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On Deferred Statistical Convergence through Regular Variations

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Abstract. In this paper, we introduce the idea of deferred statistical convergence via the concept of regular variations. In fact, we study the convergence of real sequences or measurable functions using ideas of variations such as regular, $O-$regular, translational regular and rapid, etc, in deferred statistical perspective. We established some relations among these different deferred statistical variations.

1. Introduction and preliminaries

Besides functional analysis, the use of summability theory and theory of sequence spaces has been entered into many other fields of mathematics, such as approximation theory, operator theory, fuzzy set theory, etc. The study of convergence and statistical convergence of a known sequence plays a vital role in sequence spaces and summbility theory. Due to its numerous applications in pure and applied fields, the study has been attracted, and subsequently been developed by several authors by various definitions. In 1935, Zygmund gave the idea of statistical convergence in the first edition of his monograph. The idea was introduced by Steinhaus [41] and Fast [22] independently in the context of sequence spaces. Schoenberg [40] developed the idea by applying it in operator theory and summability theory. The theory of statistical convergence was used in the convergence of trigonometric and Fourier series by Zygmund [43] and the theory of matrix characterization by Fridy and Miller [25]. Its applications have also been emerged in various fields such as in number theory by Erdős and Tenenbaum [17], spaces of locally convex sets by Maddox [28], integral summability theory by Connor and Swardson [11], theory of lacunary summability by Fridy and Orhan [24], measure theory by Connor and Swardson [12]. Later on, it was further reintroduced and applied in approximation theory, single and double sequence spaces and different areas of functional analysis by Mursaleen et al. ([30–33]), Çakallı et al. ([8, 9]), Maio and Kočinac ([29]), Et et al. ([18–21]), Salat [38], Baliarsingh et al. [4], Baliarsingh [5], Nuray [36], Nuray and Aydın [37], Çolak [10], and many others.

The idea of statistical convergence depends on the natural density of subsets of the set $\mathbb{N}$ of natural numbers. The natural density of E, a subset of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k), \tag{1}$$

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provided the limit in (1) exists. Here $\chi_{\mathbb{E}}$ denotes the characteristic function of the set E. For any finite subset E of $\mathbb{N}$, it is noted that $\delta(E) = 0$ and for its complement set E^c, $\delta(E^c) = 1$.

Definition 1.1. A sequence $x = (x_k)$ is said to be statistically convergent to L if, for every $\epsilon > 0$, we have

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

In this case, we also say that $|x_k - L| \geq \epsilon$ for almost all k and write $st - \lim x_k = L$.

Definition 1.2. A sequence $x = (x_k)$ is said to be statistically Cauchy if, for every $\epsilon > 0$, there exists a number N depending on ϵ, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - x_N| \geq \epsilon\}| = 0.$$

In this case, we also say that $|x_k - x_N| \geq \epsilon$ for almost all k.

Definition 1.3. A sequence $x = (x_k)$ of real numbers is said to be statistically bounded if there exists a number K such that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k| > K\}| = 0.$$

Also, we say that $\delta(\{k : |x_k| \geq K\}) = 0$.

Generally, every convergent sequence is statistical convergent, but the converse is not true. Similarly, every Cauchy sequence is statistical Cauchy, but the converse is not true.

The analogous definition of limit superior and limit inferior of a sequence in the statistical sense were given in Fridy [25]. Let $x = (x_k)$ be a sequence of real numbers. Then the statistical limit superior and limit inferior of the sequence x are respectively, defined by

$$st - \limsup x = \begin{cases} \sup P_x, & \text{if } P_x \neq \phi \\ -\infty, & \text{if } P_x = \phi, \end{cases}$$

and

$$st - \liminf x = \begin{cases} \sup Q_x, & \text{if } Q_x \neq \phi \\ +\infty, & \text{if } Q_x = \phi, \end{cases}$$

where $P_x = \{p \in \mathbb{R} : \delta(\{k : x_k > p\}) \neq 0\}$ and $Q_x = \{q \in \mathbb{R} : \delta(\{k : x_k < q\}) \neq 0\}$.

