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Demicompactness, Selection of Linear Relation and Application to Multivalued Matrix

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Abstract. In this paper, we investigate the concept of demicompactness and we establish some new results in Fredholm theory connected with the existence of selections of a given linear relation and we explore the possibility of finding a selection demicompact for some linear relation demicompact. Moreover, we give the relationship of the resolvent set between the linear relation with its selection. Furthermore, we give an application to matrix linear relation.

1. Introduction

Let X is a infinite dimensional vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. T multivalued linear operator or simply a linear relation $T : X \to X$ is a mapping from a subspace $\mathcal{D}(T)$ of X, called the domain of T, into $P(X)\backslash\{\emptyset\}$ (the collection of non empty subsets of X) such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all non zero scalars $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator, which is equivalent to $T(0) = \{0\}$. We denote by $\mathcal{L}R(X)$ the class of linear relations everywhere defined. Let $T \in L\mathcal{R}(X)$ is uniquely determined by its graph $G(T)$, which is defined by:

$$G(T) := \Big\{(x, y) \in X \times X : x \in \mathcal{D}(T), y \in Tx\Big\},$$

so that we can identify T with $G(T)$. The closure of T, denoted by $\overline{T}$, is the linear relation defined by $G(\overline{T}) := \overline{G(T)}$. We denote by $C\mathcal{R}(X)$ the class of all closed linear relations on X, and we denote by $\mathcal{C}(X)$ the set of all closed, densely defined linear operators on X. The inverse of T is a linear relation T^{-1} given by $G(T^{-1}) := \{(x, y) \in X \times X : (x, y) \in G(T)\}$. If $G(T)$ is closed, then T is said to be closed. We design by $R(T) = T(\mathcal{D}(T))$ the range of T. T is called surjective if $R(T) = Y$. The subspace $N(T) := T^{-1}(0)$ is called the null space of T. T is called injective if $N(T) = \{0\}$, that is, if T^{-1} is a single valued linear operator. Notice that when $x \in \mathcal{D}(T)$,

$$y \in Tx \text{ if, and only if, } Tx = y + T(0).$$

For T and $S \in L\mathcal{R}(X)$, the notation $T \subset S$ means that $G(T) \subset G(S)$. The linear relation $T + S$ is defined by:

$$G(T + S) := \Big\{(x, y) \in X \times X : y = u + v \text{ with } (x, u) \in G(T), (x, v) \in G(S)\Big\}.$$

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Let $T \in L\mathcal{R}(X)$ and $S \in L\mathcal{R}(Y,Z)$ where $R(T) \cap \mathcal{D}(S) \neq \emptyset$. The product of ST is defined by:

$$G(ST) := \Big\{(x,z) \in X \times Z : (x,u) \in G(T) \text{ and } (u,z) \in G(S) \text{ for some } u \in Y\Big\}.$$

Let Q_T denote the quotient map from X onto $X/\overline{T(0)}$. We shall denote $Q_{\overline{T(0)}}$ by Q_T. Clearly $Q_T T$ is a single valued operator and the norm of T is defined by $\|T\| := \|Q_T T\|$. We say that T is continuous if for each neighborhood V in $R(T)$, $T^{-1}(V)$ is a neighborhood in $\mathcal{D}(T)$ (equivalently $\|T\| < \infty$); bounded if its continuous with $\mathcal{D}(T) = X$; open if T^{-1} is continuous equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) := \sup\Big\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\| \text{ for } x \in \mathcal{D}(T)\Big\}$$

where $d(x, N(T))$ is the distance between x and $N(T)$. We denote the class of all bounded linear relations from X by $B\mathcal{R}(X)$ and we denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X. We denote the class of compact linear relations from X by $K\mathcal{R}(X)$. We denote by $\mathcal{K}(X)$ the subspace of compact operators on X.

If M and N are subspaces of X and of the dual space X' respectively, then

$$M^{\perp} := \Big\{x' \in X' : \ x'(x) = 0 \ \text{ for all } x \in M\Big\}$$

and

$$N^{\top} := \Big\{x \in X : \ x^{'}(x) = 0 \ \text{ for all } x^{'} \in N\Big\}.$$

The conjugate of $T \in L\mathcal{R}(X,Y)$ is the linear relations T' defined by $G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'$, so that $(y', x') \in G(T')$ if and only if $y'(y) = x'(x)$ for all $(x,y) \in G(T)$. For $T \in L\mathcal{R}(X)$, we write $\alpha(T) := \dim N(T)$, $\beta(T) := \dim X/R(T)$, $\overline{\beta}(T) := \dim Y/\overline{R(T)}$ and the index of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite. The classes of upper semi-Fredholm and lower semi-Fredholm from X into Y are defined respectively by

$$\Phi_+(X) := \Big\{T \in C\mathcal{R}(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed}\Big\},$$

and

$$\Phi_-(X) := \Big\{T \in C\mathcal{R}(X) : \beta(T) < \infty \text{ and } R(T) \text{ is closed}\Big\}.$$

$\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ is the set of Fredholm relations from X into X.

In this paper, we are concerned with the following essential spectrum of a closed linear relation T

$$\begin{aligned} \sigma_e(T) &:= \Big\{\lambda \in \mathbb{C} : \lambda - T \in \Phi(X)\Big\}. \\ \sigma_{e1}(T) &:= \Big\{\lambda \in \mathbb{C} : \lambda - T \in \Phi_+(X)\Big\}. \end{aligned}$$

Definition 1.1. *A bounded operator S is called a essential inverse of the closed operator T if*
(i) $R(S) \subset \mathcal{D}(T)$ and $TS = I + K_1$, where $K_1 \in \mathcal{K}(X)$.
(ii) $ST = I + K_2$, $K_2 \in \mathcal{K}(X)$.

Let $T_\lambda(T) := T_i[(\lambda - \lambda_i)T_i + I]^{-1}$ for all $\lambda \in \Phi_i(T)\backslash\Phi_i^0(T)$ such that

$$\Phi_i^0(T) = \Big\{\mu \in \mathbb{C} : \frac{-1}{\mu - \lambda_i}\sigma(T_i)\Big\} \cap \Phi_i(T)$$

and T_i is the bounded operator satisfying:

$$\begin{aligned} T_i(\lambda_i - T) &= I - P_{1i} \ \text{ on } \mathcal{D}(T) \\ (\lambda_i - T)T_i &= I - P_{2i} \ \text{ on } X, \end{aligned}$$

where P_{1i} and P_{2i} are two projection bounded finite rank operators. The operator $T_\lambda(T)$ is shown in [38] to be a essential inverse of $(\lambda - T)$.

The concept of Fredholm operators is one of the attempts to understand the classical Fredholm theory of integral equations. Further important contributions were due to A. Jeribi [33] who gave a simple and unified treatment of this theory which covered all the basic points while avoiding some of the involved concepts (see also [2]-[22]). Recently, W. Chaker, A. Jeribi and B. Krichen [32] have utilized demicompact operators in order to investigate the essential spectra of closed linear operators. In [34] B. Krichen introduced the relative demicompactness class with respect to a given closed linear operator as a generalization of the demicompactness notion. Lately, in [29] A. Ammar, H. Daoud and A. Jeribi defined the demicompact of a linear relation by $T : \mathcal{D}(T) \subseteq X \to X$ is said to be demicompact if for every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $Q_{I-T}(I-T)x_n \to y \in X/\overline{(I-T)(0)}$, there is a convergent subsequence of $\{Q_T x_n\}$. Then, is to generalize some results given in [32] to multivalued linear operators.

In [24] T. Alvarez, A. Ammar and A. Jeribi extended some properties of Fredholm relations that we need to study the concept of essential spectra. Linear selections have been investigated in R. W. Cross [31] and have found several applications. In [26, 27] A. Ammar, A. Jeribi and B. Saadaoui introduced the phenomenon of linear selection to ensure certain matrix decomposition (example: the Frobenius-Schur decomposition for multivalued matrices operator). The development of spectral theory for linear relations was the aim of recent paper in 2012 D. Gheorghe and F.-H. Vasilescu [35]. It has to be mentioned that D. Gheorghe and F.-H. Vasilescu study in paper [36] linear maps defined between spaces of the form $X = X_0$, where X is a vector space and X_0 is a vector subspace of X. The strong connection between linear relations and quotient range operators is well known and easily explained (see [37]). In [28] A. Ammar, A. Jeribi and B. Saadaoui studied some perturbation results and some relations between the essential pseudospectra of the sum of two multivalued linear operator and the essential pseudospectra of each of this multivalued linear operator.

This work is devoted to extend the results started in [23, 39] to various essential spectra of bounded multivalued operator.

The general objectives of the study to characterize the spectrum of the sum and the product of two linear relations. We organize our paper in the following way. In section 2, we recall some definitions and results needed in the rest of the paper. In section 3, we give some sufficient conditions for the linear relation demicompact that must be Fredholm. In section 4, we gather some results and notations from Fredholm theory connected with the third section. In section 5, we obtain a result equivalent to a special case of Theorems 3.1 and 3.2 in [23]. In the last section we apply the results of section 3 to describe the matrix $\mathcal{A} = \begin{pmatrix} A & B \\ C + CA^{-1} - CA^{-1} & D + CA^{-1}B - CA^{-1}B \end{pmatrix}$ of a linear relation in the form of a Fredholm linear relations.

2. Preliminary and auxiliary results

In this section, we recall some preliminary results from the theory of linear relation in Banach spaces which will be needed In the sequel (see [1]).

Lemma 2.1. [31] *Let X and Y be two vector spaces and let $T \in L\mathcal{R}(X)$. Then*
(i) $\mathcal{D}(T^{-1}) = R(T)$ and $\mathcal{D}(T) = R(T^{-1})$.
(ii) T injective if, and only if, $T^{-1}T = I_{\mathcal{D}(T)}$.
(iii) T is single valued if, and only if, $T(0) = \{0\}$.
(iv) $TT^{-1}y = y + T(0)$ and $T^{-1}Tx = x + T^{-1}(0)$.

