uncertainty, many researchers have focused their attention on investigating and solving these problems. Based on the fuzzy perspective, Veeramani and Sumathi [19] used α-cut for fuzzy parameters of the objective function and r-cut for fuzzy parameters of the constraints to transform the problem of fuzzy linear fractional programming (with triangular fuzzy numbers) into two sub-models. They further created the membership function of the optimal value using the range obtained for the optimal value of the objective function. However, Ebrahimnejad et al. [8] showed that this method has some deficits and does not always lead to non-negative fuzzy optimal solutions. As such, they modified their method to generate non-negative fuzzy optimal solutions and extended it to trapezoidal fuzzy numbers. In another independent method, Nayak and Ojha [15] again used α-cut for fuzzy parameter of the objective functions and r-cut for fuzzy parameters of the constraints for the fuzzy linear fractional multi-objective programming problem (with triangular fuzzy numbers). They transformed each objective of the fuzzy linear fractional programming problem into two sub-models and used the Charnes and Cooper method to linearize the sub-models. Then, based on Taylor's expansion around the solutions obtained by the Charnes and Cooper method and the weighted sum method, they were able to transform the fuzzy linear fractional multi-objective programming problem into two linear models with real data and obtain the range of the optimal value of the objective function.

A part from the above methods, one of the sets of the fractional programming which will be solved in this article is the Interval Linear Fractional Programming (ILFP) for which few solutions exist, where only the range of the optimal value of the objective function can be obtained. Including Hladik [10] who developed a method for calculating the range of optimal value of the objective function of the generalized ILFP so that, in order to calculate each bound, one of the two generalized value real linear fractional programming problems must be solved. Using the convex combination, Borza et al. [5] obtained the optimal value of the objective function of a linear fractional programming problem with coefficients of the interval objective function. Using strong optimization, Jeyakumar et al. [11] developed some dual theorems for the ILFP minimax. Using strong optimization, Sun and Chai [17] applied strong dual to solve the ILFP. Mostafaee and Hladik [14] proposed a method to determine the range of the real value of the objective function of ILFP where some of the coefficients of numerator and denominator had a particular dependence.

A point is feasible if it is applied to all the constraints of the largest region and is optimal if it is the solution of at least one characteristic model. In this paper, three algorithms are proposed to determine the optimal solution (OS) of ILFP as in the first and second algorithms, where a point is introduced as an OS using the definition of weak and strong feasible solutions for the inequalities on the largest feasible region (LFR). Since ILFP is an interval model, we seek an algorithm where for the first time a solution set is obtained as the OS set by solving two sub-models. Hence, we transform the ILFP model into two pessimistic and optimistic sub-models, as one is in the smallest feasible region (SFR) and the other on the LFR. Then, we obtain an OS set by introducing the pessimistic and optimistic model algorithms. According to Example 3.11, some points in this OS set do not apply to LFR. We add constraints to the optimistic model to ensure that the OS set is feasible. Then we introduce an algorithm called pessimistic and modified optimistic model (PMOM) algorithm. However, in this algorithm, we obtain an OS set with the feasibility condition.

2. Symbols and Definitions

This section introduces some of the important theorems and definitions used in this article.

Definition 2.1. *[1] Consider two numbers $a^-, a^+ \in \mathbb{R}$ such that $a^- \leq a^+$. We introduce the real value interval $a^\pm = [a^-, a^+]$ as $\{a \in \mathbb{R} | a^- \leq a \leq a^+\}$. We call values a^-, a^+ as the upper and lower bounds of interval $a^\pm$, respectively. If $a^- = a^+$ in this case $a^\pm$ is degenerate and $a^\pm$ is positive (negative) iff $a^- \geq 0 (a^+ \leq 0)$.*

Definition 2.2. *[1] Consider two matrices $\boldsymbol{A}^- = (a^-_{ij}), \boldsymbol{A}^+ = (a^+_{ij})$ for all i, j with real components such that $a^-_{ij} \leq a^+_{ij}$ for all i, j. We define the interval matrix with real components as $\boldsymbol{A}^\pm = [\boldsymbol{A}^-, \boldsymbol{A}^+]$ such that $\{\boldsymbol{A} \in \mathbb{R}^{m\times n} | \boldsymbol{A}^- \leq \boldsymbol{A} \leq \boldsymbol{A}^+\}$. We call values $\boldsymbol{A}^-, \boldsymbol{A}^+$ the upper and lower bounds of interval $\boldsymbol{A}^\pm$, respectively. A special case of the interval matrix is the interval vector such that $\boldsymbol{b}^- = (b^-_i), \boldsymbol{b}^+ = (b^+_i)$ for all i and $b^-_i \leq b^+_i$ for all i. We define the interval vector with real components as $\boldsymbol{b}^\pm = [\boldsymbol{b}^-, \boldsymbol{b}^+]$ such that $\{\boldsymbol{b} \in \mathbb{R}^m | \boldsymbol{b}^- \leq \boldsymbol{b} \leq \boldsymbol{b}^+\}$.*

We formulate the ILFP model as follows:

$$\begin{aligned} &max \qquad z^\pm = \frac{\sum_{j=1}^{n} c^\pm_j x^\pm_j + \alpha^\pm}{\sum_{j=1}^{n} d^\pm_j x^\pm_j + \beta^\pm} \\ &\text{subject to}: \sum_{j=1}^{n} a^\pm_{ij} x^\pm_j \leq b^\pm_i, \quad \forall i, \\ &\qquad\qquad x^-_j \geq 0, \quad \forall j. \end{aligned} \tag{1}$$

The model characteristic of model (1) is as follows:

$$\begin{aligned} &max \qquad z^\circ = \frac{\sum_{j=1}^{n} c^\circ_j x_j + \alpha^\circ}{\sum_{j=1}^{n} d^\circ_j x_j + \beta^\circ} \\ &\text{subject to}: \sum_{j=1}^{n} a^\circ_{ij} x_j \leq b^\circ_i, \quad \forall i, \\ &\qquad\qquad x_j \geq 0, \quad \forall j, \end{aligned} \tag{2}$$

such that $c^\circ_j \in c^\pm_j, d^\circ_j \in d^\pm_j, a^\circ_{ij} \in a^\pm_{ij}, b^\circ_i \in b^\pm_i, \alpha^\circ \in \alpha^\pm$ and $\beta^\circ \in \beta^\pm$.

Theorem 2.3. *[18] In model* (1), *the SFR and LFR are $\sum_{j=1}^{n} a^-_{ij} x_j \leq b^+_i, \sum_{j=1}^{n} a^+_{ij} x_j \leq b^-_i$ for all i, respectively.*

Whereas the denominator of model (1) is polynomial in terms of $x^\pm_j$ for all j and with interval coefficients. It is difficult to determine the region where the sign of denominator is positive, negative or zero. For prevention and interdiction of the denominator from becoming zero, we suppose that in model (1), for each feasible solution $x^\pm_j$ for all j, $\sum_{j=1}^{n} d^\pm_j x^\pm_j + \beta^\pm$ be positive.

The following lemma shows that this assumption does not lead to loss of generality.

Lemma 2.4. *For each feasible solution $x_j^{\pm}$, $\sum_{j=1}^{n} d_j^{\pm} x_j^{\pm} + \beta^{\pm} >_s 0$ iff for each x_j, in LFR, $\sum_{j=1}^{n} d_j^{-} x_j + \beta^{-} > 0$.*

Proof. Suppose for each feasible solution $x_j^{\pm}$, $\sum_{j=1}^{n} d_j^{\pm} x_j^{\pm} + \beta^{\pm} >_s 0$ including for β^{-} , d_j^{-} and x_j . Then, for each feasible solution x_j , $\sum_{j=1}^{n} d_j^{-} x_j + \beta^{-} > 0$.

Conversely, suppose for each x_j, in the LFR , $\sum_{j=1}^{n} d_j^{-} x_j + \beta^{-} > 0$. Therefore we have $\sum_{j=1}^{n} d_j^{\circ} x_j + \beta^{\circ} > \sum_{j=1}^{n} d_j^{-} x_j + \beta^{-} > 0$, thus for each x_j, $\sum_{j=1}^{n} d_j^{\circ} x_j + \beta^{\circ} > 0$. So for each feasible solution $x_j^{\pm}$, $\sum_{j=1}^{n} d_j^{\pm} x_j^{\pm} + \beta^{\pm} >_s 0$. □

Also for prevention of the denominator from becoming zero, suppose all the interval coefficients of the the denominator are the same sign. So

1) If all the coefficients of the denominator are considered to be positive, ($\beta^{-} > 0, d_j^{-} \geq 0$ for all j). Then we have according to Lemma 2.4, $\sum_{j=1}^{n} d_j^{\pm} x_j^{\pm} + \beta^{\pm} >_s 0$.

2) If all the coefficients of the denominator are considered to be negative, ($\beta^{+} < 0, d_j^{+} \leq 0$ for all j), the negative sign factors and the ILFP maximization problem change to an ILFP minimization problem and so we have according to Lemma 2.4, $\sum_{j=1}^{n} d_j^{\pm} x_j^{\pm} + \beta^{\pm} >_s 0$.

Suppose the optimal value of the objective function of the following model is equal to η.

$$
\begin{aligned}
min \quad & \sum_{j=1}^{n} d_j^{-} x_j + \beta^{-} \\
\text{subject to :} \quad & \sum_{j=1}^{n} a_{ij}^{-} x_j \leq b_i^{+}, \quad \forall i, \\
& x_j \geq 0, \quad \forall j.
\end{aligned}
\tag{3}
$$

So according to case 1 and 2, for each feasible solution $x_j^{\pm}$, $\sum_{j=1}^{n} d_j^{\pm} x_j^{\pm} + \beta^{\pm} >_s 0$ iff $\eta > 0$.

Throughout this article, suppose $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n | \sum_{j=1}^{n} a_{ij}^{-} x_j \leq b_i^{+}, x_j \geq 0, \forall j\}$.

Lemma 2.5. *Without loss of generality, the following condition can be considered for each $x_j^{\pm}$:*

$$\sum_{j=1}^{n} d_j^{\pm} x_j^{\pm} + \beta^{\pm} \geq_s 1.$$

Proof. An ILFP model consists of the union of unlimited linear fractional programming models. Thus, we prove this lemma for the arbitrary characteristic model (2).

According to Lemma 2.4, for each x_j, in LFR, $\sum_{j=1}^{n} d_j^{-} x_j + \beta^{-} > 0$. Since model (2) is a maximization problem, so by multiplying $\sum_{j=1}^{n} d_j^{-} x_j + \beta^{-}$ in the objective function, the OSs of model (2) are the same as the model below.

$$
\begin{aligned}
max \qquad & (\sum_{j=1}^{n} d_j^- x_j + \beta^-) z^\circ = (\sum_{j=1}^{n} d_j^- x_j + \beta^-) \frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ} \\
\text{subject to : } & \sum_{j=1}^{n} a_{ij}^\circ x_j \leq b_i^\circ, \ \ \forall i, \\
& x_j \geq 0, \ \ \forall j,
\end{aligned}
\tag{4}
$$

We now prove that the denominator in the objective function of model (4) is greater than or equal to one. If we call minimum $\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ$ in LFR as M°, thus $M^\circ = \min_{\mathbf{x} \in \mathbf{X}} \sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ$, so $0 < M^\circ \leq \sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ$, hence $\dfrac{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ}{M^\circ} \geq 1$. Therefore $0 < \sum_{j=1}^{n} d_j^- x_j + \beta^- \leq M^\circ$, so $\dfrac{1}{\sum_{j=1}^{n} d_j^- x_j + \beta^-} \geq \dfrac{1}{M^\circ}$. Since $\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ > 0$, thus $\dfrac{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ}{\sum_{j=1}^{n} d_j^- x_j + \beta^-} \geq \dfrac{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ}{M^\circ} \geq 1$, therefore $\dfrac{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ}{\sum_{j=1}^{n} d_j^- x_j + \beta^-} \geq 1$. So model (4) can be written as follows, where the denominator is greater than or equal to one.

$$
\begin{aligned}
max \qquad & (\sum_{j=1}^{n} d_j^- x_j + \beta^-) z^\circ = \frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ \big/ \sum_{j=1}^{n} d_j^- x_j + \beta^-} \\
\text{subject to : } & \sum_{j=1}^{n} a_{ij}^\circ x_j \leq b_i^\circ, \ \ \forall i, \\
& x_j \geq 0, \ \ \forall j.
\end{aligned}
$$

□

Hence by Lemma 2.5, without loss of generality, suppose the denominator of model (2) is greater than or equal to one.