The theory of variations explains the idea of the rate at which a sequence or a function converges or diverges. This is more useful and necessary in the theory of summability and sequence spaces as it deals with the asymptotic analysis of convergent and divergent processes. The theory was initially developed by Karamata [27] in the year 1930 while working on the Tauberian theorems of Hardy and Littlewood. Later de Haan [26] introduced the idea of regular variations and applied it to the weak convergence theory. In 1973, Bojanic and Seneta [7] (see also [39]) unified the definitions of regular variations and applied then in functions theory. The initial developments of $O-$regular variations can be found in the research works of Aljancic and Arandjelovic [2]. Later these results have been extended by Djurcic and Bozin [13], and Durcic [14]. The idea of $O-$regular variations has been applied in the theory of uniform convergence by Arandjelovic [3], function theory by Taskovic [42], the theory of sequence spaces by Djurcic et al. [15] and the theory of statistical convergence by Dutta and Das [16]. The main aim of this paper is to redefine the idea in deferred statistical context and study some relations among the newly defined classes. Now, we provide some primary definitions of variations given by [6] (see also, [42] and [13]).

Definition 1.4. A positive real sequence $x = (x_k)$ is said to be regularly varying if it satisfies

$$\lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = k(\alpha) < \infty, \quad \text{for all } \alpha > 0.$$

Note that if $k(\alpha) = 1$, for each $\alpha > 0$, then the sequence $x = (x_k)$ is called slowly varying. If the function $k(\alpha)$ is of the form α^ρ for some $\rho \in \mathbb{R}$, the number ρ is called the index of variability of x. By RV, SV, and RV_ρ we denote the classes of all regularly, slowly and regularly with index ρ varying sequences, respectively.

Definition 1.5. A sequence $x = (x_k)$ of positive real numbers is said to be $O-$regularly varying if for each $\alpha > 0$,

$$\limsup_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = u(\alpha) < \infty.$$

It is remarked that every regular varying sequence is $O-$regularly varying but the converse is not true in general. The set ORV denotes the class of all $O-$regularly varying sequences.

Definition 1.6. A positive real sequence $x = (x_k)$ is said to be translationally regularly varying if for each $\alpha > 0$,

$$\lim_{k\to\infty} \frac{x_{[k+\alpha]}}{x_k} = r(\alpha) < \infty.$$

Here the function $r(\alpha)$, for each $\alpha > 0$ is of the form $e^{\rho[\alpha]}$ for some $\rho \in \mathbb{R}$, where ρ is the index of variability of x. We denote the set TRV for the class of all translationally regularly varying sequences.

Definition 1.7. A positive real sequence $x = (x_k)$ is said to be rapidly varying sequence (of index of variability ∞) if for each $\alpha > 1$,

$$\lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = \infty,$$

and also, for each $0 < \alpha < 1$

$$\lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = 0.$$

Here we denote the set RV_∞ for the class of all rapidly varying sequences of index of variability ∞. A sequence $x = (x_k)$ of positive real numbers is said to be rapidly varying sequence of index of variability $-\infty$ if for each $\alpha > 1$,

$$\lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = 0,$$

and we denote these classes of sequences as the set $RV_{-\infty}$

Now, extend the above definitions in statistical sense by using deferred Cesàro mean.

Deferred Cesàro mean:. Let $p = (p_n)$ and $q = (q_n)$ be two sequences of non-negative integers (see Agnew [1]) satisfying

(i) $p_n < q_n$ for all $n \in \mathbb{N}_0$.

(ii) $\lim_{n\to\infty} q_n = \infty$,

Then, the deferred Cesàro mean of the sequence $x = (x_k)$ is defined by

$$\begin{aligned}(D_{p,q}x)_n &= \frac{x_{p_n+1} + x_{p_n+2} + \cdots + x_{q_n}}{q_n - p_n} \\ &= \sum_{k=0}^{\infty} d_{nk} x_k,\end{aligned}$$

where

$$d_{nk} = \begin{cases} \frac{1}{q_n - p_n}, & (p_n < k \leq q_n) \\ 0, & (\text{ otherwise}). \end{cases}$$

It is known that (i) and (ii) are the regularity conditions for the deferred Cesàro mean $D_{p,q}$. Indeed, the $D_{p,q}-$transform (see Nayak et al. [34, 35]) is the natural extensions various transforms such as

- $D_{n-1,n} = I$, the identity transform,
- $D_{0,n} = (C, 1)$, the Cesàro transform
- $D_{n-\lambda_n+1,n} = (V, \lambda)$, de la Vallée Poussin transform,

where $\lambda = (\lambda_k)$ being a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_k \leq \lambda_k + 1$ and $\lambda_0 = 1$.