Lemma 2.2. [31, Lemma V.2.9] *If $T \in L\mathcal{R}(X,Y)$ and $S \in L\mathcal{R}(Y,Z)$ such that $\overline{T(0)} \subset \mathcal{D}(S)$ and S is a continuous, then $Q_{ST}ST = Q_{ST}SQ_T^{-1}Q_T T$.*

Lemma 2.3. [29] *Let D be a compact linear subspace of a space X. Let $\{x_n\}$ in X be a sequence such that $\{Q_D x_n\}$ is a convergent sequence, then $\{x_n\}$ has a convergent subsequence.*

Corollary 2.4. *Let D a linear subspace of a space X with* $\dim(D) < \infty$. *Let $\{x_n\}$ in X be a sequence such that $\{Q_D x_n\}$ is a convergent sequence, then $\{x_n\}$ has a convergent subsequence.*

Theorem 2.5. [31, Theorem III.5.3] *Let X, Y be Banach spaces and let $T \in C\mathcal{R}(X,Y)$. Then, T is open if, and only if, $R(T)$ is closed.*

Proposition 2.6. (i) [30, Lemma 2.4] *Let $T \in \mathcal{L}R(X)$ and $S, R \in L\mathcal{R}(Y,Z)$. If $T(0) \subset N(S)$ or $T(0) \subset N(R)$, then $(R+S)T = RT + ST$.*
(ii) [31, Proposition I.4.2] *Let $R, S, T \in L\mathcal{R}(X)$. Then,*

(ii_1) *$(R+S)T \subset RT + ST$ with equality if T is single valued.*

(ii_2) *$T(R+S)$ is an extension of $TR + TS$ and $TR + TS = T(R+S)$ if*

$\mathcal{D}(T)$ is the whole space.
(iii) [24, Theorem 2.2] *Let $S, T \in L\mathcal{R}(X)$ be closed. If S and T are everywhere defined such that $TS \in \Phi(X)$ and $ST \in \Phi(X)$, then $S \in \Phi(X)$ and $T \in \Phi(X)$.*

Theorem 2.7. *Let $T \in C\mathcal{R}(X)$ and $\mu \in \mathbb{C}^*$. If $\frac{1}{\mu}T$ is demicompact, then $\mu - T \in \Phi_+(X)$.*

Lemma 2.8. (i) ([31, Lemma V.7.8]) *Let $T \in L\mathcal{R}(X,Y)$ have $dim T(0) < \infty$. Then, $S + T - T \in \Phi_+(X,Y)$ if, and only if, $S \in \Phi_+(X)$.*

(ii) ([31, Lemma VII.1.4]) *Let the relation S satisfy $\mathcal{D}(F) \supset \mathcal{D}(T)$ and $dim T(0) < \infty$. Then, $T + S \in \Phi(X,Y)$ if, and only if, $T \in \Phi(X,Y)$.*

Theorem 2.9. [31, Theorem V.10.3] *Let $T \in L\mathcal{R}(X)$. Then, the following are equivalent:*
(i) *$T \in \Phi_+(X)$.*
(ii) *There exists $A \in B\mathcal{R}(X)$ and a finite rank projection K such that $AT = I - K$.*

Lemma 2.10. [31, Corollary V.15.7] *Let X and Y be complete and T closed. Then, for any linear operator S satisfying $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| < \gamma(T)$ we have*

$$i(T+S) = i(T).$$

Lemma 2.11. [25, Lemma 2.5]*Let $S, T, A \in L\mathcal{R}(X,Y)$. If $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$, then $T - S + S = T$.*

Theorem 2.12. [24, Theorem 2.2] *Let X be a Banach space and let $S, T \in C\mathcal{R}(X)$. Then*
(i) *$T \in \Phi_+(X)$ if, and only if, $Q_T T \in \Phi_+(X)$. In such case $i(T) = i(Q_T T)$.*
(ii) *If $S, T \in \Phi_+(X)$, then $ST \in \Phi_+(X)$ and $TS \in \Phi_+(X)$.*
(iii) *If S and T are everywhere defined and $TS \in \Phi_+(X)$, then $S \in \Phi_+(X)$.*

Lemma 2.13. [30, Lemma 2.4] *Let $T \in L\mathcal{R}(X,Y)$ and $S, R \in L\mathcal{R}(Y,Z)$. If $T(0) \subset N(S)$ or $T(0) \subset N(R)$, then $(R+S)T = RT + ST$.*

Proposition 2.14. [26, Proposition 2.2] *Let $T \in L\mathcal{R}(X,Y)$ be closed and $S \in L\mathcal{R}(X,Y)$ be continuous. We have*

(i) *If $T \in \Phi_+(X,Y)$ and $S \in \mathcal{P}_+(X,Y)$, then $T + S \in \Phi_+(X,Y)$ and $i(T+S) = i(T)$.*

(ii) *If $T \in \Phi_-(X,Y)$ and $S \in \mathcal{P}_-(X,Y)$, then $T + S \in \Phi_-(X,Y)$ and $i(T+S) = i(T)$.*

(iii) *If $T \in \Phi(X,Y)$ and $S \in \mathcal{P}(X,Y)$, then $T + S \in \Phi(X,Y)$ and $i(T+S) = i(T)$.*

Proposition 2.15. [31, Exercise I.2.14(b)] *$T, S \in L\mathcal{R}(X,Y)$. If $\mathcal{D}(T) = \mathcal{D}(S)$ and $T(0) = S(0)$, then $T = S$ or the graphs of T and S are incomparable.*

Theorem 2.16. [26, Theorem 3.1] *Let $T \in B\mathcal{R}(X,Y)$ be a single valued bijective and assume that $R \in B\mathcal{R}(Z,W)$ is bijective with $R(0)$ closed.*
(i) *If $S \in L\mathcal{R}(Y,Z)$ is closable, then RST is closable and $\overline{RST} = R\overline{S}T$.*
(ii) *If R is bounded single valued bijective, then $S \in L\mathcal{R}(Y,Z)$ is closable if, and only if, RST is closable and $\overline{RST} = R\overline{S}T$.*

Definition 2.17. [26, Definition 2.1] *Let $S \in L\mathcal{R}(X)$ be continuous.*
S is called a Fredholm perturbation if $T + S \in \Phi(X)$ whenever $T \in \Phi(X)$ with $\dim(S(0)) < \infty$ and $S(0) \subset T(0)$.
The sets of Fredholm perturbation is denoted by $\mathcal{P}(X)$.

3. Demicompact linear relation and selection

In this section, we define a demicompact linear relation and give few properties and results.

Definition 3.1. [31, Definition I.5.1]*Let $T \in L\mathcal{R}(X)$. A linear operator S is called a selection of T if $T = S + T - T$ and $\mathcal{D}(T) = \mathcal{D}(S)$.*

Remark 3.2. *It's clear that if S is a selection of T, then we have*

$$Tx = Sx + T(0) \text{ for all } x \in \mathcal{D}(T).$$

Theorem 3.3. *Let S be selection of $T \in L\mathcal{R}(X)$ and $\dim(T(0)) < \infty$. If T is a demicompact linear relation, then S is a demicompact operator.*

Proof. Let S be selection of T. Suppose that T is a demicompact linear relation. It suffices to show that, S is demicompact operator. Let $\{x_n\}$ be a bounded sequence of X such that:

$$(I - S)x_n \longrightarrow y.$$

Therefore, $Q_T(I - S)x_n \longrightarrow Q_T y \in Y/T(0)$. Since $Q_T T(0) = 0$, we deduce that

$$Q_T(I - S)x_n + Q_T T(0) \longrightarrow Q_T y \in X/T(0).$$

Hence $Q_T(I - T)x_n \longrightarrow Q_T y \in X/T(0)$. This implies that $\{Q_T x_n\}$ has a convergent subsequence. Since $\dim(T(0)) < \infty$, then the result follows directly from Corollary 2.4. □

Theorem 3.4. *Let S be selection of $T \in L\mathcal{R}(X)$ and $dim(T(0)) < \infty$. Then, T is a demicompact linear relation if, and only if, S is a demicompact operator.*

Proof. Let T is a demicompact linear relation by using Theorem 2.7, we get $I - T \in \Phi_+(X)$. Since $I - S$ be selection of $I - T$ with $dim(T(0)) < \infty$, then by Lemma 2.8 we obtain $I - S \in \Phi_+(X)$. From Theorem 2.9, it follows that there exists $A \in B\mathcal{R}(X)$ and a finite rank projection K such that $A(I - S) = I - K$. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(S)$ such that $(I - S)x_n \longrightarrow y$, then

$$A(I - S)x_n \longrightarrow Ay.$$

We conclude that

$$(I - K)x_n \longrightarrow Ay.$$

Since K is compact, then $(Kx_n)_n$ has a convergent subsequence and so $\{x_n\}$ has also a convergent subsequence. Conversely, let S is a demicompact operator, then by using Theorem 2.7, we get $I - S \in \Phi_+(X)$. Since $I - S$ be selection of $I - T$ with $dim(T(0)) < \infty$, then by Lemma 2.8 we obtain $I - T \in \Phi_+(X)$. From Theorem 2.9, it follows that there exists $A \in B\mathcal{R}(X)$ and a finite rank projection K such that $A(I - T) = I - K$. Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $Q_T(I - T)x_n \longrightarrow y$. Since $Q_{A(I-T)}AQ_T^{-1}$ is a bounded operator, then

$$Q_{A(I-T)}AQ_T^{-1}Q_T(I - T)x_n \longrightarrow Q_{A(I-T)}AQ_T^{-1}y.$$

Since $\overline{T(0)} \subset \mathcal{D}(A)$ and A is continuous, then by using Lemma 2.2, we obtain $Q_{I-K}A(I - T)x_n \longrightarrow Q_{I-K}AQ_T^{-1}y$. Equivalently to $Q_K(I - K)x_n \longrightarrow Q_K AQ_T^{-1}y$. Since $\dim K(0) < \infty$, then $\{x_n\}$ has also a convergent subsequence. □

Theorem 3.5. *Let $T \in C\mathcal{R}(X)$ and $dimT(0) < \infty$. If $\frac{1}{\mu}T$ is demicompact for each $\mu \in [1,+\infty[$, then $\mu - T \in \Phi(X)$.*

Proof. Let S is selection of T. Since $\frac{1}{\mu}T$ is demicompact for each $\mu \in [1,+\infty[$, then by Theorem 2.7 we get $\mu - T \in \Phi_+(X)$. By using Lemma 2.8 (i) and Theorem 3.3, we get $\mu - S \in \Phi(X)$. We shall prove that the map

$$\begin{array}{llll} \varphi: & [1,+\infty[& \longrightarrow & \mathbb{Z} \\ & \mu & \longrightarrow & i(\mu - S) \end{array}$$

is continuous in μ. For this, let $\mu, \mu_0 \in [1,+\infty[$ arbitrary but fixed such that $|\mu - \mu_0| < \gamma(\mu - S)$. By using Lemma 2.10, we have

$$i(\mu - S) = i(\mu - S - \mu + \mu_0) = i(\mu_0 - S).$$

Let $\varepsilon > 0$ there exists $\delta := \gamma(\mu - S)$ such that, if $\mu, \mu_0 \in [1,+\infty[$ with $|\mu - \mu_0| < \delta$, then $|i(\mu - S) - i(\mu_0 - S)| = |0| = 0 < \varepsilon$. So, that $\varphi(\mu)$ is continuous. Now, we know that every continuous mapping of a connected in $\mathbb{Z}$ is constant.