Theorem 2.6. *[1] The inequality of the interval* $\sum_{j=1}^{n} a_{ij}^\pm x_j \leq_s b_i^\pm$ *has a strong feasible solution iff for all* i*, the inequality* $\sum_{j=1}^{n} a_{ij}^+ x_j \leq b_i^-$ *is feasible.*

Theorem 2.7. *[1] The inequality of the interval* $\sum_{j=1}^{n} a_{ij}^{\pm} x_j \leq_w b_i^{\pm}$ *has a weak feasible solution iff for all* i*, the inequality* $\sum_{j=1}^{n} a_{ij}^{-} x_j \leq b_i^{+}$ *is feasible.*

Noted in this article, assume the all interval coefficients of the ILFP are non-positive or non-negative and also the LFR should be non-empty.

3. Iterative algorithms for determining OS set of ILFP

In this section, we first present the iterative algorithm based on the strong feasible solution to find the ILFP solution so that the OS obtained from this algorithm would be a feasible point. Then, we introduce another algorithm based on the weak feasible solution to find the ILFP solution, so that the OS of this algorithm would also be a feasible point. Finally, for the first time, we propose an algorithm for the ILFP model to obtain an OS set. Thus, to determine the OS set of ILFP, ILFP is first transformed into two sub-models, as one on SFR determines the most pessimistic value of the objective function while the other on the LFR specifies the most optimistic value of the objective function. Hence, the OS obtained from these two sub-models will be as one interval.

3.1. SFOS algorithm

The purpose of this section is to introduce an iterative algorithm to obtain the feasible solution of ILFP as such, with this algorithm introducing a point as a feasible solution.

Consider model (1); as it is a maximization model so we consider a lower bound for the objective function. To this end, suppose x^0 is an arbitrary point in the LFR. Name the value of the objective function of model (1) for this point $\psi^{\pm}$. Suppose that the objective function of model (1) is strongly greater than or equal to the interval number $\psi^{\pm}$, that is

$$\frac{\sum_{j=1}^{n} c_j^{\pm} x_j + \alpha^{\pm}}{\sum_{j=1}^{n} d_j^{\pm} x_j + \beta^{\pm}} \geq_s \psi^{\pm}. \tag{5}$$

In this paper, model (1) is in a maximization form and we have considered a lower bound for the objective function. Thus, if model (1) is in a minimization form, then we should consider an upper bound for the objective function and consider the following remark:

Remark 3.1. *If model (1) is a minimization model, then we must assume that the objective function of model (1) is strongly smaller than or equal to the interval number* $\psi^{\pm}$*, then we have:* $\dfrac{\sum_{j=1}^{n} c_j^{\pm} x_j + \alpha^{\pm}}{\sum_{j=1}^{n} d_j^{\pm} x_j + \beta^{\pm}} \leq_s \psi^{\pm}$ *.*

Here, it is assumed that the denominator in the LFR is greater than or equal to one, there to cases with this assumption.

The first is that, the minimum lower bound of the numerator in the LFR is non-negative ($\min_{\mathbf{x}\in\mathbf{X}} \sum_{j=1}^{n} c_j^- x_j + \alpha^- \geq 0$). Then, given the strong feasible solution and Theorem 2.6 in the LFR, we have:

$$\left(\frac{\sum_{j=1}^{n} c_j^{\pm} x_j + \alpha^{\pm}}{\sum_{j=1}^{n} d_j^{\pm} x_j + \beta^{\pm}} \right)^- \geq \psi^+. \tag{6}$$

Now using the interval arithmetic we have:

$$\frac{\sum_{j=1}^{n} c_j^- x_j + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j + \beta^+} \geq \psi^+ \geq \psi^-, \tag{7}$$

and

$$\sum_{j=1}^{n} c_j^- x_j + \alpha^- - \psi^-(\sum_{j=1}^{n} d_j^+ x_j + \beta^+) \geq 0. \tag{8}$$

The second is that, the maximum upper bound of the numerator in the LFR is non-positive ($\max_{\mathbf{x}\in\mathbf{X}} \sum_{j=1}^{n} c_j^+ x_j + \alpha^+ \leq 0$). Then given the strong feasible solution and Theorem 2.6 in the LFR and using the interval arithmetic, we have:

$$\frac{\sum_{j=1}^{n} c_j^- x_j + \alpha^-}{\sum_{j=1}^{n} d_j^- x_j + \beta^-} \geq \psi^+ \geq \psi^-, \tag{9}$$

and so

$$\sum_{j=1}^{n} c_j^- x_j + \alpha^- - \psi^-(\sum_{j=1}^{n} d_j^- x_j + \beta^-) \geq 0. \tag{10}$$

Based on the two modes, by parameterizing the objective function as (8)(or (10)) and adding (8)(or (10)) to the LFR, model (1) turns into the following parametric model:

$$\begin{aligned} &\mathit{max} && G_s(\mathbf{x}) = \sum_{j=1}^{n} c_j^- x_j + \alpha^- - \psi^-(\sum_{j=1}^{n} d_j^{\diamond} x_j + \beta^{\diamond}) \\ &\text{subject to :} && \sum_{j=1}^{n} a_{ij}^- x_j \leq b_i^+, \;\; \forall i, \\ &&& \sum_{j=1}^{n} c_j^- x_j + \alpha^- - \psi^-(\sum_{j=1}^{n} d_j^{\diamond} x_j + \beta^{\diamond}) \geq 0, \\ &&& x_j \geq 0, \;\; \forall j, \end{aligned} \tag{11}$$

so that

$$\beta^{\diamond}=\begin{cases}\beta^{+} & \min\limits_{\mathbf{x}\in\mathbf{X}}\sum_{j=1}^{n}c_j^{-}x_j+\alpha^{-}\geq 0,\\ \beta^{-} & \max\limits_{\mathbf{x}\in\mathbf{X}}\sum_{j=1}^{n}c_j^{+}x_j+\alpha^{+}\leq 0,\end{cases}, d_j^{\diamond}=\begin{cases}d_j^{+} & \min\limits_{\mathbf{x}\in\mathbf{X}}\sum_{j=1}^{n}c_j^{-}x_j+\alpha^{-}\geq 0,\\ d_j^{-} & \max\limits_{\mathbf{x}\in\mathbf{X}}\sum_{j=1}^{n}c_j^{+}x_j+\alpha^{+}\leq 0.\end{cases} \tag{12}$$

In this article, suppose

$$\mathbf{X}_s(\psi^{-})=\{\mathbf{x}\in\mathbb{R}^n|\sum_{j=1}^{n}a_{ij}^{-}x_j\leq b_i^{+},\sum_{j=1}^{n}c_j^{-}x_j+\alpha^{-}-\psi^{-}(\sum_{j=1}^{n}d_j^{\diamond}x_j+\beta^{\diamond})\geq 0,x_j\geq 0,\forall i,j\},\mathbf{X}_s(\psi^{-})\subseteq\mathbf{X}. \tag{13}$$

In the strong feasible optimal solution (SFOS) algorithm, which we will describe, you will find that the optimal value of the objective function of the model (11) is non-negative in the first iteration, while in subsequent iterations, $\psi^{\pm}$ is found such that following a finite number of iterations, the optimal value of model (11) reaches zero. The following lemma now shows that the optimal value of the objective function of model (11) is always non-negative.

Lemma 3.2. *If the feasible region of $\mathbf{X}_s(\psi^{-})$ is non-empty then the optimal value of model* (11) *is non-negative.*

Proof. The result is derived from the second constraint of model (11). □

Theorem 3.3. *Suppose $\hat{\mathbf{x}}\in\mathbf{X}_s(\psi^{-})$, $z^{-}(\hat{x})=\hat{\psi}^{-}$ and $z^{+}(\hat{x})=\hat{\psi}^{+}$. $\hat{\mathbf{x}}$ is the OS for model* (1) *iff $\hat{\mathbf{x}}$ is the OS of model* (11) *with the optimal value of zero.*

Proof. Suppose $\hat{\mathbf{x}}\in\mathbf{X}_s(\psi^{-})$ is the OS for model (1), so for each $\mathbf{x}\in\mathbf{X}_s(\psi^{-})$:

$$\hat{\psi}^{\pm}\geq_s\frac{\sum\limits_{j=1}^{n}c_j^{\pm}x_j+\alpha^{\pm}}{\sum\limits_{j=1}^{n}d_j^{\pm}x_j+\beta^{\pm}}.$$

Therefore according to Theorem 2.6, we have:

$$\left(\frac{\sum\limits_{j=1}^{n}c_j^{\pm}x_j+\alpha^{\pm}}{\sum\limits_{j=1}^{n}d_j^{\pm}x_j+\beta^{\pm}}\right)^{-}\leq\left(\frac{\sum\limits_{j=1}^{n}c_j^{\pm}x_j+\alpha^{\pm}}{\sum\limits_{j=1}^{n}d_j^{\pm}x_j+\beta^{\pm}}\right)^{+}\leq\hat{\psi}^{-},$$

so

$$\left(\frac{\sum\limits_{j=1}^{n}c_j^{\pm}x_j+\alpha^{\pm}}{\sum\limits_{j=1}^{n}d_j^{\pm}x_j+\beta^{\pm}}\right)^{-}\leq\hat{\psi}^{-}.$$

Therefore, according to the numerator sign and using the interval arithmetic, for each $\mathbf{x}\in\mathbf{X}_s(\psi^{-})$, $\frac{\sum\limits_{j=1}^{n}c_j^{-}x_j+\alpha^{-}}{\sum\limits_{j=1}^{n}d_j^{\diamond}x_j+\beta^{\diamond}}\leq\hat{\psi}^{-}$, so that $\beta^{\diamond}$ and $d_j^{\diamond}$ are defined in (12). Thus

$$\sum_{j=1}^{n}c_j^{-}x_j+\alpha^{-}-\hat{\psi}^{-}(\sum_{j=1}^{n}d_j^{\diamond}x_j+\beta^{\diamond})\leq 0, \tag{14}$$

then we have:

$$\max_{\mathbf{x}\in\mathbf{X}_s(\psi^-)} \sum_{j=1}^{n} c_j^- x_j + \alpha^- - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\diamond x_j + \beta^\diamond) = 0. \tag{15}$$

On the other hand, given the problem assumption, we have:

$$\hat{\psi}^- = z^-(\hat{\mathbf{x}}) = \frac{\sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^-}{\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond},$$

and so

$$\sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^- - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond) = 0. \tag{16}$$

So from (15) and (16) we conclude:

$$\max_{\mathbf{x}\in\mathbf{X}_s(\psi^-)} \{\sum_{j=1}^{n} c_j^- x_j + \alpha^- - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\diamond x_j + \beta^\diamond)\} = \sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^- - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond) = 0.$$

So $\hat{\mathbf{x}}$ is the OS of model (11) with the optimal value of zero.

Conversely, suppose $\hat{\mathbf{x}} \in \mathbf{X}_s(\psi^-)$ is the OS of model (11) with the optimal value of zero, i.e. $\sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^- - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond) = 0$. With reduction ad absurdum, suppose $\hat{\mathbf{x}}$ is not the OS for none of characteristic models, i.e. for each $a_{ij}^\circ \in a_{ij}^\pm$, $b_i^\circ \in b_i^\pm$, $d_j^\circ \in d_j^\pm$, $c_j^\circ \in c_j^\pm$, $\alpha^\circ \in \alpha^\pm$, $\beta^\circ \in \beta^\pm$, $\hat{\mathbf{x}}$ is not the OS for model (2). So there is $\tilde{\mathbf{x}} \in \mathbf{X}_s(\psi^-)$ so that $z^\circ(\tilde{\mathbf{x}}) \geq z^\circ(\hat{\mathbf{x}}) = \hat{\psi}^\circ$. Therefore $\dfrac{\sum_{j=1}^{n} c_j^\circ \tilde{x}_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ \tilde{x}_j + \beta^\circ} \geq \hat{\psi}^\circ \geq \hat{\psi}^-$. Thus $\sum_{j=1}^{n} c_j^\circ \tilde{x}_j + \alpha^\circ - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\circ \tilde{x}_j + \beta^\circ) \geq 0$.