Definition 1.8. A sequence $x = (x_k)$ is said to be deferred-statistically convergent to L if, for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \leq q_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $dst - \lim x_k = L$. and the natural density with respect to deferred Cesàro mean $\delta_{pq}(k : |x_k - L|) = 0$. A convergent sequence is always deferred-statistically convergent but converse is not true.

Definition 1.9. A sequence $x = (x_k)$ is said to be deferred-statistically Cauchy if, for every $\epsilon > 0$, there exists a number N depending on ϵ, we have

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \leq q_n : |x_k - x_N| \geq \epsilon\}| = 0.$$

Definition 1.10. A sequence $x = (x_k)$ of real numbers is said to be deferred-statistically bounded if there exists a number K such that

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \leq q_n : |x_k| > K\}| = 0.$$

Definition 1.11. A positive real sequence $x = (x_k)$ is said to be deferred-statistically regularly varying if it satisfies

$$dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = k_{ds}(\alpha) < \infty, \quad \text{for all } \alpha > 0.$$

By $DSRV, DSSV$, and $DSRV_\rho$, we denote the classes of all deferred-statistically regularly, slowly and regularly with index ρ varying sequences, respectively.

Definition 1.12. A positive real sequence $x = (x_k)$ is said to be deferred-statistically $O-$regularly varying if for each $\alpha > 0$,

$$dst - \limsup_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = u_{ds}(\alpha) < \infty.$$

The set $DSORV$ denotes such class of all deferred-statistically $O-$regularly varying sequences.

Definition 1.13. A positive real sequence $x = (x_k)$ is said to be deferred-statistically translationally regularly varying if for each $\alpha > 0$,

$$dst - \lim_{k\to\infty} \frac{x_{[k+\alpha]}}{x_k} = r_{ds}(\alpha) < \infty.$$

We denote the sets $DSTRV$ and $DSTRV_\rho$ for the classes of all deferred-statistically translationally regularly and with index ρ varying sequences, respectively.

Definition 1.14. A positive real sequence $x = (x_k)$ is said to be deferred-statistically rapidly varying sequence(of index of variability ∞) if for each $\alpha > 1$,

$$dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = \infty,$$

and also, for each $0 < \alpha < 1$

$$dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = 0.$$

Here we denote the set $DSRV_\infty$ for the class of all deferred-statistically rapidly varying sequences of index of variability ∞. The sequence $x = (x_k)$ is said to be deferred-statistically rapidly varying sequence of index of variability $-\infty$ if for each $\alpha > 1$,

$$dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = 0,$$

and we denote such class of sequences as the set $DSRV_{-\infty}$

2. Main theorems

In this section, we provide some results on different types of deferred-statistically varying sequences and also the relationships among themselves.

Theorem 2.1. *For a positive real sequence $x = (x_k)$, if*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| = 0,$$

then $x \in DSORV$ or $x \in DSTRV$ and

$$\delta_{p,q}\left(k : \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right|\right) = 0.$$

Proof. We prove for the class $DSORV$ and for other it may use similar argument. Suppose that the sequence $x = (x_k)$ of positive real numbers satisfies

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| = 0, \text{ for } \alpha > 0.$$

Then, we have

$$\begin{aligned} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| &= \frac{1}{q_n - p_n} \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| < \epsilon}}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| + \frac{1}{q_n - p_n} \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| \geq \epsilon}}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| \\ &\geq \frac{1}{q_n - p_n} \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| \geq \epsilon}}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| \end{aligned}$$

Taking $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| \geq \epsilon}}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| = 0,$$

which implies that $x \in DSORV$ and

$$\delta_{p,q}\left(k : \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right|\right) = 0.$$

□

The converse of Theorem 2.1 is not true in general. However, we present the next theorem for its converse part.