If $\mu \longrightarrow +\infty$, then $i\left(\left(I - \frac{1}{\mu}S\right)\mu\right) = i\left(I - \frac{1}{\mu}S\right) = i(I) = 0$. Showing that

$i(\mu - S) = 0$ for each $\mu \in [1,+\infty[$. We conclude that $\alpha(\mu - S) = \beta(\mu - S) < \infty$, then $\mu - S$ is a Fredholm linear operator. Now, by Lemma 2.8 we conclude that $\mu - T \in \Phi(X)$. □

4. Some properties of linear selections

Lemma 4.1. *Let T_1 is selection of T and S_1 is selection of S, then:*
(i) *$T_1 + S_1$ is selection of $T + S$.*
(ii) *If $\mathcal{D}(T)$ containing the ranges of both S and S_1, then T_1S_1 is selection of TS.*

Proof. (i) Let T_1 is selection of T and S_1 is selection of S, then

$$\mathcal{D}(T + S) = \mathcal{D}(T) \cap \mathcal{D}(S) = \mathcal{D}(T_1) \cap \mathcal{D}(S_1) = \mathcal{D}(T_1 + S_1).$$

It is easy to prove that

$$T + S = T_1 + T - T + S_1 + S - S = T_1 + S_1 + (T + S) - (T + S).$$

(ii) Let T_1 is selection of T and S_1 is selection of S, then

$$TS = T(S_1 + S - S).$$

Since $\mathcal{D}(T)$ containing the ranges of both S, then by Proposition 2.6 and Proposition 4.7 we have $T(S_1+S-S) = TS_1 + TS - TS$. This implies that $TS = (T_1 + T - T)S_1 + TS - TS$. Hence, S_1 is single valued, then

$$TS = T_1S_1 + TS_1 - TS_1 + TS - TS.$$

Let $x \in \mathcal{D}(TS)$, then $TSx = T_1S_1x + TS_1(0) + TS(0) = T_1S_1x + T(0) + TS(0)$. Since $T(0) \subset TS(0)$, then

$$TSx = T_1S_1x + TS_1(0) + TS(0) = T_1S_1x + TS(0).$$

We still have to show that $\mathcal{D}(T_1S_1) = \mathcal{D}(TS)$. Indeed,

$$\begin{array}{lll} \mathcal{D}(T_1S_1) & = & \left\{x \in \mathcal{D}(S_1) : D(T_1) \cap S_1x \neq 0\right\} \\ & = & \left\{x \in \mathcal{D}(S) : D(T) \cap S_1x \neq 0\right\}. \end{array}$$

We have $\mathcal{D}(T)$ containing the ranges of both S, then

$$\mathcal{D}(T_1S_1) \quad \subset \quad \left\{x \in \mathcal{D}(S) : D(T) \cap Sx \neq 0\right\}.$$

Conversely,

$$\begin{aligned}\mathcal{D}(TS) &= \Big\{x \in \mathcal{D}(S) : D(T) \cap Sx \neq 0\Big\}\\ &= \Big\{x \in \mathcal{D}(S_1) : D(T_1) \cap Sx \neq 0\Big\}\\ &\subset S_1^{-1}\mathcal{D}(T_1) = \mathcal{D}(T_1S_1).\end{aligned}$$

We conclude that T_1S_1 is selection of TS. □

Proposition 4.2. *Let $T \in C\mathcal{R}(X, Y)$. If S is selection of T, then $S \in C\mathcal{R}(X, Y)$.*

Remark 4.3. *Let S is a selection of T, then for all $\lambda \in \mathbb{C}$ we have $\lambda - S$ is a selection of $\lambda - T$. Indeed, for all $x \in \mathcal{D}(T)$ we have $(\lambda - T)x = (\lambda - S)x + T(0) = (\lambda - S)x + (\lambda - T)(0)$ and $\mathcal{D}(\lambda - T) = \mathcal{D}(T) = \mathcal{D}(S) = \mathcal{D}(\lambda - S)$.*

Proposition 4.4. *Let $T \in L\mathcal{R}(X)$ and K is a demicompact linear relation, then $I - K + T - T \in \Phi(X)$.*

Proof. By using Theorem 3.5, we have $I - K \in \Phi(X)$ for all K is a demicompact linear relation. Since $0 = \|T - T\| < \gamma(I - K)$, then by applying [31, Theorem V.5.12] we get $I - K + T - T \in \Phi(X)$. □

Proposition 4.5. *Let $T \in \mathcal{L}R(X)$. Let S is a selection of T it is assumed there exist $S_1 \in B\mathcal{R}(X)$ and $S_2 \in \mathcal{L}(X)$ with K_1 and K_2 are demicompact such that $S_1S = I - K_1$ and $SS_2 = I - K_2$, then $T \in \Phi(X)$.*

Proof. By using Proposition 2.6, it is clear that $S_1T = S_1(S + T - T) = I - K_1 + S_1T - S_1T$. By Theorem 3.5 we get $I - K_1 \in \Phi(X)$. Since $0 = \|S_1T - S_1T\| < \gamma(I - K_1)$, then by [31, Theorem V.5.12] we have $S_1T = I - K_1 + S_1T - S_1T \in \Phi(X)$. In the same way to find the following result $TS_2 \in \Phi(X)$. By applying Proposition 2.6 (*iii*) we conclude that $T \in \Phi(X)$. □

Corollary 4.6. *Let $T \in \mathcal{L}R(X)$. Let S is a selection of T it is assumed there exist $S_1, S_2 \in \mathcal{L}(X)$ and $K_1, K_2 \in \mathcal{K}(X)$ such that $S_1S = I - K_1$ and $SS_2 = I - K_2$, then $T \in \Phi(X)$.*

Proposition 4.7. *If S is a selection of T then we have:*
(*i*) *$N(S) \subset N(T)$,*
(*ii*) *$R(S) \subset R(T)$,*
(*iii*) *If $T(0) \subset R(S)$, then $R(S) = R(T)$.*

Proof. (*i*) Let $x \in N(S)$ if and only if $x \in \mathcal{D}(S)$ such that $S(x) = 0$. Then we have $Sx + T(0) = Tx = T(0)$ for all $x \in \mathcal{D}(S) = \mathcal{D}(T)$. So, $x \in N(T)$.
(*ii*) Let $y \in R(S)$ if and only if there exists $x \in \mathcal{D}(S)$ such that $Sx = y$, then $Sx + T(0) = y + T(0)$. Therefore $Tx = y + T(0)$ equivalent to $y \in R(T)$.
(*iii*) Since $R(T) = R(S) + T(0)$, we conclude that $R(T) = R(S)$. □

Proposition 4.8. *Let $T \in C\mathcal{R}(X)$ is injective. If $R(T)$ is closed, then there exists an injective selection S such that $R(S)$ is closed.*

Proof. If T is closed, then by Proposition 4.2 we have a closed selection S. Since T is injective by Proposition 4.7 we have $N(T) = N(S) = \{0\}$. Let $R(T)$ is closed, then by Theorem 2.5 we have T is open equivalent sense $\gamma(T) > 0$. So,

$$\gamma(T)\|d(x, N(T)\| \leq \|Tx\|$$

since T is injective, then

$$\gamma(T)\|x\| \leq \|Tx\| \leq \|Sx\|.$$

Let $y_n \in R(S)$ such that $y_n \longrightarrow y$.
There exists $x_n, x_m \in \mathcal{D}(S)$ such that $y_n = Sx_n$ and $y_m = Sx_m$ for all $n, m \geq 1$. As,

$$\|x_n - x_m\| \leq \frac{1}{\gamma(T)}\|S(x_n - x_m)\| \leq \frac{1}{\gamma(T)}\|y_n - y_m\|.$$

Since y_n is converge, then it is Cauchy, consequently $(x_n)_n$ is Cauchy, therefore $x_n \longrightarrow x$, and we have $x_n \in \mathcal{D}(S)$, $Sx_n \longrightarrow y$ and S is closed, then $x \in \mathcal{D}(S)$ and $Sx = y$. Consequently $y \in R(S)$. $\square$

Proposition 4.9. *Let $T \in L\mathcal{R}(X, Y)$ is injective and S is the selection of T, then*

$$\gamma(T) \leq \gamma(S).$$

Proof. We have $\gamma(T)d(x, N(T)) \leq \|Tx\|$, by Proposition 4.7 we get $N(S) = N(T) = \{0\}$, then $\gamma(T)\|x\| \leq \|Tx\|$. Equivalently $\gamma(T)\|x\| \leq \|Tx\| \leq \|Sx\|$, so

$$\gamma(T)d(x, N(S)) \leq \|Sx\|.$$

Consequently, $\gamma(T) \leq \gamma(S)$. $\square$

Corollary 4.10. *Let $T \in L\mathcal{R}(X, Y)$ is injective and S is the selection of T.*
(i) If T is open, then S is open.
(ii) If $R(T)$ is closed, then $R(S)$ is closed.
(iii) If $dimR(T) < \infty$, then $R(S)$ is closed.