With problem assumption has contradiction. Because $\hat{\mathbf{x}} \in \mathbf{X}_s(\psi^-)$ is the OS of model (11) with the optimal value of zero. Therefore

$$\sum_{j=1}^{n} c_j^\circ \tilde{x}_j + \alpha^\circ - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\circ \tilde{x}_j + \beta^\circ) \leq \sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^- - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond) = 0,$$

and so

$$\sum_{j=1}^{n} c_j^\circ \tilde{x}_j + \alpha^\circ - \hat{\psi}^-(\sum_{j=1}^{n} d_j^\circ \tilde{x}_j + \beta^\circ) \leq 0.$$

□

Accordingly, to obtain SFOS, we introduce an iterative algorithm which finds $\psi^{\pm^r}$ where the optimal value of the objective function of model (11) becomes zero according to Theorem 3.3. In each iteration, the algorithm solves a linear programming model such that its feasible region is a subset of the feasible region of the preceding stage, whose algorithm is described below:

SFOS algorithm

1. Suppose x^0 is an arbitrary point in the LFR and select the permissible tolerance $\varepsilon > 0$ and set $r = 0$.
2. Obtain $\psi^{\pm^r}$ and form $\mathbf{X}_s(\psi^{-^r})$ according to (13).
3. Solve the following linear programming problem and name the OS and the optimal value of the objective function $\mathbf{x}^{(r+1)}$ and $G_s^{(r+1)}(\mathbf{x})$, respectively.

$$\begin{array}{ll} max & \sum_{j=1}^{n} c_j^- x_j + \alpha^- - \psi^{-^r}(\sum_{j=1}^{n} d_j^\diamond x_j + \beta^\diamond) \\ \text{subject to :} & \mathbf{x} \in \mathbf{X}_s(\psi^{-^r}). \end{array} \tag{17}$$

4. If $G_s^{(r+1)}(\mathbf{x}) < \varepsilon$, go to step 5, otherwise set $r = r + 1$ and go to step 2.
5. If $G_s^{(r+1)}(\mathbf{x}) = 0$ then introduce $\mathbf{x}^{(r+1)}$ as SFOS of model (1), otherwise introduce $\mathbf{x}^{(r+1)}$ as approximation of the SFOS of model (1).

The solving steps for the SFOS algorithm are shown in Figure 1.

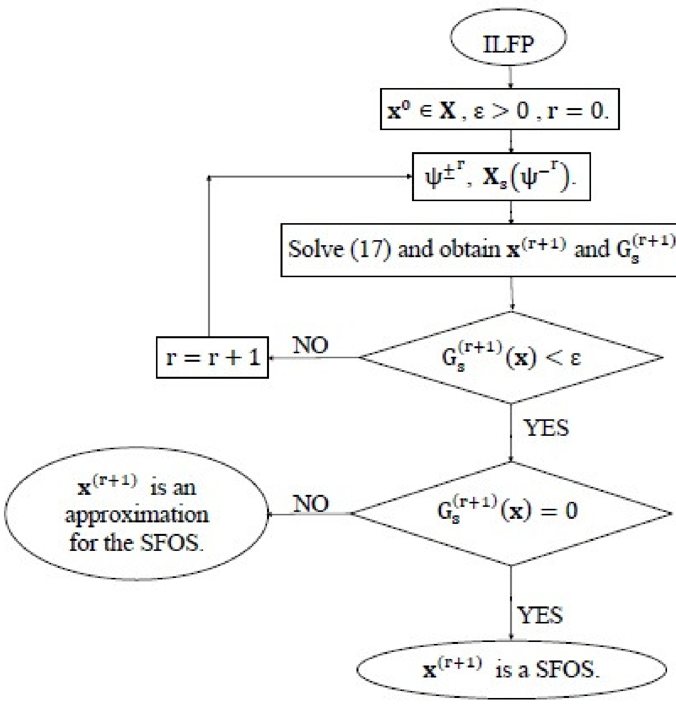

Figure 1: The solving process of the SFOS algorithm.

3.2. *WFOS algorithm*

In this section, we will propose another iterative algorithm which, similar to the SFOS algorithm, introduces a point as the feasible OS.

First, we obtain $\psi^{\pm}$, similar to the SFOS algorithm. Using the Theorem 2.7, we have: $\left(\dfrac{\sum_{j=1}^{n} c_j^{\pm} x_j + \alpha^{\pm}}{\sum_{j=1}^{n} d_j^{\pm} x_j + \beta^{\pm}}\right)^{+} \geq_w$ ψ^{-}, thus

$$\frac{\sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+}}{\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}} \geq \psi^{-}, \tag{18}$$

so that

$$\beta^{\triangleright} = \begin{cases} \beta^{-} & \min_{\mathbf{x}\in\mathbf{X}} \sum_{j=1}^{n} c_j^{-} x_j + \alpha^{-} \geq 0, \\ \beta^{+} & \max_{\mathbf{x}\in\mathbf{X}} \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} \leq 0, \end{cases}, d_j^{\triangleright} = \begin{cases} d_j^{-} & \min_{\mathbf{x}\in\mathbf{X}} \sum_{j=1}^{n} c_j^{-} x_j + \alpha^{-} \geq 0, \\ d_j^{+} & \max_{\mathbf{x}\in\mathbf{X}} \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} \leq 0. \end{cases} \tag{19}$$

Then we rewrite inequality (18) as follows:

$$\sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi^{-}\Big(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}\Big) \geq 0. \tag{20}$$

We now use the following model to obtain WFOS:

$$\begin{aligned} max \quad & G_w(\mathbf{x}) = \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi^{-}\Big(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}\Big) \\ \text{subject to:} \quad & \sum_{j=1}^{n} a_{ij}^{-} x_j \leq b_i^{+}, \ \ \forall i, \\ & \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi^{-}\Big(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}\Big) \geq 0, \\ & x_j \geq 0, \ \ \forall j, \end{aligned} \tag{21}$$

so that $\beta^{\triangleright}$ and $d_j^{\triangleright}$ are defined in (19).

We show the reduced region based on the weak feasible optimal solution (WFOS):

$$\mathbf{X}_w(\psi^{-}) = \{\mathbf{x} \in \mathbb{R}^n | \sum_{j=1}^{n} a_{ij}^{-} x_j \leq b_i^{+}, \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi^{-}\Big(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}\Big) \geq 0, x_j \geq 0, \forall i, j\}, \mathbf{X}_w(\psi^{-}) \subseteq \mathbf{X}. \tag{22}$$

Thus, to obtain WFOS, we introduce an algorithm where the stopping condition is such that the OS of model (21) has the same values in two consecutive iterations. The algorithm in each iteration solves a linear programming model where its feasible region is a subset of the previous iterative region.

Lemma 3.4. *If the feasible region of $\mathbf{X}_w(\psi^{-})$ is non-empty then the optimal value of model* (21) *is non-negative.*

Proof. The result is derived from the second constraint of model (21). □

Remark 3.5. *The following items are established in the stopping condition of WFOS algorithm:*

1. If $\boldsymbol{x}^r = \boldsymbol{x}^{(r-1)}$ then $\psi^{\pm^r} = \psi^{\pm^{(r-1)}}$.

2. If $\psi^{\pm^r} = \psi^{\pm^{(r-1)}}$ then $\max_{\boldsymbol{x}\in\boldsymbol{X}_w(\psi^{-^r})} G_w^{(r+1)}(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\boldsymbol{X}_w(\psi^{-^r})} G_w^{(r)}(\boldsymbol{x})$.

Theorem 3.6. *Suppose $\mathbf{x}^r = \mathbf{x}^{(r-1)}$, $\psi^{-^r} = z^-(\mathbf{x}^r)$ and $\psi^{+^r} = z^+(\mathbf{x}^r)$. $\mathbf{x}^r$ is the OS for model* (1) *iff $\mathbf{x}^r$ is the OS of model* (21) *with the optimal value of non-negative.*

Proof. Suppose $\mathbf{x}^r$ is the OS of model (21) with the optimal value of non-negative. Now, we want to prove that $\mathbf{x}^r$ is the OS of model (1), so it is enough to prove $\mathbf{x}^r$ is the OS of characteristic model (2). According to (19), we will have two results:

i) $\min\limits_{\mathbf{x}\in\mathbf{X}} \sum\limits_{j=1}^{n} c_j^- x_j + \alpha^- \geq 0$,

ii) $\max\limits_{\mathbf{x}\in\mathbf{X}} \sum_{j=1}^{n} c_j^+ x_j + \alpha^+ \leq 0$.

Considering the part i and that $\mathbf{x}^r$ is the OS for model (21), we have:

$$\max_{\mathbf{x}\in\mathbf{X}_w(\psi^{-^r})} \{\sum_{j=1}^{n} c_j^+ x_j + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^- x_j + \beta^-)\} = \sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^- x_j^r + \beta^-) \geq 0, \text{ thus}$$

$$\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^- x_j^r + \beta^-) \geq \sum_{j=1}^{n} c_j^+ x_j + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^- x_j + \beta^-). \tag{23}$$

On the other hand, given the problem assumption, we have:

$$\psi^{-^r} = z^-(\mathbf{x}^r) = \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+}. \tag{24}$$

By replaceing (24) in the inequality (23), we have:

$$\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+}(\sum_{j=1}^{n} d_j^- x_j^r + \beta^-) \geq \sum_{j=1}^{n} c_j^+ x_j + \alpha^+ - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+}(\sum_{j=1}^{n} d_j^- x_j + \beta^-). \tag{25}$$

According to Lemma 2.5, $\sum\limits_{j=1}^{n} d_j^- x_j + \beta^-$ in LFR is greater than or equal to one, from inequality (23), we have: $\sum\limits_{j=1}^{n} d_j^- x_j^r + \beta^- \leq \sum\limits_{j=1}^{n} d_j^- x_j + \beta^-$, so

$$\frac{\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+}{\sum_{j=1}^{n} d_j^- x_j^r + \beta^-} - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+} = \left(\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+}(\sum_{j=1}^{n} d_j^- x_j^r + \beta^-)\right) \times \frac{1}{\sum_{j=1}^{n} d_j^- x_j^r + \beta^-} \geq$$

$$\left(\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+}(\sum_{j=1}^{n} d_j^- x_j^r + \beta^-)\right) \frac{1}{\sum_{j=1}^{n} d_j^- x_j + \beta^-} \geq$$

$$\left(\sum_{j=1}^{n} c_j^+ x_j + \alpha^+ - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+}(\sum_{j=1}^{n} d_j^- x_j + \beta^-)\right)\frac{1}{\sum_{j=1}^{n} d_j^- x_j + \beta^-} =$$

$$\frac{\sum_{j=1}^{n} c_j^+ x_j + \alpha^+}{\sum_{j=1}^{n} d_j^- x_j + \beta^-} - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+} \geq \frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ} - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+},$$

so

$$\frac{\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+}{\sum_{j=1}^{n} d_j^- x_j^r + \beta^-} - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+} \geq \frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ} - \frac{\sum_{j=1}^{n} c_j^- x_j^r + \alpha^-}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+}.$$

Now, by simplifying the above inequality and maximizing the inequality sides on the region $\mathbf{X}_w(\psi^{-^r})$, we have:

$$\max_{\mathbf{x}\in\mathbf{X}_w(\psi^{-^r})} \left\{\frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ}\right\} = \frac{\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+}{\sum_{j=1}^{n} d_j^- x_j^r + \beta^-},$$

hence $\mathbf{x}^r$ is the OS for model (2).

Considering the part ii and that $\mathbf{x}^r$ is the OS of model (21), we have:

$$\max_{\mathbf{x}\in\mathbf{X}(\psi^{-^r})} \{\sum_{j=1}^{n} c_j^+ x_j + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j + \beta^+)\} = \sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+) \geq 0,$$

so

$$\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+) \geq \sum_{j=1}^{n} c_j^+ x_j + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j + \beta^+). \tag{26}$$

Considering the part ii and that $\sum_{j=1}^{n} d_j^- x_j + \beta^-$ in the LFR is greater than or equal to one, we have:

$$\frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ} \leq \frac{\sum_{j=1}^{n} c_j^+ x_j + \alpha^+}{\sum_{j=1}^{n} d_j^+ x_j + \beta^+} \leq \frac{\sum_{j=1}^{n} c_j^+ x_j + \alpha^+}{\sum_{j=1}^{n} d_j^+ x_j + \beta^+} - \psi^{-^r}. \tag{27}$$

On the other hand, from inequality (26) we have: for each $\mathbf{x} \in \mathbf{X}$, $\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+ \geq \sum_{j=1}^{n} d_j^+ x_j + \beta^+ \geq 1$ and for each $\mathbf{x} \in \mathbf{X}$, $0 \geq \sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ \geq \sum_{j=1}^{n} c_j^+ x_j + \alpha^+$. Thus

$$\begin{aligned} &\frac{\sum_{j=1}^{n} c_j^+ x_j + \alpha^+}{\sum_{j=1}^{n} d_j^+ x_j + \beta^+} - \psi^{-^r} \leq \frac{\sum_{j=1}^{n} c_j^+ x_j + \alpha^+}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^-} - \psi^{-^r} \leq \frac{\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+} - \psi^{-^r} = \\ &\left(\sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+)\right) \frac{1}{\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+} \leq \sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+). \end{aligned} \tag{28}$$

Thus from inequalities (27) and (28), we have:

$$\frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ} \leq \sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+). \tag{29}$$

Therefore we have:

$$\max_{\mathbf{x} \in \mathbf{X}_w(\psi^{-^r})} \left\{ \frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ} \right\} = \sum_{j=1}^{n} c_j^+ x_j^r + \alpha^+ - \psi^{-^r}(\sum_{j=1}^{n} d_j^+ x_j^r + \beta^+). \tag{30}$$

Considering the problem assumption, the OS of models (21) and (30) are the same. So $\mathbf{x}^r$ is OS of model (2).

