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Some Results on Linear Operators: Norm Equivalence and Closed Range

Seyed Kamel Hosseini[a], Jahangir Cheshmavar[a], Héctor Camilo Chaparro[b], Sanar Khalil Sarbaz[a]

[a]*Department of Mathematics, Payame Noor University (PNU), P.O.Box: 19395-3697, Tehran, Iran*
[b]*Universidad Militar Nueva Granada, Department of Mathematics, Cajicá 250247, Colombia*

Abstract. In this paper we develop some properties about bounded linear operators. We investigate relationships between bounded below and norm equivalent operators. Finally, we study conditions under which those operators become Fredholmn, Weyl and Browder type operators, respectively.

1. Introduction

The study of properties of operators such as closed range has always been of interest for many mathematicians. Just to mention a few, Kulkarni and Nari [7] characterized the closed range bounded linear operators between Banach spaces. Moorthy and Johnson [8] studied the composition of linear operators with closed range. Later, Barnes [1] gave conditions under which the closed range of an operator implies the closed range of its restriction, and also the converse. The closed range of topologically mutually dominated operators was studied in [6].

The closed range of some specific operators has also been of interest. For example, the closed range of $*$-multiplication operators defined on L^P and Orlicz spaces which was characterized in [4, 5], and also the closed range of multiplication operators defined on Orlicz spaces and weighted composition operators acting on multidimensional Lorentz spaces, which was studied in [2, 3].

In this paper, we want to establish some results in a general setting. We deal bounded linear operators between normed spaces, and we study properties such as boundedness below and closed range. We also define an equivalence relation (norm equivalence) and then we show that some properties, such as boundedness below and closed range (among others), are transferred by this equivalence relation. Moreover, we do a similar study in the context of direct sum of operators.

This paper is organized as follows. In Section 2, we give some basic definitions and results that will be used throughout the paper. Then, in Section 3, we define the norm equivalence of operators and we investigate common properties to norm equivalent operators, such as boundedness below, closed range, being a Fredholm operator, etc., and we also investigate the ascent and descent of norm equivalent operators. Finally, Section 4 is devoted to the study of the closed range operators. There, we state results regarding the closed range of direct sums of operators, as well as results about closed range of pseudo-inverse operators.

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Corresponding author: Jahangir Cheshmavar
Email addresses: kamelhosseini@chmail.ir (Seyed Kamel Hosseini), j_cheshmavar@pnu.ac.ir (Jahangir Cheshmavar), hector.chaparro@unimilitar.edu.co (Héctor Camilo Chaparro), skhsarbaz@gmail.com (Sanar Khalil Sarbaz)

2. Preliminaries and Auxiliary Results

2.1. Bounded Below and Closed Range Operators

Let X and Y be normed spaces and $\mathcal{B}(X, Y)$ be the set of all continuous linear operators from X into Y. We recall the definition of a bounded below operator.

Definition 2.1. *Let $T \in \mathcal{B}(X, Y)$. We say that T is bounded below if there exists $\varepsilon > 0$ such that $\|Tx\| \geq \varepsilon\|x\|$, for all $x \in X$.*

For $T \in \mathcal{B}(X, Y)$, we denote by $R(T)$ and $N(T)$ the range and the null space of T, respectively. T is said to be a *closed range operator* if $R(T)$ is a closed set in Y.

By $\mathcal{CL}(X, Y)$ and $\mathcal{BB}(X, Y)$ we denote the set of all continuous closed range linear operators and all continuous bounded below linear operators from X to Y, respectively. For every normed space X, we denote $\mathcal{B}(X, X) = \mathcal{B}(X)$, $\mathcal{CL}(X, X) = \mathcal{CL}(X)$ and $\mathcal{BB}(X, X) = \mathcal{BB}(X)$.

Now, we state the following theorem which we will use throughout this article.