Proof. (i) Since T is open, then $\gamma(T) > 0$. Using Proposition 4.9, we obtain $\gamma(S)$ is open.
(ii) Using Theorem 2.5 and (i).
(iii) Using [31, Proposition II.3.2 (d)] and (ii). $\square$

Example 4.11. *If T is open, then there exists an open selection.*
Indeed, if P is a linear projection with domain $R(T)$ and kernel $T(0)$, then PT is a selection of T. Using [31, Theorem II.3.11] *we get*

$$\gamma(P)\gamma(T) \leq \gamma(PT).$$

Since T is open, then $\gamma(T) > 0$. If $R(P) \not\subseteq \overline{N(P)}$, then by Example [31, Example II.3.3] *we get $\gamma(P) > 0$, therefore $\gamma(PT) > 0$.*

Theorem 4.12. *Let $T \in C\mathcal{R}(X)(X)$, then T has a closed selection S and if $T(0) \subset R(S)$ we get*

$$\rho(T) \subset \rho(S).$$

Proof. Let $T \in C\mathcal{R}(X)(X)$, by Lemma 4.2 we get T has a selection S is closed.
Let $\lambda \in \rho(T)$, then $\lambda - T$ is bijective. Show that $\lambda - S$ is bijective. Indeed, let $x \in N(\lambda - S)$ if, and only if, $\begin{cases} x \in \mathcal{D}(\lambda - S) = \mathcal{D}(S) = \mathcal{D}(T) \\ (\lambda - S)x = 0 \end{cases}$, therefore $(\lambda - T)x = (\lambda - S)x + T(0) = T(0)$, then $x \in N(\lambda - T)$. Hence, $N(\lambda - S) \subset N(\lambda - T)$, that is $\lambda - S$ is injective. Show that $R(\lambda - S) = X$, indeed, since $T(0) \subset R(S)$. Therefore by Proposition 4.7 we get $R(\lambda - T) = R(\lambda - S) = X$. We conclude that $\lambda \in \rho(S)$. $\square$

Lemma 4.13. *Let $T \in \Phi(X)$, $S \in B\mathcal{R}(X)$ and $F \in \mathcal{P}(X)$, suppose that $TS = F$. If $S(0) \subset N(T)$, then $S \in \mathcal{P}(X)$.*

Proof. Let T_1 is a selection of T with $T(0) \subset R(T_1)$. Since $T \in \Phi(X)$, then $T_1 \in \Phi(X)$. So, there exists $A \in \mathcal{L}(X)$ such that $AT_1 = I - K$ where $K \in \mathcal{K}(X)$. Hence, $ATS = A(T_1 + T - T)S = AF$ by using Proposition 2.6 (ii) we gate $ATS = (AT_1 + AT - AT)S = (I - K + AT - AT)S$. Since $S(0) \subset N(AT - AT)$, then we applied Proposition 2.6 (i) we find $ATS = (I - K)S + (AT - AT)S$, since $\mathcal{D}(A) = X$, then by Proposition 2.6 (ii) we have $ATS = (I - K)S + A(T - T)S$. Hence $S(0) \subset N(T)$, then by Proposition 2.6 (i) we get

$$ATS = (I - K)S + A(TS - TS) = (I - K)S + ATS - ATS.$$

Therefore, $(I - K)Sx \in ATSx$ for all $x \in X$. Obviously, $ATSx = AFx = F_1x$ where $AF = F_1$, then we get $F_1 \in \mathcal{P}(X)$, then $T + F_1 \in \Phi(X)$. We deduce that

$$T + (I - K)S + F_1 - F_1 = T + S - KS + F_1 - F_1 \in \Phi(X).$$

Now using the fact that the linear relation $KS + F_1 - F_1 \in \mathcal{P}(X)$. Hence $T + S \in \Phi(X)$, since $T \in \Phi(X)$, then it is clear that S is a Fredholm perturbation. $\square$

Lemma 4.14. *Let $T \in B\mathcal{R}(X)$, $S \in \mathcal{L}(X)$, $\lambda \in \Phi_T(X)\backslash\Phi^0(T)$, $\mu \in \Phi_S(X)\backslash\Phi^0(T)$ and T_1 is a selection of T with $T(0) \subset R(T_1)$. If there exist a Fredholm perturbation F_1 is single valued, such that $TS = ST + F_1$. Then there exists a Fredholm perturbation single valued F depending analytically on λ and μ such that*

$$(\lambda - T)\Big(ST_\lambda(T_1) - T_\lambda(T_1)S\Big) = T(0) + F.$$

If $\lambda - T$ is injective, then $T_\lambda(T_1)T_\mu(S) = T_\mu(S)T_\lambda(T_1) + F$.

Proof. Let $x \in X$, then by Proposition 2.6 (i) we have

$$\begin{aligned}(\lambda - T)ST_\lambda(T_1)x &= (\lambda S - TS)T_\lambda(T_1)x \\ &= \Big(\lambda S - (ST - F_1)\Big)T_\lambda(T_1)x.\end{aligned}$$

Since $T_\lambda(T_1)$ is single valued, therefore by Proposition 2.6 (ii) we have

$$\begin{aligned}(\lambda - T)ST_\lambda(T_1)x &= \Big(\lambda ST_\lambda(T_1) - (ST - F_1)T_\lambda(T_1)\Big)x \\ &= \Big(\lambda ST_\lambda(T_1) - STT_\lambda(T_1) - F_1T_\lambda(T_1)\Big)x \\ &= \Big(S(\lambda - T)T_\lambda(T_1) - F_1T_\lambda(T_1)\Big)x.\end{aligned}$$

Since $(\lambda - T)T_\lambda(T_1) = (\lambda - T_1 + T - T)T_\lambda(T_1) = I - K_1 + (T - T)T_\lambda(T_1)$. Hence, we get

$$(\lambda - T)ST_\lambda(T_1)x = \Big(S - SK_1 + S(T - T)T_\lambda(T_1) - F_1T_\lambda(T_1)\Big)x.$$

Let $F_2 = -SK_1 - F_1T_\lambda(T_1)$ and we have $F_2 \in \mathcal{P}(X)$, then $(\lambda - T)ST_\lambda(T_1)x = Sx + ST(0) + F_2x$. Moreover,

$$\begin{aligned}(\lambda - T)T_\lambda(T_1)Sx &= \Big(I - K_1 + (T - T)T_\lambda(T_1)\Big)Sx \\ &= (I - K_1)Sx + T(0) \\ &= Sx - K_1Sx + T(0) \\ &= Sx + K_2x + T(0), \ \ (\text{where} - K_1Sx = K_2x).\end{aligned}$$

This make us conclude that

$$\begin{aligned}(\lambda - T)\Big(ST_\lambda(T_1) - T_\lambda(T_1)S\Big)x &= Sx + ST(0) + F_2x - Sx - K_2 + T(0) \\ &= S(0) + ST(0) + F_3x + T(0) \\ &= (TS + F_1)(0) + F_3x \\ &= T(0) + F_3x.\end{aligned}$$

If $\lambda - T$ is injective we get,

$$\begin{aligned}(\lambda - T)^{-1}(\lambda - T)\Big(ST_\lambda(T_1) - T_\lambda(T_1)S\Big)x &= (\lambda - T)^{-1}\Big(T(0) + F_3x\Big) \\ &= (\lambda - T)^{-1}T(0) + (\lambda - T)^{-1}F_3x \\ &= (\lambda - T)^{-1}(0) + (\lambda - T)^{-1}F_3x \\ &= (\lambda - T)^{-1}F_3x.\end{aligned}$$

By using Lemma 2.1, we get

$$\Big(ST_\lambda(T_1) - T_\lambda(T_1)S\Big)x + (\lambda - T)^{-1}(0) = (\lambda - T)^{-1}F_3x.$$

Since $\lambda - T$ is injective we obtain that $\Big(ST_\lambda(T_1) - T_\lambda(T_1)S\Big) = F_4$, where $F_4 = (\lambda - T)^{-1}F_3$. Therefore, $ST_\lambda(T_1) = T_\lambda(T_1)S + F_4$. Thus, we obtain

$$\begin{aligned}(\mu - S)[T_\mu(S)T_\lambda(T_1) - T_\lambda(T_1)T_\mu(S)] &= (I - K_3)T_\lambda(T_1) - (\mu - S)T_\lambda(T_1)T_\mu(S)\\ &= T_\lambda(T_1) - K_4 - [T_\lambda(T_1)(\mu - S) + F_5]T_\mu(S)\\ &= T_\lambda(T_1) - K_4 - T_\lambda(T_1)(I - K_5) - F_6\\ &= -K_4 + K_6 - F_6\\ &= F_7.\end{aligned}$$

where $K_i \in \mathcal{K}(X)$ for $i = 3, 4, 5, 6$ and $F_i \in \mathcal{P}(X)$ for $i = 6, 7$. Hence,

$$T_\mu(S)T_\lambda(T_1) - T_\lambda(T_1)T_\mu(S) = F \text{ where } F \in \mathcal{P}(X).$$

Furthermore, the analyticity of F in λ and μ follows from the analyticity of $T_\mu(S)$ and $T_\lambda(T_1)$. $\square$

5. Some perturbations results

In this section, we give some perturbation results and some relations between the essential spectrum of the sum of two linear relation.