Conversely, suppose $\mathbf{x}^r$ is the OS of model (1), i.e. $\mathbf{x}^r$ is the OS a characteristic model (2). Therefore

$$\frac{\sum_{j=1}^{n} c_j^\circ x_j^r + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j^r + \beta^\circ} = \psi^{\circ^r} \geq \frac{\sum_{j=1}^{n} c_j^\circ x_j + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j + \beta^\circ}.$$

Then we will have the following two results:

i) $\dfrac{\sum_{j=1}^{n} c_j^\circ x_j^r + \alpha^\circ}{\sum_{j=1}^{n} d_j^\circ x_j^r + \beta^\circ} = \psi^{\circ^r}$, so $\sum_{j=1}^{n} c_j^\circ x_j^r + \alpha^\circ - \psi^{\circ^r}(\sum_{j=1}^{n} d_j^\circ x_j^r + \beta^\circ) = 0$.

ii) $\dfrac{\sum_{j=1}^{n} c_j^{\circ} x_j + \alpha^{\circ}}{\sum_{j=1}^{n} d_j^{\circ} x_j + \beta^{\circ}} \leq \psi^{\circ^r}$, so $\sum_{j=1}^{n} c_j^{\circ} x_j + \alpha^{\circ} - \psi^{\circ^r}(\sum_{j=1}^{n} d_j^{\circ} x_j + \beta^{\circ}) \leq 0$. Therefore we have:

$$\max_{\mathbf{x}\in\mathbf{X}(\psi^{-^r})} \{\sum_{j=1}^{n} c_j^{\circ} x_j + \alpha^{\circ} - \psi^{\circ^r}(\sum_{j=1}^{n} d_j^{\circ} x_j + \beta^{\circ})\} = 0.$$

So considering the parts i and ii, $\mathbf{x}^r$ is the OS of model (21). □

Considering the above theorem, the WFOS algorithm would be as follows:

WFOS algorithm

1. Suppose x^0 is an arbitrary point in the LFR and set r = 0.
2. Obtain $\psi^{\pm^r}$ and form $\mathbf{X}_w(\psi^{-^r})$ according to (22).
3. Solve the following linear programming problem and name the OS and the optimal value of the objective function $\mathbf{x}^{(r+1)}$ and $G_w^{(r+1)}(\mathbf{x})$, respectively.

$$\begin{aligned} & max \quad \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi^{-^r}(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}) \\ & \text{s.t.} \quad \mathbf{x} \in \mathbf{X}_w(\psi^{-^r}). \end{aligned} \tag{31}$$

4. If the OS of model (31) is identical in two consecutive iterations then introduce $\mathbf{x}^{(r+1)}$ as a WFOS, otherwise set $r = r + 1$ and go to step 2.

The solving steps for the WFOS algorithm are shown in Figure 2.

In the two SFOS and WFOS iterative algorithms, each algorithm introduces a point as the feasible OS. Since model (1) is an interval model, we attempt to find the OS set for the first time. Generating an OS set requires solving two sub-models. In the following, we will introduce the PMOM algorithm as it introduces an OS set as such, this obtained OS set is feasible. The corresponding algorithm will be described below.

3.3. PMOM algorithm

Here, to determine the OS set of model (1), we first propose two sub-models, one on the SFR and the other on the LFR; we introduce them as the pessimistic and optimistic models, respectively and then we propose the pessimistic and optimistic model algorithm. As a part of the OS set may be infeasible, by adding constraints to the optimistic model, the feasibility of the OS set is guaranteed, and we introduce the PMOM algorithm where the OS set obtained from this algorithm is completely feasible. The following theorem has been presented in order to obtain the objective functions and constraints corresponding to the two pessimistic and optimistic models.

Theorem 3.7. *The two pessimistic and optimistic sub-models of model* (1) *are as follows, respectively:*

$$\begin{aligned} & max \qquad z^{-} = \frac{\sum_{j=1}^{n} c_j^{-} x_j + \alpha^{-}}{\sum_{j=1}^{n} d_j^{\diamond} x_j + \beta^{\diamond}} \\ & \text{subject to:} \quad \sum_{j=1}^{n} a_{ij}^{+} x_j \leq b_i^{-}, \ \forall i, \\ & \qquad x_j \geq 0, \ \forall j, \end{aligned} \tag{32}$$

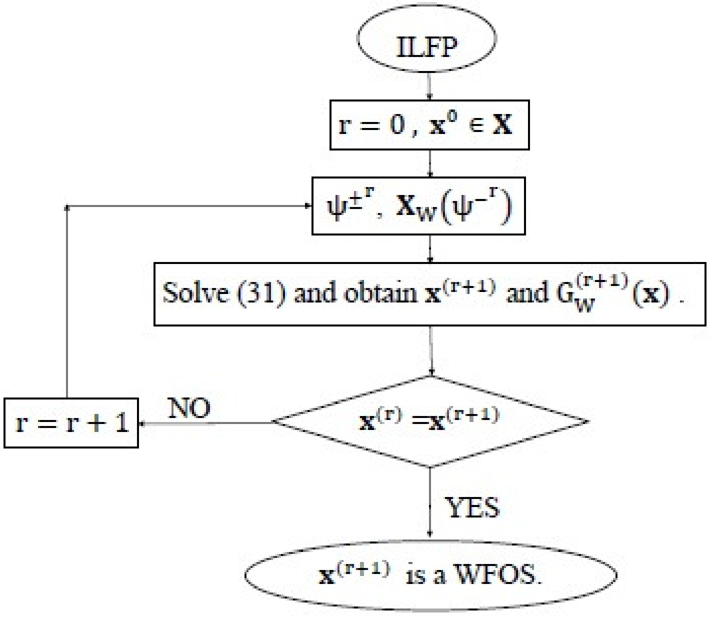

Figure 2: The solving process of the WFOS algorithm.

$$
\begin{aligned}
&max && z^{+} = \frac{\sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+}}{\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}} \\
&\text{subject to:} && \sum_{j=1}^{n} a_{ij}^{-} x_j \le b_i^{+}, \ \ \forall i, \\
& && x_j \ge 0, \ \ \forall j,
\end{aligned}
\tag{33}
$$

such that $d_j^{\diamond}$, $d_j^{\triangleright}$, $\beta^{\diamond}$ *and* $\beta^{\triangleright}$ *are defined in* (12) *and* (19).

Proof. Suppose that z_{opt}°, z_{opt}^{-} and z_{opt}^{+} are the optimal values of the objective functions of models (2), (32) and (33), respectively. We now prove $z_{opt}^{-} \le z_{opt}^{\circ} \le z_{opt}^{+}$.

Suppose $\mathbf{x}'$, $\mathbf{x}''$ and $\mathbf{x}'''$ are the OSs of models (2), (32) and (33), respectively. So model (33) has the LFR among all the feasible regions of characteristic models, therefore every feasible solution of model (2) (for example $\mathbf{x}'$) is a feasible solution of model (33). On the other hand $\mathbf{x}'''$ is the OS of model (33).

$$
z_{opt}^{+} = \frac{\sum_{j=1}^{n} c_j^{+} x_j''' + \alpha^{+}}{\sum_{j=1}^{n} d_j^{\triangleright} x_j''' + \beta^{\triangleright}} \ge \frac{\sum_{j=1}^{n} c_j^{+} x_j' + \alpha^{+}}{\sum_{j=1}^{n} d_j^{\triangleright} x_j' + \beta^{\triangleright}} \ge \frac{\sum_{j=1}^{n} c_j^{\circ} x_j' + \alpha^{\circ}}{\sum_{j=1}^{n} d_j^{\circ} x_j' + \beta^{\circ}} = z_{opt}^{\circ}.
$$

On the other hand, model (32) has the SFR among all the feasible regions of characteristic models, therefore every feasible solution of model (32) (for example $\mathbf{x}''$) is a feasible solution of model (2). On the other hand $\mathbf{x}'$ is the OS of model (2).

$$z^{\circ}_{opt} = \frac{\sum_{j=1}^{n} c^{\circ}_j x'_j + \alpha^{\circ}}{\sum_{j=1}^{n} d^{\circ}_j x'_j + \beta^{\circ}} \geq \frac{\sum_{j=1}^{n} c^{\circ}_j x''_j + \alpha^{\circ}}{\sum_{j=1}^{n} d^{\circ}_j x''_j + \beta^{\circ}} \geq \frac{\sum_{j=1}^{n} c^{-}_j x'''_j + \alpha^{-}}{\sum_{j=1}^{n} d^{\diamond}_j x'''_j + \beta^{\diamond}} = z^{-}_{opt}.$$

Consequently, $z^{-}_{opt} \leq z^{\circ}_{opt} \leq z^{+}_{opt}$. □

Using Theorem 3.7, model (1) has been transformed into two real linear fractional programming models. We are looking for an algorithm for solving models (32) and (33) where the OS set of model (1) is obtained by solving both models simultaneously. We obtain a lower bound for each of the objective functions of the two models. For this purpose, we select two arbitrary points from the feasible regions of models (32) and (33).

First, consider arbitrary point of x^0 from the feasible region (32) (i.e. SFR). Name the value of the objective function of model (32) for this arbitrary point, ψ_p and set $\frac{\sum_{j=1}^{n} c^{-}_j x_j + \alpha^{-}}{\sum_{j=1}^{n} d^{\diamond}_j x_j + \beta^{\diamond}} \geq \psi_p$, therefore

$$\sum_{j=1}^{n} c^{-}_j x_j + \alpha^{-} - \psi_p(\sum_{j=1}^{n} d^{\diamond}_j x_j + \beta^{\diamond}) \geq 0. \tag{34}$$

By parameterizing the objective function as (34) and adding it to the SFR, it causes diminished feasible region in each iteration where the resulting OS will not be iterative. Thus, the pessimistic model is transformed into the following parametric model:

$$\begin{array}{ll} max & G_p(\mathbf{x}) = \sum_{j=1}^{n} c^{-}_j x_j + \alpha^{-} - \psi_p(\sum_{j=1}^{n} d^{\diamond}_j x_j + \beta^{\diamond}) \\ \text{subject to:} & \sum_{j=1}^{n} a^{+}_{ij} x_j \leq b^{-}_i, \ \ \forall i, \\ & \sum_{j=1}^{n} c^{-}_j x_j + \alpha^{-} - \psi_p(\sum_{j=1}^{n} d^{\diamond}_j x_j + \beta^{\diamond}) \geq 0 \\ & x_j \geq 0, \ \ \forall j. \end{array} \tag{35}$$

Now, consider arbitrary point of x^0 from the feasible region (33). Name the value of the objective function of model (33) for this arbitrary point, ψ_o and set $\frac{\sum_{j=1}^{n} c^{+}_j x_j + \alpha^{+}}{\sum_{j=1}^{n} d^{\triangleright}_j x_j + \beta^{\triangleright}} \geq \psi_o$, thus

$$\sum_{j=1}^{n} c^{+}_j x_j + \alpha^{+} - \psi_o(\sum_{j=1}^{n} d^{\triangleright}_j x_j + \beta^{\triangleright}) \geq 0. \tag{36}$$

By parameterizing the objective function as (36) and adding it to the LFR, it causes diminished feasible region in each iteration where the resulting OS will not be iterative. Thus, the optimistic model is transformed into the following parametric model:

$$
\begin{aligned}
&max && G_O(\mathbf{x}) = \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi_O(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright})\\
&\text{subject to:} && \sum_{j=1}^{n} a_{ij}^{-} x_j \le b_i^{+}, \ \ \forall i,\\
& && \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi_O(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}) \ge 0,\\
& && x_j \ge 0, \ \ \forall j.
\end{aligned} \tag{37}
$$

We now show the reduced region for the two models (35) and (37) as shown below.