Theorem 2.2. *If $x = (x_k)$ is a bounded sequence of positive real numbers and $x \in DSTRV$ or $x \in DSORV$, then*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| = 0.$$

Proof. Suppose $x = (x_k)$ is a bounded sequence. Then for given $\epsilon > 0$ and $\alpha > 0$, we have

$$\begin{aligned} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| &= \frac{1}{q_n - p_n} \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| < \epsilon}}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| + \frac{1}{q_n - p_n} \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| \geq \epsilon}}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| \\ &\leq \frac{\epsilon}{q_n - p_n} \sup_k \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| < \epsilon}}^{q_n} (1) + \frac{1}{q_n - p_n} \sum_{\substack{k=p_n+1, \\ \left|\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right| \geq \epsilon}}^{q_n} \left| \frac{x_{[\alpha k]}}{x_k} - r(\alpha) \right| \end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality and using the boundedness of the sequence $\left(\frac{x_{[\alpha k]}}{x_k} - r(\alpha)\right)$, we complete the proof. □

Theorem 2.3. *For a sequence $x = (x_k)$ of positive real numbers, if $x \in STRV$(the class of statistically translationally regularly varying sequences) or $x \in SORV$(the class of statistically $O-$regularly varying sequences), then $x \in DSTRV$ or $x \in DSORV$, respectively provided the sequence $\left(\frac{p_n}{q_n - p_n}\right)$ is bounded.*

Proof. The proof is straightforward, hence omitted. □

Theorem 2.4. *For a sequence $x = (x_k)$ of positive real numbers, the following statements are equivalent:*

(i) *$x \in DSRV$*

(ii) *$x \in DSRVC$*

(iii) *For some $y \in RV$ such that*

$$\delta_{p,q}\left(k : \left|\frac{x_{[\alpha k]}}{x_k} \neq \frac{y_{[\alpha k]}}{y_k}\right|\right) = 0,$$

where DSRVC stands for the set of all deferred-statistical regular varying Cauchy sequence.

Proof. We divide the proof into three parts as follows:

Let us assume that **(i)** holds and use a notation $x[\alpha k]$ for the sequence $\frac{x_{[\alpha k]}}{x_k}$. Since $x \in DSRV$, then

$$dst - \lim_{k\to\infty} x[\alpha k] = k_{ds}(\alpha), \quad \text{for all } \alpha > 0,$$

which is equivalent to that, for every $\epsilon > 0$ and $\alpha > 0$,

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left| \left\{ p_n < k \leq q_n : |x[\alpha k] - k_{ds}(\alpha)| \geq \frac{\epsilon}{2} \right\} \right| = 0.$$

Choose a number N such that for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left| \left\{ p_n < k \leq q_n : |x[\alpha N] - k_{ds}(\alpha)| \geq \frac{\epsilon}{2} \right\} \right| = 0.$$

By Using triangle inequality to the above equations, we have

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{p_n < k \le q_n : |x[\alpha k] - x[\alpha N]| \ge \epsilon\}\right| = 0.$$

This implies **(ii)**.
Secondly, assuming that **(ii)** holds. Choose a number N such that the closed interval $I_0 = [x[\alpha N]-1, x[\alpha N]+1]$ contains $x[\alpha k]$ for almost all k. Similarly, choose another number N_1 such that $I' = [x[\alpha N_1]-1/2, x[\alpha N_1]+1/2]$ contains $x[\alpha k]$ for almost all k. Therefore, the interval $I_1 = I_0 \cap I'$ is of length less than or equal to 1, contains $x[\alpha k]$ for almost all k. By choosing the number N_2, construct a closed interval $I_1' = [x[\alpha N_1]-1/4, x[\alpha N_1]+1/4]$ containing $x[\alpha k]$ for almost all k. Proceeding the similar techniques, we claim that the interval $I_2 = I_1 \cap I_1'$ is of the length not greater than 1/2 contains $x[\alpha k]$ for almost all k. By inductive principle, for a natural number m, we can construct an interval I_m of length not greater than $\frac{1}{2^{m-1}}$ contains $x[\alpha k]$ for almost all k. By virtue of nested interval theorem, we can able to find a number σ, such that

$$\sigma = \bigcap_{j=1}^{\infty} I_j.$$

Since I_m contains $x[\alpha k]$ for almost all k, we choose an increasing sequence of positive integers $\gamma = (\gamma_m)$ such that

$$\frac{1}{q_n - p_n} \left|\{p_n < k \le q_n : x[\alpha k] \notin I_m\}\right| < \frac{1}{m} \text{ for each } n > \gamma_m. \tag{2}$$