Let $T \in C\mathcal{R}(X)$. We define the sets $\Psi(X)$ and $\Pi_T(X)$ by:

$$\begin{aligned}\Psi(X) &= \Big\{T \in L\mathcal{R}(X) : \mu T \text{ is demicompact for every } \mu \in [0,1]\Big\},\\ \Pi_T(X) &= \left\{ K \in L\mathcal{R}(X) : \begin{array}{c}\mathcal{D}(T) \subset \mathcal{D}(K),\\ K(0) \subset T(0) \text{ and } \forall \mu \in \rho(T+K),\\ -(\mu - T - K)^{-1}K \in \Psi(X)\end{array} \right\}.\end{aligned}$$

We denote

$$\sigma_r(T) := \bigcap_{K \in \Pi_T(X)} \sigma(T + K).$$

Theorem 5.1. *For each $T \in C\mathcal{R}(X)$, we have*

$$\sigma_e(T) \subseteq \sigma_r(T).$$

Proof. Let $T \in C\mathcal{R}(X)$ and $\mu \notin \sigma_r(T)$, then $\mu \notin \cap_{K\in\Pi_T(X)}\sigma(T + K)$. Therefore, $\mu \in \cup_{K\in\Pi_T(X)}\rho(T + K)$. Hence, there exists $K \in \Pi_T(X)$ such that $\mu \in \rho(T + K)$. We conclude that $-(\mu - T - K)^{-1}K$ is demicompact and $\mu - T - K$ is bijective. Hence, $\mu - T - K \in \Phi(X)$ and by applying Theorem 3.5 we get $I + (\mu - T - K)^{-1}K \in \Phi(X)$. Moreover,

$$\begin{aligned}R((\mu - T - K)^{-1}K) &= \mathcal{D}(K^{-1}(\mu - T - K))\\ &= (\mu - T - K)^{-1}R(K)\\ &= \Big\{x \in \mathcal{D}(\mu - T - K) : R(K) \cap (\mu - T - K)x \neq \emptyset\Big\}\\ &= \Big\{x \in \mathcal{D}(T) : R(K) \cap (\mu - T - K)x \neq \emptyset\Big\}.\end{aligned}$$

Let $x \in \mathcal{D}(T)$, since $\mathcal{D}(T)$ contain the ranges of $(\mu - T - K)^{-1}K$, then by Proposition 2.6 (*ii*) we get

$$\begin{aligned}(\mu - T - K)(x + (\mu - T - K)^{-1}Kx) &= (\mu - T - K)x\\ &= +(\mu - T - K)(\mu - T - K)^{-1}Kx.\end{aligned}$$

By Lemma 2.1 (*iv*), we have

$$\begin{aligned}(\mu - T - K)(x + (\mu - T - K)^{-1}Kx) &= (\mu - T - K)x + Kx\\ &\quad +(\mu - T - K)(0)\\ &= (\mu - T - K)x + Kx.\end{aligned}$$

Since $\mathcal{D}(T) \subset \mathcal{D}(K)$ and $K(0) \subset T(0)$. By Lemma 2.11, we get

$$\mu - T = \mu - T - K + K.$$

Therefore, $(\mu - T - K)(I + (\mu - T - K)^{-1}K) = \mu - T$. Hence, applying Theorem 2.12, we get $\mu - T \in \Phi(X)$. We deduce that $\mu \notin \sigma_e(T)$. $\square$

Theorem 5.2. *Let $T, S \in B\mathcal{R}(X)$, $T(0)$ is closed and $S(0) \subset T(0)$. If for every $\lambda \notin \sigma_{e1}(T)$, there exists A_λ a left inverse modulo compact of $\lambda - T$ such that SA_λ is demicompact and $T(0) \subset N(SA_\lambda)$, then*

$$\sigma_{e1}(T + S) \subset \sigma_{e1}(T).$$

Proof. Let $\lambda \notin \sigma_{e1}(T)$, then $\lambda - T \in \Phi_+(X)$. By Theorem 2.9, there exists $A_\lambda \in B\mathcal{R}(X)$ and a finite rank projection K such that $A_\lambda(\lambda - T) = I - K$. Since $SK(0) = S(0) \subset T(0)$ and $\mathcal{D}(SK) = \mathcal{D}(T) = X$, then by Lemma 2.11 we have

$$\lambda - T - S = \lambda - T - S + SK - SK.$$

By using Proposition 2.6 (*ii*) we get

$$\begin{aligned} \lambda - T - S + SK - SK &= \lambda - T - S(I - K) - SK \\ &= \lambda - T - SA_\lambda(\lambda - T) - SK. \end{aligned}$$

Since $(\lambda - T)(0) = T(0) \subset N(SA_\lambda)$, then by Lemma 2.13 we get

$$\lambda - T - S = (I - SA_\lambda)(\lambda - T) - SK.$$

As SA_λ it follows from Theorem 2.7 that $I - SA_\lambda \in \Phi_+(X)$ and we have $\lambda - T \in \Phi_+(X)$, then by Theorem 2.12 we get $(I - SA_\lambda)(\lambda - T) - SK \in \Phi_+(X)$. Since $SK \in \mathcal{P}(X)$, then by Proposition 2.14 we get $\lambda - T - S \in \Phi_+(X)$. Hence $\lambda \notin \sigma_{e1}(T + S)$. We conclude that $\sigma_{e1}(T + S) \subset \sigma_{e1}(T)$. $\square$

Theorem 5.3. *Let $T, S \in B\mathcal{R}(X)$, $T(0)$ is closed and $S(0) \subset T(0)$. If for every $\lambda \notin \sigma_e(T)$, there exists A_λ a left inverse modulo compact of $\lambda - T$ such that $SA_\lambda \in \Psi(X)$ and $T(0) \subset N(SA_\lambda)$, then*

$$\sigma_e(T + S) \subset \sigma_e(T).$$

Proof. Let $\lambda \notin \sigma_e(T)$, then $\lambda - T \in \Phi_+(X)$. Since A_λ a left inverse modulo compact operator of $\mu - T$ such that $SA_\lambda \in \Psi(X)$, then

$$\lambda - T - S = (I - SA_\lambda)(\lambda - T) - SK.$$

As, $SA_\lambda \in \Psi(X)$, then by Theorem 3.5 we get $I - SA_\lambda \in \Phi(X)$. By using Lemma 2.12 we get $(I - SA_\lambda)(\lambda - T) \in \Phi(X)$. According to Proposition 2.14, we conclude that $\lambda - T - S \in \Phi(X)$. Then $\lambda \notin \sigma_e(T + S)$. $\square$

Theorem 5.4. *Let $T, S \in B\mathcal{R}(X)$, $T(0)$ is closed and $S(0) \subset T(0)$ with $\lambda - T - S \in \Phi_+(X)$.*
(H_1) *If there exists H_λ a left inverse modulo compact of $\lambda - T - S$ such that $-\lambda^{-1}(T)SH_\lambda$ is demicompact, then*

$$\Big[\sigma_{e1}(T + S)\Big]\backslash\{0\} \subset \Big[\sigma_{e1}(T) \cup \sigma_{e1}(S)\Big]\backslash\{0\}.$$

(H_2) *Moreover, if there exists G_λ a left inverse modulo compact of $\lambda - T - S$ such that $-\lambda^{-1}S(T)G_\lambda$ is demicompact, then then*

$$\Big[\sigma_{e1}(T) \cup \sigma_{e1}(S)\Big]\backslash\{0\} = \Big[\sigma_{e1}(T + S)\Big]\backslash\{0\}.$$

Proof. (i) Let $\lambda \in \mathbb{C}\backslash\{0\}$. If there exists H_λ a left inverse modulo compact of $\lambda - T - S$, then by Theorem 2.9 we get $H_\lambda(\lambda - T - S) = I - K$ where $K \in \mathcal{K}(X)$. Since $T \in B\mathcal{R}(X)$, then by Proposition 2.6 (ii) we have $(\lambda - T)(\lambda - S) = \lambda(\lambda - T) - (\lambda - T)S$ and by Lemma 2.13 we have

$$\lambda(\lambda - T) - (\lambda - T)S \quad = \quad \lambda(\lambda - T - S) + TS$$

Hence, $TSK(0) = TS(0) \subset \Big(\lambda(\lambda - T - S) + TS\Big)(0)$ and $\mathcal{D}(\lambda(\lambda - T - S) + TS) = \mathcal{D}(TSK)$, then by lemma 2.11 we get

$$(\lambda - T)(\lambda - S) = \lambda(\lambda - T - S) + TS + TSK - TSK.$$

Clearly, $\mathcal{D}(TS) = X$, then

$$\begin{aligned}(\lambda - T)(\lambda - S) &= \lambda(\lambda - T - S) + TS(I - K) + TSK \\ &= \lambda(\lambda - T - S) + TSH_\lambda(\lambda - T - S) + TSK.\end{aligned}$$

By using Lemma 2.13 we get $\lambda(\lambda - T - S) + TSH_\lambda(\lambda - T - S) + TSK = \lambda\Big(I + \lambda^{-1}TSH_\lambda\Big)(\lambda - T - S) + TSK$. we conclude

$$(\lambda - T)(\lambda - S) = \lambda\Big(I + \lambda^{-1}TSH_\lambda\Big)(\lambda - T - S) + TSK.$$

Since $\mathcal{D}(TSK) = \mathcal{D}\Big(\lambda\Big(I + \lambda^{-1}TSH_\lambda\Big)(\lambda - T - S)\Big)$ and $TSK(0) \subset \lambda(\lambda - T - S) + TS(I - K)(0) = \lambda\Big(I + \lambda^{-1}TSH_\lambda\Big)(\lambda - T - S)(0)$. Then, by using Lemma 2.11 we prove that

$$(\lambda - T)(\lambda - S) - TSK = \lambda\Big(I + \lambda^{-1}TSH_\lambda\Big)(\lambda - T - S).$$

Let $\lambda \notin \Big[\sigma_{e1}(T) \cup \sigma_{e1}(S)\Big]\backslash\{0\}$, then $\lambda - T \in \Phi_+(X)$ and $\lambda - S \in \Phi_+(X)$. Hence, $TSK \in \mathcal{P}(X)$, then by Lemma 2.12 (iii) and Proposition 2.14 we get

$$(\lambda - T)(\lambda - S) - TSK \in \Phi_+(X).$$

Again, by using Lemma 2.12 (v), we show that $\lambda - T - S \in \Phi_+(X)$. We deduce that, $\lambda \notin \Big[\sigma_{e1}(T + S)\Big]\backslash\{0\}$.

(ii) Continuing in the same way, we can find

$$(\lambda - S)(\lambda - T) = \lambda(\lambda - T - S)\Big(I + \lambda^{-1}STG_\lambda\Big) + STK.$$

Let $\lambda \notin \Big[\sigma_{e1}(T + S)\Big]\backslash\{0\}$, then $\lambda - T - S \in \Phi_+(X)$. Since $-\lambda^{-1}TSH_\lambda$ and $-\lambda^{-1}STG_\lambda$ are demicompact, then by Theorem 2.7 we have

$$I + \lambda^{-1}TSH_\lambda \in \Phi_+(X) \text{ and } I + \lambda^{-1}STG_\lambda \in \Phi_+(X).$$

Therefore, by Lemma 2.12 we have $\lambda(\lambda - T - S)\Big(I + \lambda^{-1}(T + D)TSH_\lambda\Big) + STK \in \Phi_+(X)$ and $\lambda\Big(I + \lambda^{-1}TSH_\lambda\Big)(\lambda - T - S) + TSK \in \Phi_+(X)$. Consequently, $TSK, STK \in \mathcal{P}(X)$, then $(\lambda - T)(\lambda - S) \in \Phi_+(X)$ and $(\lambda - S)(\lambda - T) \in \Phi_+(X)$. Applying Lemma 2.12, then $\lambda - T \in \Phi_+(X)$ and $\lambda - S \in \Phi_+(X)$. We deduce that

$$\lambda \notin \Big[\sigma_{e1}(T) \cup \sigma_{e1}(S)\Big]\backslash\{0\}. \quad \square$$

5.1. *The essential spectra of the sum and the product*

In this subsection, we determined the description of the essential spectra of two essentially commuting multivalued operators.