$$
\begin{aligned}
\mathbf{X}_O(\psi_O) = \{\mathbf{x} \in \mathbb{R}^n | \sum_{j=1}^{n} a_{ij}^{-} x_j \le b_i^{+}, \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi_O(\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}) \ge 0,\\
x_j \ge 0, \forall i, j\}, \mathbf{X}_O(\psi_O) \subseteq \mathbf{X},
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
\mathbf{X}_p(\psi_p) = \{\mathbf{x} \in \mathbb{R}^n | \sum_{j=1}^{n} a_{ij}^{+} x_j \le b_i^{-}, \sum_{j=1}^{n} c_j^{-} x_j + \alpha^{-} - \psi_p(\sum_{j=1}^{n} d_j^{\diamond} x_j + \beta^{\diamond}) \ge 0,\\
x_j \ge 0, \forall i, j\}.
\end{aligned} \tag{39}
$$

Lemma 3.8. *If the feasible regions of $\mathbf{X}_O(\psi_O)$ and $\mathbf{X}_p(\psi_p)$ are non-empty then the optimal value of models* (35) *and* (37) *are non-negative, respectively.*

Proof. The results are derived from the second constraint of models (35) and (37). □

Theorem 3.9. *Suppose $\hat{\mathbf{x}} \in \mathbf{X}_p(\psi_p)$ and $\hat{\psi}_p = z^{-}(\hat{\mathbf{x}})$. $\hat{\mathbf{x}}$ is the OS for model* (32) *iff $\hat{\mathbf{x}}$ is the OS of model* (35) *with the optimal value of zero.*

Proof. Suppose $\hat{\mathbf{x}} \in \mathbf{X}_p(\psi_p)$ is the OS of model (32), so for each $\mathbf{x} \in \mathbf{X}_p(\psi_p)$,

$$
\frac{\sum_{j=1}^{n} c_j^{-} \hat{x}_j + \alpha^{-}}{\sum_{j=1}^{n} d_j^{\diamond} \hat{x}_j + \beta_j^{\diamond}} = \hat{\psi}_p \ge \frac{\sum_{j=1}^{n} c_j^{-} x_j + \alpha^{-}}{\sum_{j=1}^{n} d_j^{\diamond} x_j + \beta_j^{\diamond}},
$$

then we will have two results:

i) $\dfrac{\sum_{j=1}^{n} c_j^{-} \hat{x}_j + \alpha^{-}}{\sum_{j=1}^{n} d_j^{\diamond} \hat{x}_j + \beta^{\diamond}} = \hat{\psi}_p$, therefore

$$
\sum_{j=1}^{n} c_j^{-} \hat{x}_j + \alpha^{-} - \hat{\psi}_p(\sum_{j=1}^{n} d_j^{\diamond} \hat{x}_j + \beta^{\diamond}) = 0. \tag{40}
$$

ii) $\dfrac{\sum_{j=1}^{n} c_j^- x_j + \alpha^-}{\sum_{j=1}^{n} d_j^\diamond x_j + \beta^\diamond} \leq \hat{\psi}_p$, hence

$$\sum_{j=1}^{n} c_j^- x_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond x_j + \beta^\diamond) \leq 0. \tag{41}$$

Thus, we have:

$$\max_{\mathbf{x}\in\mathbf{X}_p(\psi_p)} \sum_{j=1}^{n} c_j^- x_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond x_j + \beta^\diamond) = 0. \tag{42}$$

So from (40) and (42) we conclude:

$$\max_{\mathbf{x}\in\mathbf{X}_p(\psi_p)} \sum_{j=1}^{n} c_j^- x_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond x_j + \beta^\diamond) = \sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond) = 0.$$

Thus $\hat{\mathbf{x}}$ is the OS of model (35) with the optimal value of zero.

Conversely, suppose $\hat{\mathbf{x}} \in \mathbf{X}_p(\psi_p)$ is the OS of model (32) with the optimal value of zero, i.e. $\sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond) = 0$. With reduction ad absurdum, suppose $\hat{\mathbf{x}}$ is not the OS of model (35). So there is $\tilde{\mathbf{x}} \in \mathbf{X}_p(\psi_p)$ so that $z^-(\tilde{\mathbf{x}}) \geq z^-(\hat{\mathbf{x}}) = \hat{\psi}_p$. Therefore $\dfrac{\sum_{j=1}^{n} c_j^- \tilde{x}_j + \alpha^-}{\sum_{j=1}^{n} d_j^\diamond \tilde{x}_j + \beta^\diamond} \geq \hat{\psi}_p$ thus $\sum_{j=1}^{n} c_j^- \tilde{x}_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond \tilde{x}_j + \beta^\diamond) \geq 0$. With problem assumption has contradiction. Because $\hat{\mathbf{x}} \in \mathbf{X}_p(\psi_p)$ is the OS of model (35) with the optimal value of zero. Therefore

$$\sum_{j=1}^{n} c_j^- \tilde{x}_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond \tilde{x}_j + \beta^\diamond) \leq \sum_{j=1}^{n} c_j^- \hat{x}_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond \hat{x}_j + \beta^\diamond) = 0,$$

so $\sum_{j=1}^{n} c_j^- \tilde{x}_j + \alpha^- - \hat{\psi}_p(\sum_{j=1}^{n} d_j^\diamond \tilde{x}_j + \beta^\diamond) \leq 0.$ □

Theorem 3.10. *Suppose $\hat{\mathbf{x}} \in \mathbf{X}_O(\psi_O)$ and $\hat{\psi_O} = z^+(\hat{x})$. $\hat{\mathbf{x}}$ is the OS for model* (33) *iff $\hat{x}$ is the OS of model* (37) *with the optimal value of zero.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.9. □

The SFOS algorithm belongs to the interval model, and we present an algorithm similar to the SFOS algorithm proposed for the interval model (the non-interval version of the SFOS algorithm). Our goal is to determine an OS set for model (1), where the OSs obtained by this algorithm generate an OS set.

(I) Pessimistic model algorithm

1. Suppose x^0 is an arbitrary point in the SFR and select the permissible tolerance $\varepsilon > 0$ and set $r = 0$.
2. Obtain $\psi_p^r = z^-(x^r)$ and form $\mathbf{X}_p(\psi_p^r)$ according to (39).

3. Solve the following linear programming problem and name the OS and the optimal value of the objective function $\mathbf{x}^{(r+1)}$ and $G_p^{(r+1)}(\mathbf{x})$, respectively.

$$\begin{array}{ll} max & \sum_{j=1}^{n} c_j^- x_j + \alpha^- - \psi_p^r \sum_{j=1}^{n} d_j^{\diamond} x_j + \beta^{\diamond} \\ \text{subject to:} & \mathbf{x} \in \mathbf{X}_p(\psi_p^r). \end{array} \tag{43}$$

4. If $G_p^{(r+1)}(\mathbf{x}) < \varepsilon$, go to step 5, otherwise set $r = r + 1$ and go to step 2.

5. If $G_p^{(r+1)}(\mathbf{x}) = 0$ then introduce $\mathbf{x}^{(r+1)}$ as OS of model (32), otherwise introduce $\mathbf{x}^{(r+1)}$ as approximation of the OS of model (32).

(II) Optimistic model Algorithm

1. Suppose x^0 is an arbitrary point in the LFR and select the permissible tolerance $\varepsilon > 0$ and set $r = 0$.

2. Obtain $\psi_O^r = z^+(x^r)$ and form $\mathbf{X}_O(\psi_O^r)$ according to (38).

3. Solve the following linear programming problem and name the OS and the optimal value of the objective function $\mathbf{x}^{(r+1)}$ and $G_O^{(r+1)}(\mathbf{x})$, respectively.

$$\begin{array}{ll} max & \sum_{j=1}^{n} c_j^+ x_j + \alpha^+ - \psi_O^r \sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright} \\ \text{subject to:} & \mathbf{x} \in \mathbf{X}_O(\psi_O^r). \end{array} \tag{44}$$

4. if $G_O^{(r+1)}(\mathbf{x}) < \varepsilon$, go to step 5 otherwise set $r = r + 1$ and go to step 2.

5. if $G_O^{(r+1)}(\mathbf{x}) = 0$ then introduce $x^{(r+1)}$ as OS of model (33), otherwise introduce $x^{(r+1)}$ as approximation of the OS of model (33).

In the following, we will solve an example via the proposed algorithm. We further show that the pessimistic and optimistic model algorithm may lead to an OS set, a part of which is not applied to the LFR and then the infeasible part will be eliminated by presenting the modified method.

Example 3.11. *Consider the the following ILFP :*

$$\begin{array}{ll} max & z^{\pm} = \dfrac{[-3.5, -3]x_1^{\pm} + [1, 1.2]x_2^{\pm} + [-5.79, -3.45]}{[0.27, 1.28]x_1^{\pm} + [1.3, 2.9]x_2^{\pm} + [0.9, 1.2]} \\ \text{s.t.} & [1, 1.1]x_1^{\pm} + [1.6, 1.8]x_2^{\pm} \leq [11.6, 12], \\ & [3, 4]x_1^{\pm} + [-3, -2]x_2^{\pm} \geq [6.5, 7], \\ & x_1^-, x_2^- \geq 0. \end{array} \tag{45}$$

First, by using Theorem 2.3, we obtain the LFR model (45). *So we have:*
$\min_{x\in X} \sum_{j=1}^{n} d_j^- x_j + \beta^- > 1$ *and* $\max_{x\in X} \sum_{j=1}^{n} c_j^+ x_j + \alpha^+ < 0$.

The process of finding a solution by SFOS algorithm is as follows:

Iteration 1:

1. Select $x^0 = (1.88, 0.5)$ *from the LFR and consider the permissible tolerance* $\varepsilon = 0.01$ *and set* $r = 0$.

2. $z^-(x^0) = \psi^{-^0} = -5.7689$ *and* $X_s(\psi^{-^0}) = \{x_1, x_2 \in \mathbb{R} \mid x_1 + 1.6x_2 \leq 12, 4x_1 - 2x_2 \geq 6.5, -1.9424x_1 + 8.4996x_2 \geq 0.5980, x_1 \geq 0, x_2 \geq 0\}$.

3.

$$\begin{array}{ll} max & -3.5x_1 + x_2 - 5.79 + 5.7689(0.27x_1 + 1.3x_2 + 0.9) \\ \text{subject to:} & x \in X_s(\psi^{-^0}). \end{array}$$

So $x^1 = (4.0952, 4.9405)$ *and* $G_s^1 = 33.4395$.

4. $G_s^1 = 33.4395 \nless 0.01$. *Set* $r = 1$ *and go to step 2.*

Iteration 2:

2. $z^-(x^1) = \psi^{-1} = -1.8014$ *and update* $\boldsymbol{X}_s(\psi^{-1})$.
3.

$$\begin{array}{ll} max & -3.5x_1 + x_2 - 5.79 + 1.8014(0.27x_1 + 1.3x_2 + 0.9) \\ \text{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_s(\psi^{-1}). \end{array}$$

So $x^2 = (4.0952, 4.9405)$ *and* $G^2 = 0.0000$.
4. $G^2 = 0.0000 < 0.01$, *go to step 5.*
5. $G^2 = 0.0000$, *thus* $x^2 = (4.0952, 4.9405)$ *is SFOS of model* (45).
The process of finding a solution by WFOS algorithm is as follows:
Iteration 1:
1. Select $x^0 = (1.88, 0.5)$ *from the LFR and set* $r = 0$.
2. $z^-(x^0) = \psi^{-0} = -5.7689$ *and* $\boldsymbol{X}_w(\psi^{-0}) = \{x_1 + 1.6x_2 \leq 12, 4x_1 - 2x_2 \geq 6.5, 4.3842x_1 + 17.9298x_2 \geq -3.4727, x_1 \geq 0, x_2 \geq 0\}$
3.

$$\begin{array}{ll} max & -3x_1 + 1.2x_2 - 3.45 + 5.7689(1.28x_1 + 2.9x_2 + 1.2) \\ \text{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_w(\psi^{-0}). \end{array}$$

So $x^1 = (4.0952, 4.9405)$ *and* $G^1 = 110.0088$.
4. $x_0 \neq x_1$. *Set* $r = 1$ *and go to step 2.*
Iteration 2:
2. $z^-(x^1) = \psi^{-1} = -1.8014$ *and update* $\boldsymbol{X}_w(\psi^{-1})$.
3.

$$\begin{array}{ll} max & -3x_1 + 1.2x_2 - 3.45 + 5.7689(1.28x_1 + 2.9x_2 + 1.2) \\ \text{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_w(\psi^{-1}). \end{array}$$