Theorem 2.2. *[8] Let $T \in \mathcal{B}(X, Y)$. Then $T \in \mathcal{CL}(X, Y)$ if and only if there exists $\varepsilon > 0$ such that for given $x \in X$, there exists $y \in X$ such that $Tx = Ty$ and $\|y\| \leq \varepsilon\|Tx\|$.*

Remember that, for a bounded linear operator T, there is a unique factorization of T as a product $T = VP$, where V is a partial isometry, P is a non-negative self-adjoint operator and the initial space of V is the closure of the range of P. The following result shows us the equivalence between the boundedness below of T and P.

Proposition 2.3. *Let H be a Hilbert space and let $T \in \mathcal{B}(H)$ having polar decomposition $T = VP$. Then $T \in \mathcal{BB}(H)$ if and only if $P \in \mathcal{BB}(H)$.*

Proof. Let $T \in \mathcal{B}(H)$ having polar decomposition $T = VP$. Then we have,

$$\|Px\|^2 = \langle Px, Px\rangle = \langle P^2x, x\rangle = \langle T^*Tx, x\rangle = \|Tx\|^2,$$

for all $x \in H$. Suppose that $T \in \mathcal{BB}(H)$. Then there exists $\varepsilon > 0$ such that $\|Tx\| \geq \varepsilon\|x\|$, for all $x \in H$. Therefore, $\|Px\| = \|Tx\| \geq \varepsilon\|x\|$, for all $x \in H$, that is, $P \in \mathcal{BB}(H)$.

Conversely, suppose that $P \in \mathcal{BB}(H)$. Then there exists $\varepsilon > 0$ such that $\|Px\| \geq \varepsilon\|x\|$, for all $x \in H$. Therefore, $\|Tx\| = \|Px\| \geq \varepsilon\|x\|$ for all $x \in H$. That is, $T \in \mathcal{BB}(H)$. □

Next, we prove that composition of bounded below operators is again a bounded below operator, as also are products between bounded below operators with non-zero scalars.

Proposition 2.4. *Let $T \in \mathcal{B}(X, Y)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Let $S \in \mathcal{B}(Y, Z)$.*

(i) If $T \in \mathcal{BB}(X, Y)$ and $S \in \mathcal{BB}(Y, Z)$, then $ST \in \mathcal{BB}(X, Z)$,

(ii) $T \in \mathcal{BB}(X, Y)$ if and only if $\lambda T \in \mathcal{BB}(X, Y)$.

Proof. (i) Suppose that $T \in \mathcal{BB}(X, Y)$ and $S \in \mathcal{BB}(Y, Z)$. Then $\|STx\| \geq \eta\|Tx\| \geq \eta\varepsilon\|x\|$, for all $x \in X$.

(ii) Let $T \in \mathcal{BB}(X, Y)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then, $\|\lambda Tx\| \geq \varepsilon|\lambda|\|x\|$, for all $x \in X$. Therefore, $\lambda T \in \mathcal{BB}(X, Y)$. Conversely, if $\lambda T \in \mathcal{BB}(X, Y)$ and $\lambda \in \mathbb{C} \setminus \{0\}$, we have $\|Tx\| \geq \frac{\varepsilon}{|\lambda|}\|x\|$, for all $x \in X$.
□

3. Norm Equivalent Operators

In this section we present some results on norm equivalent operators. We begin with the following definition.

Definition 3.1. *Let $T, S \in \mathcal{B}(X, Y)$. S and T are said to be norm equivalent, and we denote this by $T \sim S$, if there exist positive real numbers k_1 and k_2 such that $k_1\|Sx\| \leq \|Tx\| \leq k_2\|Sx\|$, for all $x \in X$.*

Norm equivalence between operators is in fact an equivalence relation, as the following proposition shows.