Theorem 5.5. *Let $T \in B\mathcal{R}(X)$ and $S \in \mathcal{L}(X)$. Suppose that there exist $F \in \mathcal{P}(X)$ such that $TS = ST + F$. Then $\sigma_e(T + S) \subseteq \sigma_e(T) + \sigma_e(S)$.*

Proof. If $\sigma_e(T)+\sigma_e(S)=\mathbb{C}$, then the theorem is trivially true. Thus, we assume in the next that $\sigma_e(T)+\sigma_e(S)$ is not the entire plane and let $\mu\notin\sigma_e(T)+\sigma_e(S)$. We define the operator Z as $Z=\mu-T$ and let Z_1 is a selection of Z with $Z(0)\subset R(Z_1)$. Hence, if $\lambda\in\sigma_e(S)$, then $\mu=(\mu-\lambda)+\lambda\notin\sigma_e(T)+\sigma_e(S)$, then $\mu-\lambda\notin\sigma_e(T)$. By using [40, Theorem 3.3], we infer that there exists a domain $\mathcal{D}$ can be chosen such $T_\lambda(Z_1)$ and $T_\lambda(S)$ are analytic on $B(\mathcal{D})$. Define the operators M_1 and M_2 as follows

$$M_1=\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}T_\lambda(Z_1)T_\lambda(S)d\lambda$$

and

$$M_2=\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}T_\lambda(S)T_\lambda(Z_1)d\lambda.$$

We have $\mu-S-T=-(\lambda-Z)+(\lambda-S)$. Since we can write

$$\begin{aligned}(\mu-S-T)M_1 &= \frac{-1}{2i\pi}\int_{+B(\mathcal{D})}-(\lambda-Z)T_\lambda(Z_1)T_\lambda(S)d\lambda\\ &\quad+\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}(\lambda-S)T_\lambda(Z_1)T_\lambda(S)d\lambda.\end{aligned}$$

Obviously,

$$(\lambda-Z)T_\lambda(Z_1)=(\lambda-Z_1+Z-Z)T_\lambda(Z_1)=I-K+(Z-Z)T_\lambda(Z_1)$$

where $K\in\mathcal{K}(X)$. Then the first integral of the above equality is of the form

$$\begin{aligned}\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\Big(I-K+(Z-Z)T_\lambda(Z_1)\Big)T_\lambda(S)d\lambda &= \frac{-1}{2i\pi}\int_{+B(\mathcal{D})}(I-K)T_\lambda(S)d\lambda\\ &\quad+Z-Z.\end{aligned}$$

By [38, Theorem 13], we get $\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}T_\lambda(S)d\lambda=I-K_1$. So, we infer that the first integral is of the form

$$(\mu-S-T)M_1=I-K_2+\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}(\lambda-S)T_\lambda(Z_1)T_\lambda(S)d\lambda+Z-Z.$$

Applying Lemma 4.14, we get $T_\lambda(Z_1)T_\lambda(S)=T_\lambda(S)T_\lambda(Z_1)+K_3$ where K_3 is compact and we have S is a single valued, then we get

$$\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}(\lambda-S)T_\lambda(S)T_\lambda(Z_1)d\lambda=I+K_4,\ \ K_4\in\mathcal{K}(X).$$

Using the fact that $\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}(\lambda-S)K_3d\lambda$ is compact, we have

$$(\mu-S-T)M_1 = I+K_5+Z-Z.$$

By a similar argument we obtain $M_2(\mu-S-T)=I+K_6+Z-Z$, where $K_6\in\mathcal{K}(X)$. By using [31, Theorem V.5.12] we have $(\mu-S-T)M_1\in\Phi(X)$. Again by applying [31, Theorem V.5.12] we get $M_2(\mu-S-T)\in\Phi(X)$. Therefore, Proposition 2.6 (*iii*) we conclude that $(\mu-S-T)\in\Phi(X)$. □

Theorem 5.6. *Let $T\in C\mathcal{R}(X)$ and $S\in\mathcal{L}(X)\cap\Phi(X)$. Suppose that there exist $F\in\mathcal{P}(X)$ such that $TSx=STx+Fx$, for all $x\in\mathcal{D}(T)$. Then $\sigma_e(TS)\subseteq\sigma_e(T)\sigma_e(S)$.*

Proof. Let $\gamma\in\sigma_e(S)\sigma_e(T)$. In what follows, we will show that $\gamma\in\sigma_e(TS)$. Observing that $\sigma_e(T)$ is closed, $\sigma_e(S)$ is compact and $0\notin\sigma_e(S)$. then there exists an open set U, with bounded boundary $B(U)$, containing $\sigma_e(S)$ and satisfying that $0\notin U$ and $\gamma-\mu T\in\Phi(X)$. we obtain the existence of a domain $\mathcal{D}$ such that $\sigma_e(S)\in\mathcal{D}\subseteq U$. Let $\gamma-\mu T$ as follows

$$\gamma-\mu T=(\mu\gamma)\Big(\frac{1}{\mu}-\frac{1}{\gamma}T\Big)=\Big(\frac{\gamma}{\lambda}\Big)\Big(\lambda-\frac{1}{\gamma}T\Big),\ \text{ where }\lambda=\frac{1}{\mu}.$$

Taking $\mathcal{D}'$ the image of $\mathcal{D}$ under the map $\lambda = \frac{1}{\mu}$, we can assume that $T_\lambda(Z_1)$ is analytic in λ on $S(\mathcal{D}')$ where $Z := \frac{1}{\gamma}T$ and Z_1 is a selection of Z with $Z(0) \subset R(Z_1)$. This assumption holds true thanks to the fact that $\lambda - Z \in \Phi(X)$. Let us define the operators M_1 and M_2 as follows

$$M_1 = \frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\frac{1}{\gamma\lambda}T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)d\lambda$$

and

$$M_2 = \frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\frac{1}{\gamma\lambda}T_{\frac{1}{\lambda}}(S)T_\lambda(Z_1)d\lambda.$$

Moreover, since S is a single valued, then we have

$$\begin{aligned}(\gamma - ST) &= (\gamma - S\gamma Z)\\ &= \gamma - S\gamma Z + \gamma\lambda S - \gamma\lambda S\\ &= \gamma S(\lambda - Z) + \gamma(I - \lambda S).\end{aligned}$$

We get

$$(\gamma - ST)M_1 = \frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\frac{1}{\lambda}S(\lambda - Z)T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S) + (\frac{1}{\lambda} - S)T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)d\lambda.$$

In light of this, the first part of the integrand can be written as follows

$$\begin{aligned}&\int_{+B(\mathcal{D})}\frac{1}{\lambda}S(\lambda - Z)T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)d\lambda\\ &= \int_{+B(\mathcal{D})}\frac{1}{\lambda}S\Big(I - K_1 + (Z - Z)T_\lambda(Z_1)\Big)T_{\frac{1}{\lambda}}(S)d\lambda\\ &= \int_{+B(\mathcal{D})}\frac{1}{\lambda}ST_{\frac{1}{\lambda}}(S)d\lambda + K_2 + Z - Z.\end{aligned}$$

Since $0 \notin D$ and S is single valued, hence using [38, Theorems 13 and 14.9] we get

$$\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\frac{1}{\lambda}ST_{\frac{1}{\lambda}}(S)d\lambda = \frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\mu ST_\mu(S)d\lambda = I + K_3.$$

Then we obtain,

$$(\gamma - ST)M_1 = \frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\frac{1}{\lambda}S(\lambda - Z)T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)d\lambda = I + K_4 + Z - Z. \tag{1}$$

Since S is a single valued, we get the second part of the integrand is equal to

$$\int_{+B(\mathcal{D})}(\frac{1}{\lambda} - S)T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)d\lambda = \int_{+B(\mathcal{D})}T_\lambda(Z)d\lambda + F_3. \tag{2}$$

Then we deduce by [38, Theorem 7.4] that

$$\frac{1}{2i\pi}\int_{+B(\mathcal{D})}T_\lambda(Z)d\lambda = K_5. \tag{3}$$

By using Eqs. (1),(2) and (3) we find

$$\begin{aligned}\frac{-1}{2i\pi}\int_{+B(\mathcal{D})}\frac{1}{\lambda}S(\lambda - Z)T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)d\lambda &= I + K_4 + K_5 + F_3 + Z - Z\\ &= I + F_4 + Z - Z.\end{aligned}$$

Since $B \in \mathcal{L}(X)$ we can easily check that

$$\begin{array}{rcl}(\gamma - ST) & = & \gamma - \gamma S\dfrac{T}{\gamma} + \gamma\lambda S - \gamma\lambda S \\ & = & \gamma S(\lambda - Z) + \gamma(I - \lambda S) \\ & = & \gamma(\lambda - Z)S + \gamma(I - \lambda S) + F_5.\end{array}$$