So $x^2 = (4.0952, 4.9405)$ *and* $G^2 = 27.6069$.
4. $x^1 = x^2$, *thus* $x^2 = (4.0952, 4.9405)$ *is WFOS of model* (45).
To determine the OS set by the pessimistic and optimistic model algorithm, we first obtain the pessimistic and optimistic model (45)*:*
pessimistic model:

$$\begin{array}{ll} max & z^- = \dfrac{-3.45x_1 + x_2 - 5.79}{0.27x_1 + 1.3x_2 + 0.9} \\ \text{subject to:} & 1.1x_1 + 1.8x_2 \leq 11.6, \\ & 3x_1 - 3x_2 \geq 7, \\ & x_1, x_2 \geq 0. \end{array} \tag{46}$$

optimistic model:

$$\begin{array}{ll} max & z^+ = \dfrac{-3x_1 + 1.2x_2 - 3.45}{1.28x_1 + 2.9x_2 + 1.2} \\ \text{subject to:} & x_1 + 1.6x_2 \leq 12, \\ & 4x_1 - 2x_2 \geq 6.5, \\ & x_1, x_2 \geq 0. \end{array} \tag{47}$$

The process of finding a solution by pessimistic model algorithm is as follows:
Iteration 1:
1. Select $x^0 = (3, 0.5)$ *from the SFR and consider the permissible tolerance* $\varepsilon = 0.01$ *and set* $r = 0$.
2. $z^-(x^0) = \psi_p^0 = -6.6907$ *and* $\boldsymbol{X}_p(\psi_p^0) = \{1.1x_1 + 1.8x_2 \leq 11.6, 3x_1 - 3x_2 \geq 7, -1.6935x_1 + 9.6979x_2 \geq -0.2316, x_1, x_2 \geq 0.\}$

3.

$$\begin{array}{ll} \mathit{max} & -3.5x_1 + x_2 - 5.79 + 6.6907(0.27x_1 + 1.3x_2 + 0.9) \\ \text{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_p(\psi_p^0). \end{array}$$

So $x^1 = (5.4483, 3.1149)$ *and* $G_p^1 = 21.2133$.
4. $G_p^1 = 21.2133 \nless 0.01$. *Set* $r = 1$ *and go to step 2.*
Iteration 2:
2. $z^-(x^1) = \psi_p^1 = -3.3867$ *and update* $\boldsymbol{X}_p(\psi_p^1)$.
3.

$$\begin{array}{ll} \mathit{max} & -3.5x_1 + x_2 - 5.79 + 3.3867(0.27x_1 + 1.3x_2 + 0.9) \\ \text{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_p(\psi_p^1). \end{array}$$

So $x^2 = (5.4483, 3.1149)$ *and* $G_p^2 = 0.0000$.
4. $G_p^2 = 0.0000 < 0.01$, *go to step 5.*
5. $G_p^2 = 0.0000$, *thus* $x^2 = (5.4483, 3.1149)$ *is OS of model* (46).
The process of finding a solution by optimistic model algorithm is as follows:
Iteration 1:
1. Select $x^0 = (2, 0.1)$ *from the LFR and consider the permissible tolerance* $\varepsilon = 0.01$ *and set* $r = 0$.
2. $z^+(x^0) = \psi_O^0 = -2.3037$ *and* $\boldsymbol{X}_O(\psi_O^0) = \{x_1 + 1.6x_2 \leq 12, 4x_1 - 2x_2 \geq 6.5, -0.0513x_1 + 7.8807x_2 \geq 0.6856, x_1, x_2 \geq 0.\}$
3.

$$\begin{array}{ll} \mathit{max} & -3x_1 + 1.2x_2 - 3.45 + 2.3037(1.28x_1 + 2.9x_2 + 1.2) \\ \text{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_O(\psi_O^0) \end{array}$$

So $x^1 = (4.0952, 4.9405)$ *and* $G_O^1 = 38.0387$.
4. $G_O^1 = 38.0387 \nless 0.01$. *Set* $r = 1$ *and go to step 2.*
Iteration 2:
2. $z^+(x^1) = \psi_O^1 = -0.4722$ *and update* $\boldsymbol{X}_O(\psi_O^1)$.
3.

$$\begin{array}{ll} \mathit{max} & -3x_1 + 1.2x_2 - 3.45 + 0.4722(1.28x_1 + 2.9x_2 + 1.2) \\ \text{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_O(\psi_O^1) \end{array}$$

So $x^2 = (4.0952, 4.9405)$ *and* $G_O^2 = 0.0001$.
4. $G_O^2 = 0.0001 < 0.01$, *go to step 5.*
5. $G_O^2 = 0.0001$, *thus* $x^2 = (4.0952, 4.9405)$ *is as approximation of the OS of model* (47).
The OS set obtained by two pessimistic and optimistic model algorithms is as follows:

$$X_{opt}^{\pm} = \begin{pmatrix} [4.0952, 5.4483] \\ [3.1149, 4.9405] \end{pmatrix}.$$

The OS set is not completely feasible as no arbitrary point in the solution region should be placed outside the LFR of model (45), *while some points of the OS set obtained by this algorithm are not included within the LFR.*
Consider model (45), *the LFR is as:*

$$\begin{cases} x_1 + 1.6x_2 \leq 12, \\ 4x_1 - 2x_2 \geq 6.5, \\ \quad x_1, x_2 \geq 0. \end{cases}$$

Consider an arbitrary point from the OS set. For example, the point (5.4483, 4.9405) *does not apply to the first LFR constraint, i.e.* $x_1 + 1.6x_2 \leq 12$. *The point* (5.4483, 4.9405) *is not feasible, while it lies in the OS set obtained by*

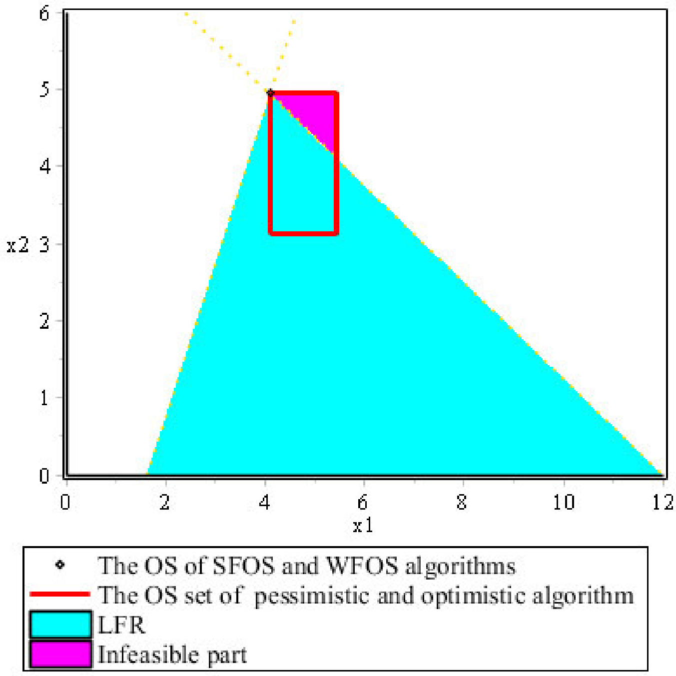

Figure 3: The OS set obtained by the proposed algorithms.

this algorithm. Based on the figure, it is clear that an infinite number of these points does not apply to the constraint $x_1 + 1.6x_2 \leq 12$. *The OS set by the pessimistic and optimistic model algorithm has been shown in Figure 3. Note that SFOS and WFOS lie in the OS set obtained by the pessimistic and optimistic model algorithm.*

Thus, to ensure the feasibility of the OS set obtained by this algorithm, we will add some constraints to model (33), as discussed below.

Theorem 3.12. *Assume* $B_1 = \{j; c_j^- \geq 0\}$, $B_2 = \{j; c_j^+ \leq 0\}$, $E_1 = \{j : j \in B_1, a_{ij}^- \geq 0\}$, $E_2 = \{j : j \in B_1, a_{ij}^+ \leq 0\}$, $E_3 = \{j : j \in B_2, a_{ij}^+ \leq 0\}$ *and* $E_4 = \{j : j \in B_2, a_{ij}^- \geq 0\}$. *To ensure that the OS set is feasible, the constraints* (48) *should be added to model* (33)*:*

$$\begin{aligned}
&\sum_{j\in E_1} a_{\eta j}^- x_j + \sum_{j\in E_2} a_{\eta j}^- x_{j_{opt}} + \sum_{j\in E_3} a_{\eta j}^- x_j + \sum_{j\in E_4} a_{\eta j}^- x_{j_{opt}} \leq b_\eta^+,\\
&x_j \geq x_{j_{opt}} \geq 0, j \in B_1,\\
&0 \leq x_j \leq x_{j_{opt}}, j \in B_2.
\end{aligned} \tag{48}$$

So that η *is index of the constraints of model* (1) *which for* $j \in E_1, E_3; sign(a_{\eta j}^\pm) = sign(c_j^\pm)$ *and for* $j \in E_2, E_4; sign(a_{\eta j}^\pm) \neq sign(c_j^\pm)$ *and also for all* $j, x_{j_{opt}}$ *is OS of pessimistic model.*

Proof. First we solve the pessimistic model algorithm and for all j, we obtain $x_{j_{opt}}$. According to Theorem 3.7 for the optimistic model, we have the LFR. So
for $j \in B_1$, we have: $c_j^- \geq 0$ Thus for all $j, x_j \geq x_{j_{opt}} \geq 0$,
for $j \in B_2$ we have $c_j^+ \leq 0$. Thus for all $j, 0 \leq x_j \leq x_{j_{opt}}$.

Suppose for ηth constraint of model (1), for $j \in E_1, E_3; sign(a_{\eta j}^\pm) = sign(c_j^\pm)$ and for $j \in E_2, E_4$; $sign(a_{\eta j}^\pm) \neq sign(c_j^\pm)$.

Therefore to ensure the feasibility, it is sufficient that $\sum_{j=1}^{n} a_{\eta j}^{-} x_j \leq b_{\eta}^{+}$, and or $\sum_{j\in E_1} a_{\eta j}^{-} x_j + \sum_{j\in E_2} a_{\eta j}^{-} x_j + \sum_{j\in E_3} a_{\eta j}^{-} x_j + \sum_{j\in E_4} a_{\eta j}^{-} x_j \leq b_{\eta}^{+}$.

Considering that for $j \in E_1, E_4$; $a_{\eta j}^{-} \geq 0$ and for $j \in E_2, E_3$; $a_{\eta j}^{+} \leq 0$. So it is sufficient that $\sum_{j\in E_1} a_{\eta j}^{-} x_j + \sum_{j\in E_2} a_{\eta j}^{-} x_{j_{opt}} + \sum_{j\in E_3} a_{\eta j}^{-} x_j + \sum_{j\in E_4} a_{\eta j}^{-} x_{j_{opt}} \leq b_{\eta}^{+}$. $\square$

Therefore, we present the modified optimistic model as follows:

$$
\begin{array}{ll}
max & z^{+} = \dfrac{\sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+}}{\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}} \\
\text{subject to:} & \sum_{j=1}^{n} a_{ij}^{-} x_j \leq b_i^{+}, \ \ \forall i, \\
& \sum_{j\in E_1} a_{\eta j}^{-} x_j + \sum_{j\in E_2} a_{\eta j}^{-} x_{j_{opt}} + \sum_{j\in E_3} a_{\eta j}^{-} x_j + \sum_{j\in E_4} a_{\eta j}^{-} x_{j_{opt}} \leq b_{\eta}^{+}, \\
& x_j \geq x_{j_{opt}} \geq 0, j \in B_1, \\
& 0 \leq x_j \leq x_{j_{opt}}, j \in B_2,
\end{array}
\tag{49}
$$

so that $d_j^{\triangleright}$ and $\beta^{\triangleright}$ are defined in (19).

Remark 3.13. *The pessimistic and modified optimistic model algorithm will be as follows:*

(I) Pessimistic model algorithm

1. Suppose x^0 is an arbitrary point in the SFR and select the permissible tolerance $\varepsilon > 0$ and set $r = 0$.

2. Obtain $z^{-}(x^r) = \psi_p^r$ and form $\boldsymbol{X}_p(\psi_p^r)$ according to (39).

3. Solve the following linear programming problem and name the OS and the optimal value of the objective function $x^{(r+1)}$ and $G_p^{(r+1)}(\boldsymbol{x})$, respectively.

$$
\begin{array}{ll}
max & \sum_{j=1}^{n} c_j^{-} x_j + \alpha^{-} - \psi_p^r \sum_{j=1}^{n} d_j^{\diamond} x_j + \beta^{\diamond} \\
\textit{subject to:} & \boldsymbol{x} \in \boldsymbol{X}_p(\psi_p^r).
\end{array}
$$

4. If $G_p^{(r+1)}(\boldsymbol{x}) < \varepsilon$, go to step 5, otherwise set $r = r + 1$ and go to step 2.

5. If $G_p^{(r+1)}(\boldsymbol{x}) = 0$ then introduce $x^{(r+1)}$ as OS of model (32)*, otherwise introduce $x^{(r+1)}$ as approximation of the OS of model* (32).