Proposition 3.2. *Let $T \in \mathcal{B}(X, Y)$ and $[T] = \{S \in \mathcal{B}(X, Y) : S \sim T\}$. Then $\sim$ is an equivalence relation and $[T]$ is the equivalence class of T.*

Proof. Let $T \in \mathcal{B}(X, Y)$. It is clear that $T \sim T$. Let $T, S \in \mathcal{B}(X, Y)$ and $T \sim S$. Then there exist positive real numbers k_1 and k_2 such that $k_1\|Sx\| \leq \|Tx\| \leq k_2\|Sx\|$, for all $x \in X$. Therefore, $\frac{1}{k_2}\|Tx\| \leq \|Sx\| \leq \frac{1}{k_1}\|Tx\|$, for all $x \in X$, so $S \sim T$. Let $T, S, V \in \mathcal{B}(X, Y)$ such that $T \sim S$ and $S \sim V$. Then there exist positive real numbers k_1 and k_2 such that $k_1\|Sx\| \leq \|Tx\| \leq k_2\|Sx\|$, for all $x \in X$. Also, there exist positive real numbers j_1 and j_2 such that $j_1\|Vx\| \leq \|Sx\| \leq j_2\|Vx\|$, for all $x \in X$. Therefore, $k_1 j_1\|Vx\| \leq \|Tx\| \leq k_2 j_2\|Vx\|$, for all $x \in X$. So $T \sim V$, i.e. $\sim$ is transitive. □

Norm equivalent operators share some of their properties, this is shown in the proposition below.

Proposition 3.3. *Let $T, S \in \mathcal{B}(X, Y)$ be norm equivalent operators. Then, the following propositions holds:*

(i) $T \in \mathcal{BB}(X, Y)$ if and only if $S \in \mathcal{BB}(X, Y)$,

(ii) $T \in \mathcal{CL}(X, Y)$ if and only if $S \in \mathcal{CL}(X, Y)$,

(iii) T is one-to-one if and only if S is one-to-one,

(iv) αT and βS are norm equivalent, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

Proof. There exist two positive real numbers k_1 and k_2 such that $k_1\|Tx\| \leq \|Sx\| \leq k_2\|Tx\|$, for all $x \in X$.

(i) Let $T \in \mathcal{B}(X, Y)$. Then there exists $\varepsilon > 0$ such that $\|Tx\| \geq \varepsilon\|x\|$, for all $x \in X$. We have $\|Sx\| \geq k_1\|Tx\| \geq k_1\varepsilon\|x\|$, for all $x \in X$. Conversely, Let $S \in \mathcal{B}(X, Y)$. Then there exists $\varepsilon > 0$ such that $\|Sx\| \geq \varepsilon\|x\|$, for all $x \in X$. We have $\|Tx\| \geq \frac{1}{k_2}\|Sx\| \geq \frac{1}{k_2}\varepsilon\|x\|$, for all $x \in X$.

(ii) It follows Remark 2.4 in [6].

(iii) Let T be one-to-one, and $Sx = 0$. Then, we have $\|Tx\| \leq \frac{1}{k_1}\|Sx\|$. Hence, $x = 0$. Conversely, Let S is onto and $Tx = 0$. By hypothesis, we have $\|Sx\| \leq k_2\|Tx\|$. Hence, $x = 0$.

(iv) Let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Then, $k_1\|\frac{\alpha}{|\alpha|}Tx\| \leq \|\frac{\beta}{|\beta|}Sx\| \leq k_2\|\frac{\alpha}{|\alpha|}Tx\|$, for all $x \in X$. Therefore, $\frac{k_1|\beta|}{|\alpha|}\|\alpha Tx\| \leq \|\beta Sx\| \leq \frac{k_2|\beta|}{|\alpha|}\|\alpha Tx\|$, for all $x \in X$, which shows that $T \sim V$, i.e., $\sim$ is transitive.

□

Next, we recall the definition of a Fredholm and also a semi-Fredholm operator.

Definition 3.4. *Let X and Y be Banach spaces. $T \in \mathcal{B}(X, Y)$ is said to be a Fredholm operator and we write $T \in \Phi(X, Y)$, if $R(T)$ is closed,* $\operatorname{nul} T := \dim N(T) < \infty$ *and* $\operatorname{def} T := \operatorname{codim}\ R(T) = \dim(Y/R(T)) < \infty$.

For $T \in \Phi(X, Y)$, we define the index of T to be

$$\operatorname{ind}\ T = \operatorname{nul}\ T - \operatorname{def} T.$$

Now, put

$$\Phi_+(X,Y) = \{T \in \mathcal{B}(X,Y) : \ R(T) \ \textit{is closed and} \ \text{nul} \ T < \infty\},$$

and

$$\Phi_-(X,Y) = \{T \in \mathcal{B}(X,Y) : \ def \ T < \infty\}.$$

We say that T is a semi-Fredholm operator if $T \in \Phi_+(X,Y)$ or $T \in \Phi_-(X,Y)$.