This implies that,

$$\begin{array}{rcl}M_2(\gamma - ST) & = & \dfrac{-1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}\dfrac{1}{\gamma\lambda}T_{\frac{1}{\lambda}}(S)T_\lambda(Z_1)\Big(\gamma(\lambda - Z)S + \gamma(I - \lambda S) + F_5\Big)d\lambda \\ & = & \dfrac{-1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}\dfrac{1}{\lambda}T_{\frac{1}{\lambda}}(S)T_\lambda(Z_1)(\lambda - Z)Sd\lambda \\ & & -\dfrac{1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}\dfrac{1}{\lambda}T_{\frac{1}{\lambda}}(S)T_\lambda(Z_1)(I - \lambda S)d\lambda + F_5 \\ & = & \dfrac{-1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}\dfrac{1}{\lambda}T_{\frac{1}{\lambda}}(S)\Big(I - K_6 + T_\lambda(Z_1)(Z - Z)\Big)Sd\lambda \\ & & -\dfrac{1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}\dfrac{1}{\lambda}(T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S) - F_1)(I - \lambda S)d\lambda + F_5 \\ & = & \dfrac{-1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}\dfrac{1}{\lambda}T_{\frac{1}{\lambda}}(S)Sd\lambda + K_7 + ZS - ZS \\ & & -\dfrac{1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}\dfrac{1}{\lambda}T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)(I - \lambda S)d\lambda + F_6 \\ & = & I + F_7 + Z - Z - \dfrac{1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}T_\lambda(Z_1)T_{\frac{1}{\lambda}}(S)(\dfrac{1}{\lambda} - S)d\lambda \\ & = & I - \dfrac{1}{2i\pi}\displaystyle\int_{+B(\mathcal{D})}T_\lambda(Z_1)(I - K_8)d\lambda. \\ & = & I + F_8 + Z - Z\end{array}$$

where, $K_i \in \mathcal{K}(X)$ and $F_i \in \mathcal{P}(X)$ for all $i \in \{1,2,3,4,5,6,7,8\}$. Now, to show that $\sigma_e(TS) \subseteq \sigma_e(S)\sigma_e(T)$. Indeed, $(\gamma T - TS)M_1 = (\gamma - ST + F)M_1 = (\gamma - ST)M_1 + F_9 = I + F_4 + F_9 + T - T = I + F_{10} + T - T$. where, $F_i \in \mathcal{P}(X)$ for all $i \in \{9,10\}$. By using [31, Theorem V.5.12] we have $(\gamma - TS)M_1 \in \Phi(X)$. Furthermore, since S single valued we have $M_2(\gamma - TS) = M_2\Big(\gamma(\lambda - Z)S + \gamma(I - \lambda S)\Big) = I + F_{11} + T - T$ where, $F_{11} \in \mathcal{P}(X)$. Again by applying [31, Theorem V.5.12] we get $M_2(\gamma - TS) \in \Phi(X)$. Therefore, by using Proposition 2.6 (*iii*) we conclude that $(\gamma - TS) \in \Phi(X)$.

□

6. Application 2 × 2 matrix linear relation

In the product space $X \times Y$, we consider an multivalued linear relation formally defined by a matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C + CA^{-1} - CA^{-1} & D + CA^{-1}B - CA^{-1}B \end{pmatrix}.$$

The block matrices multivalued linear realtion is defined with domain

$$\mathcal{D}(\mathcal{A}) = \Big[\mathcal{D}(A) \cap \mathcal{D}(C + CA^{-1} - CA^{-1})\Big] \times \Big[\mathcal{D}(B) \cap \mathcal{D}(D + CA^{-1}B - CA^{-1}B)\Big].$$

Remark 6.1. *If $CA^{-1}(0) \subset C(0)$ and $CA^{-1}B(0) \subset D(0)$ with $\mathcal{D}(C) \subset \mathcal{D}(CA^{-1})$ and $\mathcal{D}(D) \subset \mathcal{D}(CA^{-1}B)$, then*

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The purpose of this section is to determine the decomposition of $\mathcal{A}$ then we will use the notion of compactness to discuss Fredholm relation.

6.1. Relationship between Fredholm linear relation of $\mathcal{A}$ and demicompactness

Theorem 6.2. *Let $\mathcal{D}(CA^{-1})$ contain the ranges of both A and B and, $\mathcal{D}(C)$ contain the ranges of A with $C(0) \subset CA^{-1}(0)$ and $D(0) \subset CA^{-1}B(0)$. Then,*

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}.$$

Proof. Let $\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] \in G\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right]$.

This is equivalent to

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We can get $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} x_1 + A^{-1}Bx_2 \\ x_2 \end{pmatrix}$

if, and only if, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A(x_1 + A^{-1}Bx_2) \\ Dx_2 - CA^{-1}Bx_2 \end{pmatrix}$. Since,

$$\begin{aligned} R(A^{-1}B) &= \mathcal{D}(B^{-1}A) \\ &= A^{-1}R(B) \\ &= \left\{x \in \mathcal{D}(A) : Ax \cap R(B) \neq \emptyset\right\}, \end{aligned}$$

then $\mathcal{D}(A)$ contain the ranges of $A^{-1}B$. According to Proposition 2.6, we obtain

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} Ax_1 + AA^{-1}Bx_2 \\ Dx_2 - CA^{-1}Bx_2 \end{pmatrix}$. Therefore,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} Ax_1 + Bx_2 + A(0) \\ Dx_2 - CA^{-1}Bx_2 \end{pmatrix}.$$

Which is equivalent to that $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} Ax_1 + Bx_2 \\ Dx_2 - CA^{-1}Bx_2 \end{pmatrix}$

if, and only if, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} Ax_1 + Bx_2 \\ CA^{-1}(Ax_1 + Bx_2) + Dx_2 - CA^{-1}Bx_2 \end{pmatrix}$. Hence, $\mathcal{D}(CA^{-1})$ contain the ranges of both A and B, then by Proposition 2.6 we get

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} Ax_1 + Bx_2 \\ CA^{-1}Ax_1 + CA^{-1}Bx_2 + Dx_2 - CA^{-1}Bx_2 \end{pmatrix}.$$

Equivalent to,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} Ax_1 + Bx_2 \\ C(x_1 + A^{-1}(0)) + Dx_2 + CA^{-1}Bx_2 - CA^{-1}Bx_2 \end{pmatrix}.$$

Since $\mathcal{D}(C)$ contain the ranges of A, then

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + CA_x^{-1}1 - CA_x^{-1}1 + Dx_2 + CA^{-1}Bx_2 - CA^{-1}Bx_2 \end{pmatrix}.$$

We conclude that

$$G\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right] \subseteq G\left[\begin{pmatrix} A & B \\ C + CA^{-1} - CA^{-1} & D + CA^{-1}B - CA^{-1}B \end{pmatrix}\right]. \quad (4)$$

Moreover, $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{aligned}
&= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} A^{-1}B(0) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} AA^{-1}B(0) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A(0) + B(0) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} A(0) + B(0) \\ CA^{-1}(B(0) + A(0)) \end{pmatrix} \\
&= \begin{pmatrix} A(0) + B(0) \\ CA^{-1}(0) + CA^{-1}B(0) \end{pmatrix}.
\end{aligned}$$

Since, $C(0) \subset CA^{-1}(0)$ and $D(0) \subset CA^{-1}B(0)$, then

$$\begin{aligned}
&\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\quad = \begin{pmatrix} A(0) + B(0) \\ C(0) + CA^{-1}(0) + D(0) + CA^{-1}B(0) \end{pmatrix} \\
&\quad = \begin{pmatrix} A & B \\ C + CA^{-1} - CA^{-1} & D + CA^{-1}B - CA^{-1}B \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (5)
\end{aligned}$$

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right]$

if, and only if, $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \neq \emptyset$

if, and only if, $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}\begin{pmatrix} x + A^{-1}By \\ y \end{pmatrix} \neq \emptyset$

if, and only if, $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} Ax + By \\ Dy - CA^{-1}By \end{pmatrix} \neq \emptyset$

if, and only if, $\begin{pmatrix} Ax + By \\ Cx + CA^{-1}(0) + CA^{-1}By + Dy - CA^{-1}By \end{pmatrix} \neq \emptyset$

if, and only if, $\begin{pmatrix} A & B \\ C + CA^{-1} - CA^{-1} & D + CA^{-1}B - CA^{-1}B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \neq \emptyset.$

We obtain that

$$\mathcal{D}\left[\begin{pmatrix} A & B \\ C+CA^{-1}-CA^{-1} & D+CA^{-1}B-CA^{-1}B \end{pmatrix}\right] = \mathcal{D}\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right]. \quad (6)$$

By Eq.s (4), (5), (6) and by using Proposition 2.15, we conclude that

$$\begin{pmatrix} A & B \\ C+CA^{-1}-CA^{-1} & D+CA^{-1}B-CA^{-1}B \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}. \quad \square$$

Theorem 6.3. *Let $\mathcal{D}(CA^{-1}) = X$ and $\mathcal{D}(C)$ contain the ranges of A with $C(0) \subset CA^{-1}(0)$ and $D(0) \subset CA^{-1}B(0)$. If $\begin{pmatrix} A^{-1}B(0) \\ 0 \end{pmatrix} \subset N\left[\begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\right]$, then*

$$\mathcal{A} = -\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & A^{-1}B \\ CA^{-1} & I+C(A^{-1})^2B \end{pmatrix}.$$

Proof. By Theorem 6.2, we get

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A+I-I & 0 \\ 0 & I-I+D-CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\right]\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}. \end{aligned}$$

Hence, $\begin{pmatrix} A^{-1}B(0) \\ 0 \end{pmatrix} \subset N\left[\begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\right]$, then by using Lemma 2.13 we get

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\left[\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} - \begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right]$$

According to Proposition 2.6, we conclude that

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \\ &\quad -\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}. \end{aligned}$$

Let $\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] \in G\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right]$. This is equivalent to $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} x_1 + A^{-1}Bx_2 \\ x_2 \end{pmatrix}$. Then, we can infer that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} x_1 + A^{-1}Bx_2 \\ CA^{-1}(x_1 + A^{-1}Bx_2) + x_2 \end{pmatrix}.$$

Therefore, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} x_1 & A^{-1}Bx_2 \\ CA^{-1}x_1 & CA^{-1}A^{-1}Bx_2 + x_2 \end{pmatrix}$. We prove that,

$$G\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right] \subset G\left[\begin{pmatrix} I & A^{-1}B \\ CA^{-1} & C(A^{-1})^2B + I \end{pmatrix}\right]. \quad (7)$$

Moreover,

$$\begin{aligned}\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A^{-1}B(0) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} A^{-1}B(0) \\ C(A^{-1})^2B(0) \end{pmatrix},\end{aligned}$$

we infer that

$$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I & A^{-1}B \\ CA^{-1} & C(A^{-1})^2B+I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{8}$$

On the one hand,

$$\mathcal{D}\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right] = \mathcal{D}\left[\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right] = X \times \mathcal{D}(A^{-1}B).$$

On the other hand,

$$\mathcal{D}\left[\begin{pmatrix} I & A^{-1}B \\ CA^{-1} & C(A^{-1})^2B+I \end{pmatrix}\right] = X \times \mathcal{D}(A^{-1}B).$$

We can also show that

$$\mathcal{D}\left[\begin{pmatrix} I & A^{-1}B \\ CA^{-1} & C(A^{-1})^2B+I \end{pmatrix}\right] = \mathcal{D}\left[\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}\right]. \tag{9}$$

By Eq.s (7), (8), (9) and by using Proposition 2.15, we deduce that

$$\begin{pmatrix} I & A^{-1}B \\ CA^{-1} & C(A^{-1})^2B+I \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}. \quad \square$$

6.2. *Fredholm linear relation of $\mathcal{A}$*

Theorem 6.4. *Assume that the hypotheses of Theorem 6.3 are satisfied. Let A and $D - CA^{-1}B$ are closed and demicompact linear relation such that $A^{-1}B$ and*

CA^{-1} are bounded single valued. If $\begin{pmatrix} I & A^{-1}B \\ CA^{-1} & C(A^{-1})^2B+I \end{pmatrix} \in \mathcal{P}(X)$, then $\mathcal{A}$ is a Fredholm linear relation.

