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On the Large $O-$Rates of Convergence in Limit Theorems for Compound Random Sums of Arrays of Row-Wise Independent Random Variables

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Abstract. Compound random sums are extensions of random sums when the random number of summands is a partial sum of independent and identically distributed positive integer-valued random variables, which assumed independent of summands. In the paper, upper bounds for the large $O-$ rates of convergence in weak limit theorems for compound random sums of arrays of row-wise independent random variables, in term of Trotter distance are studied. The main results are approximation theorems which give the Trotter distance between normalized compound random sums of the given independent random variables and the compound $\varphi-$ decomposable random variables. By these results the converging rates in central limit theorem, weak law of large numbers and stable limit theorem for compound random sums are then established. The obtained results in this paper are closely related to the classical ones.

1. Introduction

Throughout this paper let $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ be an array of row-wise independent (not necessarily identically distributed) random variables, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with several characterizations of a random variable $X_{n,j}$, such as probability distribution $F_{n,j}(x) := \mathbb{P}(X_{n,j} \leq x)$, expected value $\mu_j := \mathbb{E}(X_{n,j})$ and finite variance $\sigma^2_{n,j} := \mathbb{V}ar(X_{n,j}) \in (0, +\infty)$. For each $n \geq 1$, we will denote by $S_{n,m_n} := \sum\limits_{j=1}^{m_n} X_{n,j}$ the partial sum of random variables $X_{n,1}, X_{n,2}, \cdots, X_{n,m_n}$ in nth row, here $\{m_n, n \geq 1\}$ is a sequence of natural numbers and $m_n \to +\infty$ when $n \to +\infty$. Let $\{Y_j, j \geq 1\}$ be a sequence of independent, identically distributed (i.i.d.) positive integer-valued random variables with common expected value $\mathbb{E}(Y_j) = \nu \in (0, +\infty)$ and finite variance $\mathbb{V}ar(Y_j) = \tau^2 \in (0, +\infty)$ for $j = 1, 2, \cdots, n; n \geq 1$. Throughout this paper we shall assume that the random variables $Y_1, Y_2, \cdots$ are independent of all $X_{n,j}, (1 \leq j \leq m_n, n \geq 1)$. Furthermore, it is to be noticed that both sequences $\{X_{n,j}; 1 \leq j \leq m_n, n \geq\}$ and $\{Y_j, j \geq 1\}$ are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For $n \geq 1$, the symbols N_n and S_{n,N_n} are used for two sums

$$N_n := Y_1 + Y_2 + \cdots + Y_n \tag{1}$$

and

$$S_{n,N_n} := X_{n,1} + X_{n,2} + \cdots + X_{n,N_n}. \tag{2}$$

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The random sum S_{n,N_n} in (2) is so-called compound random sum since the number of summands is a partial sum (1) of random variables $Y_j, j \geq 1$. The compound random summation comes from the actual requirements, e.g. a negative-binomial random sum is a compound random sum structured from several geometric random sums. Actually, suppose that $Y_j, j = 1, 2, \cdots n$, are n independent, geometric distributed random variables with parameter $p \in (0, 1)$. Then, the sum $S_{Y_1} = X_1 + X_2 + \cdots + X_{Y_1}$ is said to be a geometric random sum. Obviously, the sum $N_n = Y_1 + Y_2 + \cdots + Y_n$ is a negative-binomial random variable with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$. Therefore, the random sum S_{N_n} is an extension of the S_{Y_1}, and it is called negative binomial random summation (see for instance [12] and [25]).

It is obvious that when $N_1 = Y_1$ almost surely (a.s.) the compound random sum S_{n,N_n} in (2) corresponds to the random sums in classical textbooks (see for instance [13] and [26]). For a deeper discussion of research achievements and applications related to random sums of independent random variables we refer the reader to Robbins [32], Rényi [31], Feller [11], Butzer and Schulz [4], Kruglov and Korolev [26], Gnedenko and Korolev [13], Hung [16], Shang [36], Hung and Thanh [18] and [19], Rao and Sreehari [30], etc. Further, the weak limit theorems for random sums of dependent random variables have been investigated like central limit theorems for random sums of m-dependent random variables (see [36], [15], [1]), the central limit theorems for random sums of martingale difference sequences (see for instance [5] and [6]), etc. The interested reader is referred to DasGupta (2008) in [10], Gut (2013) in [14] and Čekanavičius (2016) in [7] for a survey of some related topics on limit theorems for random sums of dependent random variables. Note that, in recent years, several results on compound random sums of strictly stationary $m-$ dependent random variables have been studied by Işlak [15] (Propositions 2.1, Proposition 2.2 and Theorem 2.3).