Define a subsequence $z = (z_k)$ of $x = (x_k)$ consisting of all the terms x_k such that $k > \gamma_1$ and if $\gamma_m < k \le \gamma_{m+1}$, then $x[\alpha k] \notin I_m$.

With the help of the subsequence z, consider the sequence $y = (y_k)$ with

$$y[\alpha k] = \begin{cases} \sigma, & \text{if } x_k \text{ is a term of } z \\ x[\alpha k], & \text{otherwise.} \end{cases}$$

It is clear that $\lim_{k\to\infty} y[\alpha k] = \sigma$. For $k > \gamma_m$ and $0 < \frac{1}{m} < \epsilon$, we have x_k is either of the form z_k or $x[\alpha k] = y\alpha k] \in I_m$ and $|y[\alpha k] - \sigma|$ is not greater than the length of I_m.

For $\gamma_m < n < \gamma_{m+1}$, using (2) we calculate that

$$\begin{aligned} \frac{1}{q_n - p_n} \left|\{p_n < k \le q_n : x[\alpha k] \ne x[\alpha k]\}\right| &< \frac{1}{q_n - p_n} \left|\{p_n < k \le q_n : x[\alpha k] \notin I_m\}\right| \\ &< \frac{1}{m}. \end{aligned}$$

Taking limit for $n \to \infty$ in the above inequality, we conclude that

$$\delta_{p,q}\left(k : \left|\frac{x_{[\alpha k]}}{x_k} \ne \frac{y_{[\alpha k]}}{y_k}\right|\right) = 0,$$

as desired in **(iii)**.

Finally, we consider **(iii)** as the hypothesis to show **(i)** is true i.e., $x \in DSRV$. Since $y \in RV$, for every $\epsilon > 0$ and $\alpha > 0$,

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{p_n < k \le q_n : |y[\alpha k] - k_{ds}(\alpha)| \ge \epsilon\}\right| = 0.$$

And also, from the hypothesis,

$$\frac{1}{q_n - p_n} \left|\{p_n < k \le q_n : x[\alpha k] \ne x[\alpha k]\}\right|.$$

Now, for every $\epsilon > 0$ and $\alpha > 0$, we have

$$\frac{1}{q_n - p_n}\left|\{p_n < k \leq q_n : |x[\alpha k] - k_{ds}(\alpha)| \geq \epsilon\}\right| < \frac{1}{q_n - p_n}\left|\{p_n < k \leq q_n : x[\alpha k] \neq x[\alpha k]\}\right| + \frac{1}{q_n - p_n}\left|\{p_n < k \leq q_n : |y[\alpha k] - k_{ds}(\alpha)| \geq \epsilon\}\right|.$$

Taking limit as $n \to \infty$, we have $\delta_{p,q}(k : x[\alpha k] - k_{ds}(\alpha)) = 0$, i.e., $x \in DSRV$. This completes the proof. □

Theorem 2.5. *For a sequence $x = (x_k)$ of positive real numbers, the following statements are equivalent:*

(i) $x \in DSTRV$

(ii) $x \in DSTRVC$

(iii) *For some $y \in TRV$ such that*

$$\delta_{p,q}\left(k : \left|\frac{x_{[\alpha k]}}{x_k} \neq \frac{y_{[\alpha k]}}{y_k}\right|\right) = 0,$$

where the class DSRVC stands for the set of all deferred-statistical translationally regular varying Cauchy sequence.