The property of being a Fredholm (semi-Fredholm) operator is shared by norm equivalent operators, as the following proposition states.

Proposition 3.5. *Let $T, S \in \mathcal{B}(X,Y)$ be norm equivalent and let $R(T) = R(S)$. Then the following claims are true:*

(i) T is a Fredholm operator if and only if S a is Fredholm operator,

(ii) T is a semi-Fredholm operator if and only if S a is Fredholm operator.

Proof. By hypothesis, $N(T) = N(S)$, $R(T) = R(S)$ and $\text{ind } T = \text{ind } S$.

(i) Let T be a Fredholm operator. Then, both $\text{nul } T$ and $\text{def } T$ are finite. Therefore, both $\text{nul } S$ and $\text{def } S$ are finite. Consequently, S is Fredholm operator. Conversely, let S be a Fredholm operator. Then, both $\text{nul } S$ and $\text{def } S$ are finite. Therefore, both $\text{nul } T$ and $\text{def } T$ are finite. Consequently, T is a Fredholm operator.

(ii) Let T be a semi-Fredholm operator. If $T \in \Phi_+(X,Y)$ then, $S \in \Phi_+(X,Y)$. If $T \in \Phi_-(X,Y)$ then, $S \in \Phi_-(X,Y)$. Therefore, S is semi-Fredholm. Conversely, Let S be a semi-Fredholm operator. If $S \in \Phi_+(X,Y)$ then, $T \in \Phi_+(X,Y)$. If $S \in \Phi_-(X,Y)$ then, $T \in \Phi_-(X,Y)$. Therefore, T is a semi-Fredholm.

□

There is a relationship regarding the closed range and the polar decomposition of norm equivalent operators. This is our next result.

Proposition 3.6. *Let H be a Hilbert space and let $T_1, T_2 \in \mathcal{B}(H)$ be norm equivalent operators. Let $T_1 = V_1P_1$ and $T_2 = V_2P_2$ be polar decompositions of T_1 and T_2. Then $R(P_1)$ is closed if and only if $R(P_2)$ is closed.*

Proof. By hypothesis, there exist two positive real numbers k_1 and k_2 such that $k_1\|T_1x\| \leq \|T_2x\| \leq k_2\|T_1x\|$, for all $x \in H$. Then for $x \in H$, $\|P_ix\| = \|T_ix\|$, for $i = 1, 2$. Therefore, $k_1\|P_1x\| \leq \|P_2x\| \leq k_2\|P_1x\|$. Hence, by Proposition 3.3, the proof is complete. □

The following two propositions tell us about compositions of norm equivalent operators with bounded operators and also with isometries.

Proposition 3.7. *Let $P \in \mathcal{B}(X,Y)$ and let $T, S \in \mathcal{B}(Y,Z)$. If T and S are norm equivalent, then $TP \in \mathcal{CL}(X,Z)$ if and only if $SP \in \mathcal{CL}(X,Z)$.*

Proof. There exist two positive real numbers k_1 and k_2 such that $k_1\|Ty\| \leq \|Sy\| \leq k_2\|Ty\|$, for all $y \in Y$. Therefore, $k_1\|TPx\| \leq \|SPx\| \leq k_2\|TPx\|$, for all $x \in X$. Consequently, TP and SP are norm equivalent. Hence, $TP \in \mathcal{CL}(X,Z)$ if and only if $SP \in \mathcal{CL}(X,Z)$. □

Proposition 3.8. *Let $T, S \in \mathcal{B}(X,Y)$ and let $P \in \mathcal{B}(Y,Z)$ be an isometry. If T and S are norm equivalent, then $PT \in \mathcal{CL}(X,Z)$ if and only if $PS \in \mathcal{CL}(X,Z)$.*