It is worth pointing out that several characterizations and asymptotic behavior of a compound random sum of the following form

$$S_{N_n} = X_1 + X_2 + \cdots + X_{N_n} \tag{3}$$

for a sequence of i.i.d random variables $X_1, X_2, \cdots$ have been studied by Giang and Hung (2018) in [12]. Further, the central limit theorem (CLT) for normalized compound random sum of i.i.d. random variables $X_1, X_2, \cdots$ is stated as follows

$$\frac{S_{N_n} - n\nu\mu}{\sqrt{n(\nu\sigma^2 + \mu^2\tau^2)}} \xrightarrow{D} X^* \quad \text{as} \quad n \to \infty, \tag{4}$$

where X^* is a standard normal distributed random variable with distribution function $\Phi(x) = (2\pi)^{-1/2} \int\limits_{-\infty}^{x} \exp(-0.5u^2)du$. The symbol $\xrightarrow{D}$ hereinafter denotes convergence in distribution. Besides, the weak law of large numbers (WLLN) for the sequence of i.i.d random variables $X_1, X_2, \cdots$ is also established in [12]. In fact,

$$n^{-1}S_{N_n} \xrightarrow{P} \nu\mu \quad \text{as} \quad n \to \infty. \tag{5}$$

In (5) and from now on, the notation $\xrightarrow{P}$ denotes convergence in probability. It should be noted that, using Stein's method, the large $O-$rate of convergence in the limiting expression (4) was established by Chen et al (2011) in [8] as follows

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\Big(\frac{S_{N_n} - n\nu\mu}{\sqrt{n(\nu\sigma^2 + \mu^2\tau^2)}} \leq x \Big) - \Phi(x) \right| \leq Cn^{-1/2}\Big(\frac{\tau^2}{\nu^2} + \frac{\mathbb{E}|Y_1|^3}{\tau^3} + \frac{\mathbb{E}|X_1|^3}{\nu^{1/2}\sigma^3} + \frac{\sigma}{\mu\nu^{1/2}} \Big), \tag{6}$$

where C is a positive constant. For a deeper discussion of characteristics and asymptotic behaviors of the compound random sum S_{N_n} for a sequence of i.i.d. random variables, we refer the reader to [8] (Theorem 10.6, page 271) and [12] (Propositions 2.1, 2.2 and Theorems 3.1, 3.2).

The main purpose of this paper is to establish upper bounds for the large $O-$rates of convergence in weak limit theorems for normalized compound random sums $\varphi(N_n)S_{n,N_n}$ of arrays of row-wise independent (not necessarily identically distributed) random variables, in term of Trotter distance [16]. In the paper, we shall assume that $\varphi(N_n) \stackrel{a.s.}{=} o(1)$ as $n \to \infty$, for a normalizing function $\varphi : \mathbb{N} \mapsto (0, +\infty)$ such that $\varphi(m_n) = o(1)$ as $n \to \infty$ (see [4] for the definition of the normalizing function φ). The results of this nature may be found in [4], [5], [9], [16], [19] and [20] for one-dimensional random variables and [35], [29] and [17] for multidimensional random vectors. The main results of this paper are concerned with the large $O-$rates of convergence in general approximation theorems for compound $\varphi-$ random sums of arrays of row-wise independent random variables (Theorem 5.1 and Theorem 5.3). In particular, the large $O-$ rates of convergence in the random-sum central limit theorem (Theorem 5.4, Theorem 5.5, Theorem 5.6 and Corollary 5.7), in random-sums weak law of large numbers (Theorem 5.8 and Corollary 5.9) and in random-sum $\gamma-$ stable limit theorem (Theorem 5.11 and Corollary 5.12) of independent random variables, depending upon corresponding normalizing functions as $\varphi(N_n) = \left(\sum_{j=1}^{N_n} a_j^2\right)^{-1/2}, a_j > 0; \varphi(N_n) = (N_n)^{-1/2}; \varphi(N_n) = (N_n)^{-1}$ and $\varphi(N_n) = (N_n)^{-1/\gamma}, (\gamma \in (0, 2), \gamma \neq 1)$, are investigated. The obtained results in this paper are closely related to the classical ones.