(II) Modified optimistic model algorithm

1. Suppose x^0 is an arbitrary point in the LFR and select the permissible tolerance $\varepsilon > 0$ and set $r = 0$.

2. Obtain $\psi_{MO}^r = z^{+}(x^r)$ and form $\boldsymbol{X}_{MO}(\psi_{MO}^r)$.

$$
\boldsymbol{X}_{MO}(\psi_{MO}^r) = \{\boldsymbol{x} \in \mathbb{R}^n | \sum_{j=1}^{n} a_{ij}^{-} x_j \leq b_i^{+}, \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi_{MO}^r (\sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright}) \geq 0, x_j \geq 0, \sum_{j\in E_1} a_{\eta j}^{-} x_j + \sum_{j\in E_2} a_{\eta j}^{-} x_{j_{opt}} + \sum_{j\in E_3} a_{\eta j}^{-} x_j + \sum_{j\in E_4} a_{\eta j}^{-} x_{j_{opt}} \leq b_{\eta}^{+}, x_j \geq x_{j_{opt}} \geq 0, j \in B_1, 0 \leq x_j \leq x_{j_{opt}}, j \in B_2\}.
\tag{50}
$$

3. Solve the following linear programming problem and name the OS and the optimal value of the objective function $x^{(r+1)}$ and $G_{MO}^{(r+1)}(x)$, respectively.

$$\begin{aligned} &max && \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} - \psi_{MO}^{r} \sum_{j=1}^{n} d_j^{\triangleright} x_j + \beta^{\triangleright} \\ &\text{subject to:} && x \in X_{MO}(\psi_{MO}^{r}). \end{aligned} \tag{51}$$

4. If $G_{MO}^{(r+1)}(x) < \varepsilon$, go to step 5, otherwise set $r = r + 1$ and go to step 2.

5. If $G_{MO}^{(r+1)}(x) = 0$ then introduce $x^{(r+1)}$ as OS of model (33), otherwise introduce $x^{(r+1)}$ as approximation of the OS of model (33).

The solving steps for the PMOM algorithm are shown in Figure 4.

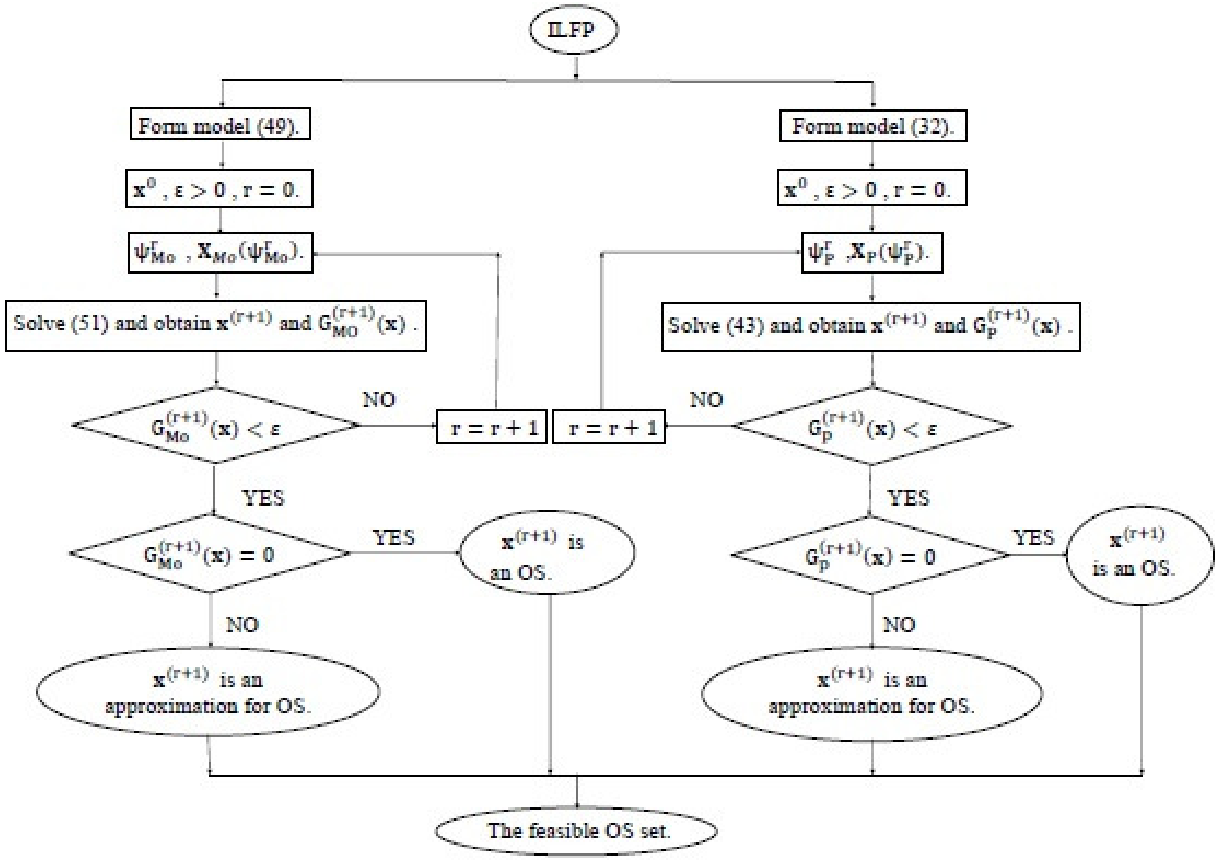

Figure 4: The solving process of the PMOM algorithm.

Consider model (45). Considering the model (49), the modified optimistic model will be as follows:

$$\begin{array}{ll} max & z^{+}=\dfrac{-3x_1+1.2x_2-3.45}{1.28x_1+2.9x_2+1.2} \\ \text{subject to:} & x_1+1.6x_2\leq 12, \\ & 4x_1-2x_2\geq 6.5, \\ & x_{1_{opt}}+1.6x_2\leq 12, \\ & x_2\geq x_{2_{opt}}=3.1149, \\ & 0\leq x_1\leq x_{1_{opt}}=5.4483. \end{array} \tag{52}$$

The process of finding a solution by modified optimistic model algorithm is as follows:
Iteration 1:
1. Select $x^0=(3.5,3.2)$ from feasible region of model (52) and consider the permissible tolerance $\varepsilon=0.01$ and set $r=0$.
2. $z^{+}(x^0)=\psi^0_{MO}=-0.6758$ and $\mathbf{X}_{MO}(\psi^0_{MO})=\{x_1+1.6x_2\leq 12, 4x_1-2x_2\geq 6.5, -2.135x_1+3.1598x_2\geq 2.639, x_{1_{opt}}+1.6x_2\leq 12, x_2\geq x_{2_{opt}}=3.1149, 0\leq x_1\leq x_{1_{opt}}=5.4483\}$
3.

$$\begin{array}{ll} max & -3x_1+1.2x_2-3.45+0.6758(1.28x_1+2.9x_2+1.2) \\ \text{subject to:} & \mathbf{x}\in\mathbf{X}_{MO}(\psi^0_{MO}). \end{array}$$

So $x^1=(3.6724,4.0948)$ and $G^1_{MO}=2.4559$.
4. $G^1_{MO}=2.4559\not<0.01$. Set $r=1$ and go to step 2.
Iteration 2:
2. $z^{+}(x^1)=\psi^1_{MO}=-0.5374$ and update $\mathbf{X}_{MO}(\psi^1_{MO})$.
3.

$$\begin{array}{ll} max & -3x_1+1.2x_2-3.45+0.5374(1.28x_1+2.9x_2+1.2) \\ \text{subject to:} & \mathbf{x}\in\mathbf{X}_{MO}(\psi^1_{MO}). \end{array}$$

So $x^2=(3.6726,4.0951)$ and $G^2=0.0000$.
4. $G^2=0.0000<0.01$ and go to step 5.
5. $G^2=0.0000$ then $x^2=(3.6726,4.0951)$ is the OS of model (47).
So the OS set obtained by PMOM algorithm is as follows:

$$X^{\pm}_{opt}=\begin{pmatrix} [3.6726,5.4483] \\ [3.1149,4.0951] \end{pmatrix}.$$

The OS set obtained by PMOM algorithm are shown in Figure 5. The OS set applies in LFR so that the OS set is feasible. Noted that SFOS and WFOS are not found in the OS set obtained by the PMOM algorithm.

We will solve two more examples in Section 5 where in the first, we will show that SFOS and WFOS are the same and not included in the OS set of the PMOM algorithm. On the other hand, in the second example, it will be observed that SFOS and WFOS are not identical; SFOS lies in the OS set obtained by the PMOM algorithm while WFOS is not included in the OS set of the PMOM algorithm.

4. Properties of three proposed algorithms

In this section, we will compare the three proposed algorithms.

In the SFOS algorithm and the PMOM algorithm, the stopping condition is that the optimal value of the objective function should be less than the permitted tolerance ε. On the other hand, in the WFOS algorithm, the stopping condition is that the OS obtained in two consecutive iterations should be identical. In all three algorithms, the number of iterations depends on the starting point and in each iteration, only one

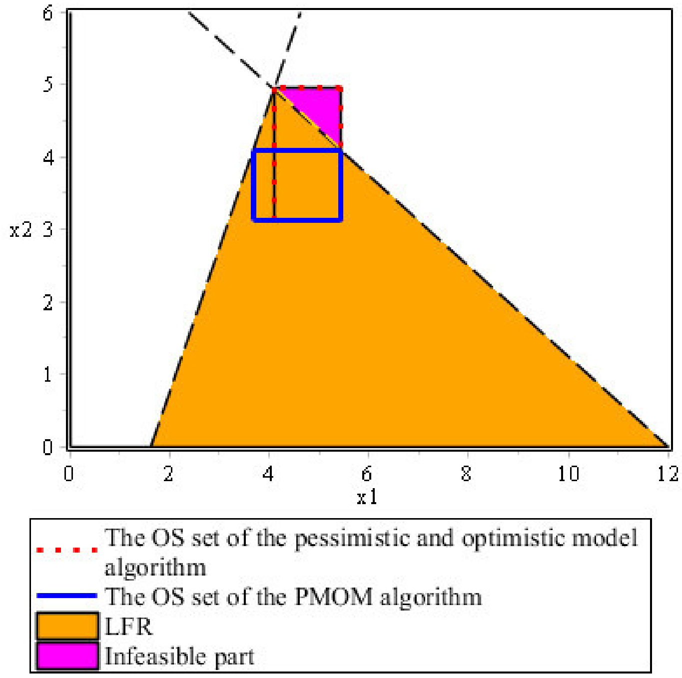

Figure 5: The OS set obtained by the PMOM algorithm.

linear model is solved. The feasible region in each iteration is a subset of the feasible region in the previous iteration. In each iteration, only one new constraint is added to the defined feasible region.

In the SFOS and WFOS algorithms, we obtain only one point as the feasible OS. On the other hand, an OS set is obtained using the PMOM algorithm where all points of this OS set are feasible. The feasible OS obtained by SFOS and WFOS algorithms may be included in the OS set found by PMOM algorithm. The union of the feasible OSs obtained from these three proposed algorithms will generate a more complete OS set.

5. Numerical examples and results analysis

To illustrate the efficiency of the proposed algorithms, we have solved two numerical examples and compared the OS obtained from the algorithms.

Example 5.1. *Consider the following ILFP:*

$$\begin{array}{ll} \max & z^{\pm} = \dfrac{[1,1.2]x_1^{\pm} + [-6,-3.2]x_2^{\pm} + [-4,-3]}{[-2.5,-1.5]x_1^{\pm} + [8,9.1]x_2^{\pm} + [4,4.3]} \\ \text{subject to:} & [3,5.1]x_1^{\pm} + [-9.6,-7]x_2^{\pm} \leq [4.1,4.6], \\ & [1.1,1.2]x_1^{\pm} + [0.5,1]x_2^{\pm} \leq [8.4,8.7], \\ & [2.7,3]x_1^{\pm} + [0.1,0.6]x_2^{\pm} \geq [10.8,12.1], \\ & x_1^{-}, x_2^{-} \geq 0. \end{array} \tag{53}$$

First, by using Theorem 2.3, we obtain the LFR model (53)*. So we have:* $\min\limits_{x\in X} \sum_{j=1}^{n} d_j^{-} x_j + \beta^{-} > 1$ *and* $\max\limits_{x\in X} \sum_{j=1}^{n} c_j^{+} x_j + \alpha^{+} < 0$*. Consider the permissible tolerance* $\varepsilon = 0.01$*.*

In the following, using the proposed algorithms, we solve model (53)*. The results have been given in Table 1.*

Table 1: The results obtained from the solving of model (53) using the proposed algorithms.