Proof. There exist positive real numbers k_1 and k_2 such that $k_1\|Sx\| \leq \|Tx\| \leq k_2\|Sx\|$, $\|PTx\| = \|Tx\|$ and $\|PSx\| = \|Sx\|$, for all $x \in X$. Therefore, $k_1\|PTx\| \leq \|PTx\| \leq k_2\|PTx\|$, for all $x \in X$. Consequently, PT and PS are norm equivalent. Hence, $PT \in \mathcal{CL}(X,Z)$ if and only if $PS \in \mathcal{CL}(X,Z)$. □

3.1. *Ascent and Descent of Norm Equivalent Operators*

The ascent and descent of an operator are defined as follows.

Definition 3.9. *Let X be a vector space and $T : X \to X$ be a linear operator. If $N(T^n) \neq N(T^{n+1})$ for all $n \in \mathbb{N}$, then T has infinite* ascent *and we set* ascent $T = \infty$. *Otherwise, we say that T has finite* ascent *and set*

$$\text{ascent } T = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}.$$

Similarly, if $R(T^n) \neq R(T^{n+1})$ for all $n \in \mathbb{N}$, then T has infinite descent *and we set* descent $T = \infty$. *Otherwise, we say that T has finite* descent *and set*

$$\text{descent } T = \min\{p \in \mathbb{N} : R(T^p) = R(T^{p+1})\}.$$

Norm equivalent operators have the same ascent and descent. We prove this for the ascent in the corollary below. Since the proof for the descent is similar, we omit it.

Corollary 3.10. *Let $T, S \in \mathcal{B}(X)$ be norm equivalent and let $TS = ST$. Then*

(i) T^n and S^n are norm equivalent operators and $N(T^n) = N(S^n)$, for all $n \in \mathbb{N}$.

(ii) ascent $T < \infty$ *if and only if* ascent $S < \infty$;

(iii) T is nilpotent if and only if S is nilpotent.

Proof. (i) Since $T \sim S$, then by definition of $\sim$ we have $T^n \sim S^{n-1}T$ and $S^n \sim TS^{n-1}$, for all $n \geq 2$. On the other hand, since $TS = ST$, then $TS^{n-1} = S^{n-1}T$. Then, by Proposition 3.2 we have $T^n \sim S^n$. Then the fact that $N(T^n) = N(S^n)$ for all $n \in \mathbb{N}$ is obvious from the definition of $\sim$.

(ii) Let ascent $T = p < \infty$. Then $N(T^n) = N(T^p)$, for all $n \geq p$. Therefore, $N(S^n) = N(S^p)$, for all $n \geq p$. Consequently, ascent $S \leq p < \infty$. The converse is true similarly.

(iii) Let T be nilpotent. Then there exists $n \in \mathbb{N}$ such that $T^n x = 0$, for all $x \in X$. Moreover, there exist positive real numbers k_1 and k_2 such that $k_1\|T^n x\| \leq \|S^n x\| \leq k_2\|T^n x\|$, for all $x \in X$. Therefore, $S^n x = 0$, for all $x \in X$. Consequently, S is nilpotent. The proof of the converse is similar.
□

4. Closed Range Operators

We begin this section by presenting some properties of closed range operators.

Proposition 4.1. *Let $T \in \mathcal{B}(X, Y)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $T \in \mathcal{CL}(X, Y)$ if and only if $\lambda T \in \mathcal{CL}(X, Y)$.*

Proof. Let T be a closed range operator. By Theorem 2.2, there exists a constant $c > 0$ such that for $x \in X$, there exists $y \in X$ such that $Tx = Ty$ and $\|y\| \leq c\|Tx\|$. Therefore, for $\lambda \in \mathbb{C} \setminus \{0\}$ we have $\lambda Tx = \lambda Ty$ and $\|y\| \leq \frac{c}{|\lambda|}\|\lambda Tx\|$.

Conversely, let $\lambda \in \mathbb{C} \setminus \{0\}$ and let $R(\lambda T)$ be closed, then $R(T) = R(\frac{1}{\lambda}\lambda T)$ is closed. □

Proposition 4.2. *Let $T \in \mathcal{B}(X, Y)$ and let $S \in \mathcal{B}(Y, Z)$. Then the following holds:*

(i) If $T \in \mathcal{CL}(X, Y)$ and $S \in \mathcal{BB}(Y, Z)$, then $ST \in \mathcal{CL}(X, Z)$.