Proof. Let $\mathcal{A} = -\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & A^{-1}B \\ CA^{-1} & I+C(A^{-1})^2B \end{pmatrix}$.

Since A and $D - CA^{-1}B$ are closed, then $\begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}$ is closed.

It is easy to notice that $\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$ and $\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$ are single valued

bounded bijective. By using Theorem 2.16 we infer that $\mathcal{A}$ is closed. Hence, A and $D - CA^{-1}B$ are demicompact, according to Theorem 3.5, we deduce that $I - A \in \Phi(X)$ and $I - D + CA^{-1}B \in \Phi(X)$. By using Theorem 2.12, we get

$\begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} I-A & 0 \\ 0 & I-D+CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$ is a Fredholm linear

relation. Since $\begin{pmatrix} I & A^{-1}B \\ CA^{-1} & C(A^{-1})^2B+I \end{pmatrix} \in \mathcal{P}(X)$, then by Theorem 2.14 (*iii*)

we get $\mathcal{A}$ is a Fredholm linear relation. $\square$

References

[1] Ammar, A., Jeribi, A.: Spectral Theory of Multivalued Linear Operators. Apple Academic Press, Oakville (2021)

[2] Ammar, A., Daoud, H., Jeribi, A., Demicompact and k-D-set-contractive multivalued linear operators. Mediterr. J. Math. 15 (2018), no. 2, Paper No. 41, 18 pp.

[3] Ammar, A., Fakhfakh, S., Jeribi, A.: Shechter spectra and relatively demicompact linear relations. Commun. Korean Math. Soc. 35 (2020), no. 2, 499–516.

[4] Ammar, A., Fakhfakh, S., Jeribi, A.: Demicompact operators and ellipsoid spectra on quaternionic Hilbert spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 114 (2020), no. 3, Paper No. 149, 10 pp.

[5] Ammar, A.; Jeribi, A.; Saadaoui, B.: On some classes of demicompact linear relation and some results of essential pseudospectra. Mat. Stud. 52 (2019), no. 2, 195–210.

[6] Ben Brahim, F., Jeribi, A., Krichen, B.: Spectral properties for polynomial and matrix operators involving demicompactness classes. Extracta Math. 33 (2018), no. 1, 11–32.

[7] Ben Brahim, F., Jeribi, A., Krichen, B.: Essential pseudospectra involving demicompact and pseudodemicompact operators and some perturbation results. Filomat 33 (2019), no. 8, 2519–2528.

[8] Ben Brahim, F., Jeribi, A., Krichen, B.: Polynomially demicompact operators and spectral theory for operator matrices involving demicompactness classes. Bull. Korean Math. Soc. 55 (2018), no. 5, 1351–1370.

[9] Ben Brahim, F., Jeribi, A., Krichen, B.: Spectral theory for polynomially demicompact operators. Filomat 33 (2019), no. 7, 2017–2030.

[10] Benkhaled, H., Elleuch, A., Jeribi, A., Demicompactness results for strongly continuous semigroups, generators and resolvents. Mediterr. J. Math. 15 (2018), no. 2, Paper No. 71, 19 pp.

[11] Benkhaled, H., Elleuch, A., Jeribi, A., The class of order weakly demicompact operators. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 114 (2020), no. 2, Paper No. 76, 8 pp.

[12] Ferjani, I., Jeribi, A., Krichen, B.: Spectral properties involving generalized weakly demicompact operators. Mediterr. J. Math. 16 (2019), no. 2, Paper No. 30, 21 pp.

[13] Ferjani, I., Jeribi, A., Krichen, B.: Invariance of essential spectrum by means of generalized weak demicompactness. (English) Zbl 07361673 Jeribi, Aref (ed.), Operator theory. Proceedings of the international conference on operator theory, Hammamet, Tunisia, April 30 – May 3, 2018. Berlin: De Gruyter. De Gruyter Proc. Math., 115-120 (2021).

[14] Ferjani, I., Jeribi, A., Krichen, B.: Generalized weak demicompactness involving measure of non-strict-singularity. Indian J. Pure Appl. Math. 52 (2021), no. 2, 529-541.

[15] Chaker, W., Jeribi, A., Krichen, B.: Demicompact linear operators, essential spectrum and some perturbation results. Math. Nachr. 288 (2015), no. 13, 1476–1486.

[16] Chaker, W., Jeribi, A., Krichen, B.: Some Fredholm theory results around relative demicompactness concept. Commun. Korean Math. Soc. 36 (2021), no. 2, 313–325.

[17] Jeribi, A.: Linear operators and their essential pseudospectra. (English) Zbl 06849046 Boca Raton, FL : CRC Press (ISBN 978-1-77188-699-4/hbk). 352 p. (2018).

[18] Jeribi, A.: Denseness, bases and frames in Banach spaces and applications. (English) Zbl 06849137 Berlin : De Gruyter (ISBN 978-3-11-048488-5/hbk ; 978-3-11-049386-3/ebook). xv, 406 p. (2018).

[19] Jeribi, A.: Perturbation theory for linear operators: Denseness and bases with applications. Springer-Verlag (ISBN 978-981-16-2527-5), Singapore (2021).

[20] Jeribi, A. Krichen, B., Salhi, M.: Characterization of relatively demicompact operators by means of measures of noncompactness. J. Korean Math. Soc. 55 (2018), no. 4, 877–895.

[21] Jeribi, A. Krichen, B., Zitouni, A.: Spectral properties for diagonally dominant operator matrices using demicompactness classes and applications. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113 (2019), no. 3, 2391–2406.

[22] Jeribi, A. Krichen, B., Zitouni, A.: Properties of demicompact operators, essential spectra and some perturbation results for block operator matrices with applications. Linear Multilinear Algebra 68 (2020), no. 12, 2506–2522.

[23] F. Abdmouleh, S. Charfi and A. Jeribi, *On a characterization of the essential spectra of the sum and the product of two operators*, J. Math. Anal. Appl. 386 (2012) 83-90.

[24] T. Alvarez, A. Ammar and A. Jeribi, *On the essential spectra of some matrix of lineair relations*, Math. Methods Appl. Sci., 37 (2014) 620-644.

[25] T. Alvarez, A. Ammar and A. Jeribi, *A characterization of some subsets of S-essential spectra of a multivalued linear operator*, Colloq. Math. 135 (2014), 171-186.

[26] A. Ammar, A. Jeribi and B. Saadaoui, *Frobenius-Schur factorization for multivalued* 2×2 *matrices linear operator*, Mediterr. J. Math. 14 (2017), no. 1, 14-29.

[27] A. Ammar, A. Jeribi and B. Saadaoui, *A charactarization of essential pseudospectra of the multivalued operator matrix*, Anal. Math. Phys. 8 (2018), no. 3, 325-350.

[28] A. Ammar, A. Jeribi and B. Saadaoui, *On some classes of demicompact linear relation and some results of essential pseudospectra*, Mat. Stud. 52 (2019), 195-210.

[29] A. Ammar, H. Daoud and A. Jeribi, *Demicompact and k-D-set-contractive Linear Relations*, Mediterr. J. Math. (2018).

[30] E. Chafai and M. Mnif, Perturbation of normally solvable linear relations in paracomplete spaces, Linear Algebra Appl. 439 (2013), no. 7, 1875-1885.

[31] R. W. Cross, *Multivalued linear operators*, Marcel Dekker, (1998).

[32] W. Chaker, A. Jeribi and B. Krichen, *Demicompact linear operators, essential spectrum and some perturbation results*, Math. Nachr. 288 (2015), no. 13, 1476-1486.

[33] A. Jeribi, *Spectral Theory and Applications of Linear operator and Block Operator Matrices*, Springer-Verlag, New York, (2015).

[34] B. Krichen, *Relative essential spectra involving relative demicompact unbounded linear operators*, Acta Math. Sci. Ser. B Engl. Ed. 34 (2014), no. 2, 546-556.
[35] D. Gheorghe and F.-H. Vasilescu, *Spectral theory for linear relations via linear operators,* Pacific Journal of Mathematics 255 (2012) 349-372.
[36] D. Gheorghe and F.-H. Vasilescu, *Normal extension of subnormal linear relations via quaternionic Cayley transforms,* Monatsh. Math. 170 (2013) no 3-4, 437-463.
[37] F.-H. Vasilescu, *Spectral and Fredholm theory for linear relations via linear operators,* Advances in mathematics, 289-306. Ed. Acad. Romane, Bucharest, 2013.
[38] J. Shapiro and M. Schechter, *A Generalized operational calculus developed from Fredholm operator theory*, Trans. Amer. Math. Soc. 175 (1973) 439-667.
[39] J. Shapiro and M. Snow, *The Fredholm spectrum of the sum and product of two operators*, Trans. Amer. Math. Soc. 191 (1974) 387-393.
[40] A. E. Taylor, *Spectral theory of closed distributive operators*, Acta Math. 84 (1951) 189-224.