Proof. The proof follows the similar lines as described in Theorem 2.4, hence omitted. □

Theorem 2.6. *Let $x = (x_k)$ be a positive real sequence. Then*

$$dst - \liminf x \leq dst - \limsup x.$$

Proof. This follows from Fridy [23]. □

Theorem 2.7. *Let $x = (x_k)$ be a positive real sequence. Then*

$$dst - \limsup x = \gamma < \infty$$

if and only if for every $\epsilon > 0$

$$\delta_{p,q}(\{k : x_k > \gamma - \epsilon\}) \neq 0 \text{ and } \delta_{p,q}(\{k : x_k > \gamma + \epsilon\}) = 0.$$

Proof. This follows from Fridy [23]. □

Theorem 2.8. *Let $x = (x_k)$ be a positive real sequence. Then $x \in DSRV$ i.e.,*

$$dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = k_{ds}(\alpha), \text{ for all } \alpha > 0,$$

if and only if there exists a subsequence $y = (y_k)$ of x such that

$$\lim_{k\to\infty} \frac{y_{[\alpha k]}}{y_k} = k_{ds}(\alpha),$$

and $k_{ds}(\alpha) = \alpha^\rho$ for some $\rho \in \mathbb{R}$.

Proof. This theorem is a direct consequence of Definition 1.4 and Theorem 2.4. □

Theorem 2.9. *Let $x = (x_k)$ be a positive real sequence. Then $x \in DSTRV$ i.e.,*

$$dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = r_{ds}(\alpha), \text{ for all } \alpha > 0,$$

If and only if there exists a subsequence $y = (y_k)$ of x such that

$$\lim_{k\to\infty} \frac{y_{[\alpha k]}}{y_k} = r_{ds}(\alpha),$$

and $r_{ds}(\alpha) = e^{\rho[\alpha]}$ for some $\rho \in \mathbb{R}$.

Proof. This follows from Theorem 2.5 along with the Definition 1.6. □

Theorem 2.10. *Let $x = (x_k)$ be sequence of positive real numbers and $x \in DSRV$ i.e., $dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = k_{ds}(\alpha)$. Then $x \in RV$ with the same limit if*

$$\Delta\left(\frac{x_{[\alpha k]}}{x_k}\right) = O((q_k - p_k)^{-1}).$$

Proof. Suppose that $x \in DSRV$ and by Theorem 2.4 there exists a sequence $y = (y_k)$ such that

$$\lim_{k\to\infty} y[\alpha k] = k_{ds}(\alpha),$$

and $\delta_{p,q}(k : x[\alpha k] \neq y[\alpha k]) = 0$.

Consider $q_k - p_k = a(k) + b(k)$, where $a(k) = \max\{p_k < n \leq q_k : |x[\alpha n] = y[\alpha n]\}$ and $b(k) = \max\{p_k < n \leq q_k : |x[\alpha n] \neq y[\alpha n]\}$. Now, we claim that

$$\lim_{k\to\infty} \frac{b(k)}{a(k)} = 0.$$

If not, possibly we take $\frac{b(k)}{a(k)} \geq \epsilon > 0$, then

$$\begin{aligned}
\frac{1}{q_k - p_k}\left|\{p_k < n \leq q_k : x[\alpha n] = y[\alpha n]\}\right| \leq & \frac{a(k)}{a(k) + b(k)} \\
\leq & \frac{a(k)}{a(k) + \epsilon a(k)} \\
= & \frac{\epsilon}{1+\epsilon}.
\end{aligned}$$

This leads to a contradiction to the fact that $\delta_{p,q}(k : x[\alpha k] \neq y[\alpha k]) = 0$. Therefore, $\lim_{k\to\infty} \frac{b(k)}{a(k)} = 0$. Since $\Delta\left(\frac{x_{[\alpha k]}}{x_k}\right) = O((q_k - p_k)^{-1})$, there exist a constant $\mathcal{M}$ such that $\Delta\left(\frac{x_{[\alpha k]}}{x_k}\right) = \frac{\mathcal{M}}{q_k - p_k}$.