(ii) If $ST \in \mathcal{CL}(X, Z)$ and T is onto, then $S \in \mathcal{CL}(Y, Z)$.

(iii) If T is invertible and $S \in \mathcal{BB}(Y, Z)$, then $ST \in \mathcal{CL}(X, Z)$.

(iv) If $ST \in \mathcal{CL}(X, Z)$ and S is one-to-one, then $T \in \mathcal{CL}(X, Y)$.

Proof. (i) By Theorem 2.2, there exists a constant $c > 0$ such that for given $x \in X$, there exists $y \in X$ such that $Tx = Ty$ and $\|y\| \le c\|Tx\|$. Also, there exists $\varepsilon > 0$ such that $\varepsilon\|Tx\| \le \|S(Tx)\|$. Therefore, $S(Tx) = S(Ty)$ and $\|y\| \le \frac{c}{\varepsilon}\|STx\|$.

(ii) For any $y_1 \in Y$, there exists $x_1 \in X$ such that $Tx_1 = y_1$. By hypothesis, there exists a constant $c > 0$ such that for $x_1 \in X$, there exists $x_2 \in X$ such that $STx_1 = STx_2$ and $\|x_2\| \le c\|STx_1\|$. Therefore, $Sy_1 = Sy_2$ where $y_2 = Tx_2$ and $\|y_2\| \le c\|T\|\|Sy_1\|$. Consequently, $S \in \mathcal{CL}(Y, Z)$.

(iii) By hypothesis, there exists a constant $c > 0$ such that for given $x_1 \in X$, there exists a $y_2 \in Y$ such that $STx_1 = Sy_2$ and $\|y_2\| \le c\|STx_1\|$. Also, There exists $\varepsilon > 0$ such that for y_2, there exists $x_2 \in X$ such that $Tx_2 = y_2$ and $\varepsilon\|x_2\| \le \|Tx_2\| = \|y_2\| \le c\|STx_1\|$. Therefore, $STx_1 = STx_2$ and $\|x_2\| \le \frac{c}{\varepsilon}\|STx_1\|$.

(iv) There exists a constant $c > 0$ such that for given $x_1 \in X$, there exists $x_2 \in X$ such that $STx_1 = STx_2$ and $\|x_2\| \le c\|STx_1\|$. Therefore, $Tx_1 = Tx_2$ and $\|x_2\| \le c\|S\|\|Tx_1\|$.
□

Corollary 4.3. *Let $T \in \mathcal{B}(X, Y)$ and let $S \in \mathcal{B}(Y, Z)$. Then the following holds:*

(i) Let T be invertible. Then $ST \in \mathcal{CL}(X, Z)$ if and only if $S \in \mathcal{CL}(Y, Z)$.

(ii) Let S be invertible. Then $ST \in \mathcal{CL}(X, Z)$ if and only if $T \in \mathcal{CL}(X, Y)$.

Proof. (i) Let $T \in \mathcal{B}(X, Y)$ be invertible. Then T is onto. If $ST \in \mathcal{CL}(X, Z)$, then $S \in \mathcal{CL}(Y, Z)$ is closed. Moreover, If $S \in \mathcal{CL}(Y, Z)$, then $ST \in \mathcal{CL}(X, Z)$.

(ii) Let $T \in \mathcal{B}(X, Y)$ and let $S \in \mathcal{B}(Y, Z)$ be invertible. Then $T \in \mathcal{BB}(X, Y)$ is one-to-one. If $ST \in \mathcal{CL}(X, Z)$, then $T \in \mathcal{CL}(X, Y)$ is closed. If $T \in \mathcal{CL}(X, Y)$, then $ST \in \mathcal{CL}(X, Z)$.
□

Corollary 4.4. *Let $T, S \in \mathcal{B}(X)$. Then the following holds:*

(i) If $T \in \mathcal{CL}(X)$ and S is invertible, then $ST, TS \in \mathcal{CL}(X)$,

(ii) If $T \in \mathcal{BB}(X)$ and $n \in \mathbb{N}$. Then $T^n \in \mathcal{CL}(X)$.