Iterations	SFOS algorithm	WFOS algorithm	model algorithm	
			pessimistic	modified optimistic
$r = 0$	$x^0 = (4, 2)$	$x^0 = (4, 2)$	$x^0 = (4.4, 3)$	$x^0 = (4.4, 2)$
	$\psi^{-^0} = -1.1010$	$\psi^{-^0} = -1.1010$	$\psi^0_p = -0.9832$	$\psi^0_{MO} = -0.2424$
$r = 1$	$x^1 = (0.2143, 16.9286)$	$x^1 = (0.2143, 16.9286)$	$x^1 = (4.3643, 3.1628)$	$x^1 = (6.4715, 1.5432)$
	$G^1_s = 48.5548$	$G^1_w = 118.2863$	$G^1_p = 0.3550$	$G^1_{MO} = 2.1875$
	$\psi^{-^1} = -0.7537$	$\psi^{-^1} = -0.7537$	$\psi^1_p = -0.9648$	$\psi^1_{MO} = -0.0177$
$r = 2$	$x^2 = (0.2143, 16.9286)$	$x^2 = (0.2143, 16.9286)$	$x^2 = (4.3643, 3.1628)$	$x^2 = (6.4715, 1.5432)$
	$G^2_s = 0.0047$	$G^2_w = 60.8815$	$G^2_p = 0.0000$	$G^2_{MO} = 0.0003$

In the SFOS algorithm, consider the arbitrary point $x^0 = (4, 2) \in \boldsymbol{X}$. *After two iterations,* $G^2_s = 0.0047 < 0.01$, *thus we consider the point* $x^2 = (0.2143, 16.9286)$ *as an approximation of SFOS. To find WFOS by the WFOS algorithm, consider the arbitrary point* $x^0 = (4, 2) \in \boldsymbol{X}$. *The OSs obtained in the first and second iterations are the same, hence the point* $x^2 = (0.2143, 16.9286)$ *is WFOS. Using the pessimistic and optimistic model algorithm for model* (53)*, the OS derived from the pessimistic model algorithm is* $(4.3643, 3.1628)$ *and the optimal value of the objective function of the pessimistic model for this point is* $z^-_{opt} = -1.0120$. *The OS of the optimistic model algorithm is* $(7.1161, 1.7446)$ *and the optimal value of the objective function of the optimistic model for this point is* $z^+_{opt} = -0.0046$. *Thus, the optimal value of the objective function of the model* (53) *is equal to* $[-1.0120, -0.0046]$. *The optimal value of the objective function of model* (53) *for SFOS and WFOS is* $[-0.7586, -0.3601]$. *Note that the pessimistic and optimistic model has the SFR and LFR among all the characteristic models. In this case, the optimal value of the objective function derived from these two sub-models includes the optimal value obtained from SFOS and WFOS algorithm and thus* $(0.2143, 16.9286)$ *has the condition of optimality. In the PMOM algorithm, consider the point* $x^0 = (4.4, 3)$ *for the pessimistic model. After two iteration steps,* $G^2_p = 0.0000 < 0.01$ *thus,* $x^2 = (4.3643, 3.1628)$ *is the feasible OS of the pessimistic model. And consider the point* $x^0 = (4.4, 2)$ *for the modified optimistic model. After two iteration steps,* $G^2_{Io} = 0.0003 < 0.01$ *thus,* $x^2 = (6.4715, 1.5432)$ *is as approximation of the feasible OS of the modified optimistic model. The OS set obtained by PMOM algorithm is as follows:*

$$\boldsymbol{X}^{\pm}_{opt} = \begin{pmatrix} [4.3643, 6.4715] \\ [1.5432, 3.1628] \end{pmatrix}.$$

The OS set by PMOM algorithm and the OS obtained by SFOS and WFOS algorithms have been shown in Figure 6.

Noted that the OS set of PMOM algorithm is feasible and the OSs obtained from SFOS and WFOS algorithms are the same and not included in the OS set obtained by PMOM algorithm. Further, the union of the OSs obtained from the three proposed algorithms will bemore complete OSs set.

In Example 5.2, we show that the SFOS and WFOS algorithms do not have similar OSs and the OS of the SFOS algorithm lies in the OS set obtained by PMOM algorithm. However, the OS of the WFOS algorithm is not found in the OS set obtained by the PMOM algorithm.

Example 5.2. *Consider the following ILFP:*

$$\begin{array}{lll} max & z^{\pm} = \dfrac{[-6, -5.2]x^{\pm}_1 + [-1.2, -1]x^{\pm}_2 + [3, 4]}{[2, 3.5]x^{\pm}_1 + [4, 5]x^{\pm}_2 + [2, 3.5]} \\ subject\ to: & [0, \frac{1}{3}]x^{\pm}_1 + [1.5, 2]x^{\pm}_2 \geq [5, 5.3], \\ & [5.1, 5.2]x^{\pm}_1 + [-4.5, 0]x^{\pm}_2 \geq [3.1, 3.2], \\ & [1, 1.1]x^{\pm}_1 + [1.6, 2]x^{\pm}_2 \leq [11.2, 12], \\ & x^-_1, x^-_2 \geq 0. \end{array} \tag{54}$$

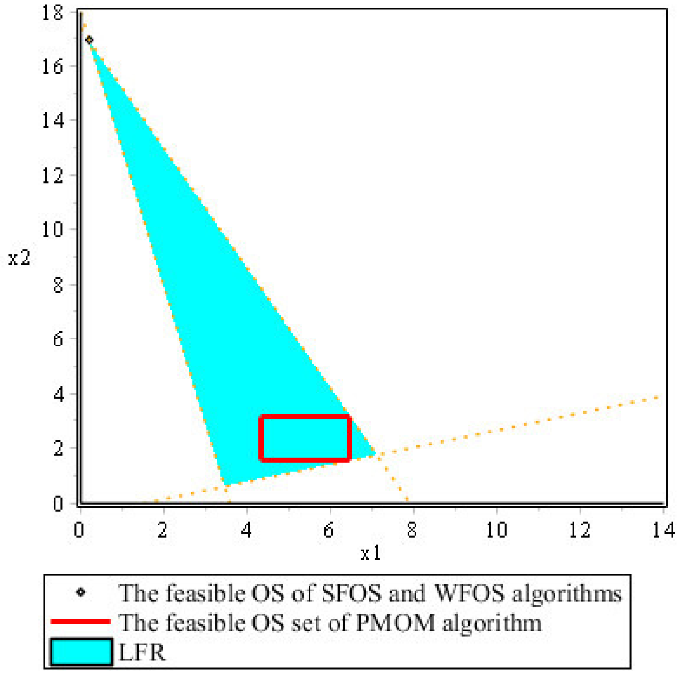

Figure 6: The OS set obtained by proposed algorithms.

First, by using Theorem 2.3, we obtain the LFR model (54). *So we have:* $\min\limits_{x\in X} \sum_{j=1}^{n} d_j^- x_j + \beta^- > 1$ *and* $\min\limits_{x\in X} \sum_{j=1}^{n} c_j^- x_j + \alpha^- < 0$. *Consider the permissible tolerance* $\varepsilon = 0.01$.

In the following, using the proposed algorithms, we solve model (54). *The results have been given in Table 2.*

Table 2: The results obtained from the solving of model (54) using the proposed algorithms.

Iterations	SFOS algorithm	WFOS algorithm	model algorithm	
			pessimistic	modified optimistic
$r = 0$	$x^0 = (3, 2)$	$x^0 = (3, 2)$	$x^0 = (3.75, 3.5333)$	$x^0 = (1, 2.5)$
	$\psi^{-^0} = -1.0876$	$\psi^{-^0} = -1.0876$	$\psi_p^0 = -1.0045$	$\psi_{MO}^0 = -0.1897$
$r = 1$	$x^1 = (0.5962, 7.1274)$	$x^1 = (0.5962, 7.1274)$	$x^1 = (3.7451, 3.5333)$	$x^1 = (0.5962, 2.4006)$
	$G_s^1 = 25.3492$	$G_w^1 = 38.6073$	$G_p^1 = 0.0192$	$G_{MO}^1 = 1.8362$
	$\psi^{-^1} = -0.2880$	$\psi^{-^1} = -0.2880$	$\psi_p^1 = -1.0037$	$\psi_{MO}^1 = -0.0853$
$r = 2$	$x^2 = (0.5962, 2.4006)$	$x^2 = (0.5962, 7.1274)$	$x^2 = (3.7451, 3.5333)$	$x^2 = (0.5962, 2.4006)$
	$G_s^2 = 0.2272$	$G_w^2 = 5.6450$	$G_p^2 = 0.0003$	$G_{MO}^2 = 0.0003$
	$\psi^{-^2} = -0.2703$			
$r = 3$	$x^3 = (0.5962, 2.4006)$			
	$G_s^2 = 0.0008$			

In the SFOS algorithm, after three iterations, we obtain the point (0.5962, 2.4006) *as an approximation of SFOS. The optimal value of the objective function of model* (54)*, for this point is* $[-0.2703, -0.0853]$. *In the WFOS algorithm, after two iteration, we obtain the point* (0.5962, 7.1274) *as WFOS. The optimal value of the objective function of model* (54) *for this point is* $[-0.2880, -0.1510]$. *We now apply the pessimistic and optimistic model algorithm for model* (54). *The optimal value of the objective function for model* (54) *is* $[-1.0037, -0.0853]$. *Forasmuch as the optimal value of the objective function derived from the pessimistic and optimistic two sub-models includes the optimal value of the* *objective function obtained from SFOS and WFOS algorithms and thus points* $(0.5962, 2.4006)$ *and* $(0.5962, 7.1274)$ *have the condition of optimality. In the PMOM algorithm, for the pessimistic model, after two iteration steps, point* $(3.7451, 3.5333)$ *is as an approximation of the feasible OS of the pessimistic model. For the modified optimistic model, after two iteration steps, point* $(0.5962, 2.4006)$ *is as an approximation of the feasible OS of the modified optimistic model. The OS set obtained by PMOM algorithm is as follows:*

$$\boldsymbol{X}^{\pm}_{opt} = \begin{pmatrix} [0.5962, 3.7451] \\ [2.4006, 3.5333] \end{pmatrix}.$$

Noted that the OS set of PMOM algorithm is feasible and the OS of the SFOS algorithm lies in the OS set obtained by PMOM algorithm. However, the OS of the WFOS algorithm is not found in the OS set obtained by the PMOM algorithm.

6. Conclusions

In this paper, an important class of mathematical programming called linear fractional programming with uncertainty data in the form of an interval was studied. Three iterative algorithms were presented to determine the feasible OS of the ILFP model. In the SFOS and WFOS algorithms, using the definition of strong and weak feasible solutions, we transformed the objective function of the ILFP model into a linear programming model on the LFR where it depended on the parameter ψ^- and a new constraint was added to the LFR. The parameter value ψ^- was updated in each iteration and so the new objective function and constraint also changed. Adding this new constraint to the LFR resulted in a reduction in the feasible region of each iteration. In the SFOS algorithm, the optimal value of the objective function became zero after the finite number of iterations, where SFOS was obtained; after a finite number of iterations in the WFOS algorithm, the feasible OS was identical in two consecutive steps where WFOS was obtained. In the SFOS and WFOS algorithms, we obtain only one point as the feasible OS of the ILFP. Mean while, the ILFP model was an interval model. To determine the feasible OS set, we first transformed the ILFP model into two pessimistic and optimistic sub-models, here one of them was defined on the SFR and the other on the LFR. Next, we added constraints to the optimistic model to ensure the feasibility of OS and called the optimistic model as the modified optimistic model. Then, using the PMOM algorithm, we linearized the objective function of each model separately, as the pessimistic model was dependent on the parameter ψ_p and the modified optimistic model was contingent upon the parameter ψ_{MO}. We further added a new constraint to the defined feasible region of that model, and the values of ψ_p and ψ_{MO} parameters were updated where the objective function and the new constraint of both models also changed. Adding a new constraint to the defined feasible region of each model resulted in diminished feasible region of each iteration and, after a finite number of iterations, the optimal value of the objective function of both models became zero where the OS was obtained. The OSs obtained from these two models form an OS set as the OS set was feasible. Noted that in each three algorithms, in each iteration, only one linear programming model is solved, the feasible OS was obtained after a finite number of iterations. The feasible OS obtained by SFOS and WFOS algorithms may be included in the OS set found by PMOM algorithm so the union of the feasible OSs obtained from these three proposed algorithms will generate a more complete OS set.

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