Now, consider the difference

$$\begin{aligned}
|y[\alpha a(k)] - x[\alpha k]| &= |a[\alpha a(k)] - x[\alpha(a(k) + b(k))]| \\
&= \sum_{i=a(k)}^{a(k)+b(k)-1} (x[\alpha k] - x[\alpha(k+1)]) \\
&= \sum_{i=a(k)}^{a(k)+b(k)-1} \left(\frac{x_{[\alpha k]}}{x_k} - \frac{x_{[\alpha(k+1)]}}{x_{k+1}}\right) \\
&= \sum_{i=a(k)}^{a(k)+b(k)-1} \Delta\left(\frac{x_{[\alpha k]}}{x_k}\right) \\
&\leq \frac{\mathcal{M}(a(k) + b(k) - a(k))}{a(k) + b(k)} \\
&= \frac{\mathcal{M}\left(\frac{b(k)}{a(k)}\right)}{1 + \frac{b(k)}{a(k)}}
\end{aligned}$$

Taking limit as $k \to \infty$ on both the sides, $\lim_{k\to\infty} x[\alpha k] = \lim_{k\to\infty} y[\alpha a(k)] = k_{ds}(\alpha)$. □

Theorem 2.11. *Let $x = (x_k)$ be a positive real sequence and $x \in DSTRV$ i.e., $dst - \lim_{\to\infty} \frac{x_{[k+\alpha]}}{x_k} = r_{ds}(\alpha)$. Then $x \in TRV$ with the same limit if*

$$\Delta\left(\frac{x_{[\alpha k]}}{x_k}\right) = O((q_k - p_k)^{-1}).$$

Proof. This is similar to that of Theorem 2.10. □

Theorem 2.12. *Let $x = (x_k)$ be a positive real sequence. Then $x \in DSTRV_\rho$ i.e., $dst - \lim_{k\to\infty} \frac{x_{[k+\alpha]}}{x_k} = r_{ds}(\alpha)$, if and only if*

$$x_k = x_1 e^{A_{k-1}},\ (k > 1),$$

where , $x_1 > 0, A_{k-1} = \sum_{j=1}^{k-1} \mu_k$ and (μ_k) is a sequence of real numbers satisfying $dst - \lim_k e^{\mu_k} = e^\rho,\ (\rho \in \mathbb{R})$.

Proof. Suppose $x \in DSTRV_\rho$ i.e., $dst - \lim_{k\to\infty} \frac{x_{[k+\alpha]}}{x_k} = r_{ds}(\alpha)$. Using Theorem 2.9 for $\alpha = 1$, we have

$$dst - \lim_{k\to\infty} \frac{x_{[k+1]}}{x_k} = r_{ds}(1) = e^\rho,\ (\rho \in \mathbb{R}).$$

From Theorem 2.5, there exist a sequence (ν_k) in TRV_ρ such that

$$dst - \lim_{k\to\infty} \nu_k = r_{ds}(1) = e^\rho,$$

and

$$\delta_{p,q}\left(k : \left|\frac{x_{k+1}}{x_k} \neq \nu_k\right|\right) = 0,$$

i.e., $\frac{x_{k+1}}{x_k} = \nu_k$ for almost all k. For this case one may write

$$\begin{aligned} x_{k+1} &= \nu_k x_k \\ &= \nu_k \nu_{k-1} \nu_{k-2} \ldots \nu_1 x_1 \\ &= e^{\mu_k} e^{\mu_{k-1}} e^{\mu_{k-2}} \ldots e^{\mu_1} x_1 \\ &= x_1 e^{\sum_{j=1}^k \mu_j} \\ &= x_1 e^{A_k}, \end{aligned}$$

where $A_k = \sum_{j=1}^k \mu_k$. If we put k in the place of $k+1$, we get the result as desired. The converse part is simple, hence omitted. □

Theorem 2.13. *Let $x = (x_k)$ be sequence of positive real numbers and $x \in DSTRV$ i.e., $dst - \lim_{k\to\infty} \frac{x_{[\alpha k]}}{x_k} = r_{ds}(\alpha)$. Then $x \in TRV$ with the same limit if*

$$\Delta\left(\frac{x_{[\alpha k]}}{x_k}\right) = O((q_k - p_k)^{-1}).$$

Proof. This is similar to that of Theorem 2.10. □

3. Conclusion

In this work, using differed Cesáro mean we have defined and studied the statistical convergence of real sequences or measurable functions through variations such as regular, $O-$regular, translational regular and rapid, etc. Subsequently, we have established the relationships among newly defined deferred statistical convergence.

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