4.1. Closed Range of Tensor Product and Direct Sum of Operators

It is a known fact that, if T_1 and T_2 are bounded linear operators on normed spaces X and Y, respectively, then there exists unique bounded linear operator T on $X \oplus Y$ such that

$$T(x \oplus y) = T_1x \oplus T_2y,$$

for all x in X and y in Y. This operator is called a *direct sum* of operators T_1 and T_2. It is denoted by $T_1 \oplus T_2$.

Closed range is preserved by operation $\oplus$. In fact, we have the following result.

Proposition 4.5. *Let $T_i \in \mathcal{B}(X_i, Y_i)$ for $i = 1, 2$. Let T_1 and T_2 be closed range operators, then $T_1 \oplus T_2$ has closed range.*

Proof. There exist constants $c_i > 0$, (i=1,2) such that for given $x_{1i} \in X_i$, there exists $x_{2i} \in X_i$ such that $T_ix_{1i} = T_ix_{2i}$ and $\|x_{2i}\| \le c_i\|T_ix_{1i}\|$. Therefore, $\|(x_{21}, x_{22})\| \le \max\{c_1, c_2\}\|(T_1 \oplus T_2)(x_{11}, x_{12})\|$. Consequently, $R(T_1 \oplus T_2)$ is closed. □

4.2. Weyl and Browder Operators

An operator $T \in \mathcal{B}(X)$ is called *Weyl operator* if T is a Fredholm operator with null index. An operator $T \in \mathcal{B}(X)$ is called a *Browder operator* if it is a Fredholm operator of finite ascent and descent.

Theorem 4.6. *Let $T, S \in \mathcal{B}(X, Y)$ and let $T \neq 0$. For all $f \in Y^*$ and $x \in X$, let $k_1|f(Tx)| \leq |f(Sx)| \leq k_2|f(Tx)|$ for some $k_1, k_2 > 0$, then the operator S is a:*

(i) Fredholm operator if and only if T is Fredholm operator.

(ii) Weyl operator if and only if T is Weyl operator.

(iii) Browder operator if and only if T is Browder operator.

(iv) compact operator if and only if T is compact.

(v) invertible operator if and only if T is invertible.

Proof. According to Theorem 2.7 in [6], $S = \alpha T$ for some scalar α. We show that $\alpha \neq 0$. Since $T \neq 0$, therefore, $Tx \neq 0$ for some $x \in X$. Then, there exists $f \in Y^*$ such that $f(Tx) \neq 0$. Hence, $Sx \neq 0$. Consequently, $\alpha \neq 0$. On the other hand, $N(T) = N(S)$, $R(T) = R(S)$, $ascent\ T = ascent\ S$ and $descent\ T = descent\ S$.

(i) Let S be a Fredholm operator, then both $nul\ T = nul\ S$ and $def\ T = def\ S$ are finite. Therefore, both $nul\ T$ and $def\ T$ are finite. Consequently, T is Fredholm operator. conversely, Let T be Fredholm operator. Then, both $nul\ S = nul\ T$ and $def\ S = def\ T$ are finite. Therefore, both $nul\ S$ and $def\ S$ are finite. Consequently, S is Fredholm operator.

(ii) Let S be a Weyl operator. Then S is Fredholm operator of index zero. Therefore, T is a Fredholm operator. Also, $ind\ T = nul\ T - def\ T = nul\ S - def\ S = ind\ S = 0$. Consequently, T is a Weyl operator. Conversely, let T be a Weyl operator. Then T is a Fredholm operator of index zero. Therefore, S is a Fredholm operator. Also, $ind\ S = nul\ S - def\ S = nul\ T - def\ T = ind\ T = 0$. Consequently, S is a Weyl operator.

(iii) Let S be a Browder operator. Then S is a Fredholm operator of finite ascent and descent. Therefore, T is a Fredholm of finite ascent and descent. In other words, T is a Browder operator. Conversely, Let T be a Browder operator. Then T is a Fredholm operator of finite ascent and descent. Therefore, S a is Fredholm operator of finite ascent and descent. In other words, S is a Browder operator.

(iv) This is trivial.

(v) Let S be an invertible operator. Then T is an invertible operator and $T^{-1} = \alpha S^{-1}$. Conversely, Let T be an invertible operator. Then S is an invertible operator and $S^{-1} = \alpha^{-1} T^{-1}$.

□

4.3. Closed Range of Pseudo-Inverse Operators

Let $T \in \mathcal{B}(X, Y)$. Recall that, a *pseudo-inverse* of T is an operator $S \in \mathcal{B}(Y, X)$ such that $TST = T$.

The next proposition tells us about direct sum of pseudo-inverse operators.

Proposition 4.7. *Let $T_i \in \mathcal{B}(X_i, Y_i)$. Let $S_i \in \mathcal{B}(Y_i, X_i)$ be pseudo-inverses of T_i, $(i = 1, 2)$. Then $S_1 \oplus S_2$ is a pseudo-inverse of $T_1 \oplus T_2$.*

Proof. Let $x_1 \in X_1$ and $x_2 \in X_2$. Then,

$$\begin{aligned}(T_1 \oplus T_2)(S_1 \oplus S_2)(T_1 \oplus T_2)(x_1, x_2) &= (T_1 \oplus T_2)(S_1 \oplus S_2)(T_1x_1, T_2x_2)\\ &= (T_1 \oplus T_2)(S_1T_1x_1, S_2T_2x_2)\\ &= (T_1S_1T_1x_1, T_2S_2T_2x_2)\\ &= (T_1x_1, T_2x_2)\\ &= (T_1 \oplus T_2)(x_1, x_2),\end{aligned}$$

as desired. □

A pseudo-inverse operator of a Fredholm operator is again a Fredholm operator, and the compactness of a pseudo-inverse operator implies the compactness of the operator. We prove this in the next Proposition.

Proposition 4.8. *Let $T \in \mathcal{B}(X, Y)$ and let $S \in \mathcal{B}(Y, X)$ be pseudo-inverse of T. Then*

(i) If T is a Fredholm operator, then so is S,

(ii) If S is a compact operator, then so is T.

Proof. (i) If T is a Fredholm operator, then so is TST. Therefore, TS is a Fredholm operator and then so is S.

(ii) If S is a compact operator, then so is TST and then so is T.
□

Pseudo-inverse operators of Fredholm type operators are also Fredholm type operators, and a similar result is valid is valid for Weyl type operators and for invertible operators. More precisely, we have the following result.

Theorem 4.9. *Let $T \in \mathcal{B}(X)$ and let $S \in \mathcal{B}(X)$ be pseudo-inverse of T. Then*

(i) If T is a Freholm operator, then so is S.

(ii) If T is a Weyl operator, then so is S.

(iii) If T is a invertible operator, then so is S and $S = T^{-1}$.

Proof. (i) It follows from the previous proposition.

(ii) Let T be a Weyl operator. Then S is a Fredholm operator. Also

$$0 = \text{ind } T = \text{ind } T + \text{ind } S + \text{ind } T = \text{ind } S.$$

Therefore, ind $S = 0$. Consequently, S is a Weyl operator.

(iii) Let T be a invertible operator. Then $TS = I$ and $ST = I$. Therefore, S is a invertible operator and $S = T^{-1}$.
□

Proposition 4.10. *Let $T \in \mathcal{B}(H)$ and let $S \in \mathcal{B}(H)$ be a pseudo-inverse of T. Then*

(i) S^* is a pseudo-inverse of T^*.

(ii) If T is unitary operator, then so is S.

Proof. (i) Let $TST = T$. Then $T^*S^*T^* = (TST)^* = T^*$. Therefore, S^* is a pseudo-inverse of T^*.

(ii) Let T be a unitary operator, then T is invertible and $S = T^*$. Therefore, $SS^* = S^*S = I$. Consequently, S is a unitary operator.
□

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