The remainder of the paper is structured as follows. Some notations, definitions and auxiliary propositions concerned with compound $\varphi-$ infinitely divisible and compound $\varphi-$ stable random variables are presented in Section 2. Section 3 is devoted to Trotter distance that is needed in this paper. The Definitions of modulus of continuity and Lipschitz classes are recalled in Section 4. Section 5 is devoted to main results of this paper concerned with estimates of the large $O-$ rates of convergence in General Approximation Theorems and related theorems with some consequences in term of Trotter distance. Section 6 closes with concluding remarks on open works.

2. Compound $\varphi-$ infinitely divisible and compound $\varphi-$ stable random variables

Following [28] (Petrov (1995), Notation and Abbreviations, page 5), for the sequence (b_n) of positive real numbers, the relation $a_n = O(b_n)$ means that $\lim_{n\to\infty} \sup(a_n/b_n) < +\infty$. Further, the relation $a_n = o(b_n)$ means that $\lim_{n\to\infty} a_n/b_n = 0$. Extending the notion of $\varphi-$ decomposability in [3], the concepts of compound $\varphi-$ decomposability, compound $\varphi-$ infinite divisibility and compound $\varphi-$ stability that are used in the paper, will be introduced as follows. Let φ denote the positive normalizing function, $\varphi : \mathbb{N} \mapsto [0, +\infty)$ such that

$$\varphi(m) = o(1) \quad \text{as} \quad m \to \infty.$$

A random variable Z is said to be $\varphi-$decomposable if for each $m \in \mathbb{N}$ there exist independent random variables $Z_j, 1 \leq j \leq m$, such that the distribution $\mathbb{P}_Z$ of Z can be represented as

$$\mathbb{P}_Z = \mathbb{P}_{\varphi(m)\sum_{j=1}^{m} Z_j}. \tag{7}$$

(see [3] and [4] for the definition of $\varphi-$decomposability). The concept of $\varphi-$decomposability in (7), can be extended to a compound $\varphi-$decomposability since the range of the partial sum $N_n := Y_1 + Y_2 + \cdots + Y_n$ is a subset of $\mathbb{N}$ (see [5] and [6] for more details). Therefore, a random variable Z is said to be compound $\varphi-$ decomposable if for each $m \in \mathbb{N}$ there exist independent random variables $Z_j, 1 \leq j \leq m$, such that the distribution $\mathbb{P}_Z$ of Z can be represented as

$$\mathbb{P}_Z = \mathbb{P}_{\varphi(N_n)\sum_{j=1}^{N_n} Z_j} = \sum_{m=1}^{\infty} \left\{ \mathbb{P}\Big(N_n = m\Big) \mathbb{P}_{\varphi(m)\sum_{j=1}^{m} Z_j} \right\}. \tag{8}$$

Further, for simplicity of notation, the expression in (8) takes the equivalent form

$$Z \stackrel{D}{=} \varphi(N_n) \sum_{j=1}^{N_n} Z_j, \tag{9}$$

where $N_n = Y_1 + Y_2 + \cdots + Y_n$, and $Y_1, Y_2, \cdots$ are independent of all $Z_j, j \geq 1$. Moreover, suppose that $\varphi(N_n) \xrightarrow{a.s.} 0$ when $n \to \infty$. Here and from now on, the symbols $\stackrel{D}{=}$ and $\xrightarrow{a.s.}$ denote the equality of distributions and convergence almost surely, respectively.

Remark 2.1. *It is worth pointing out that when $N_n \stackrel{a.s.}{=} Y_1$ the notation in (9) reduces to concept of $\varphi-$ decomposability in [5] and [6].*

Note that, if the desired components $Z_j, j \geq 1$ in (8) and (9) are i.i.d. random variables, then notion of compound $\varphi-$ infinitely divisible ($\varphi-$ ID) random variable will be defined as follows.

Definition 2.2. *A $\varphi-$decomposable random variable Z (and its probability distribution) is said to be compound $\varphi-$ infinitely divisible ($\varphi-$ ID), if for each $m \in \mathbb{N}$ there exist i.i.d. random variables $Z_j, 1 \leq j \leq m$ such that*

$$\mathbb{P}_Z = \sum_{m=1}^{\infty} \left\{ \mathbb{P}\Big(N_n = m\Big) \mathbb{P}_{\varphi(m) \sum\limits_{j=1}^{m} Z_j} \right\},$$

or in equivalent form

$$Z \stackrel{D}{=} \varphi(N_n) \sum_{j=1}^{N_n} Z_j, \tag{10}$$

where $Y_1, Y_2, \cdots, Y_n$ are i.i.d positive integer-valued random variables, independent of all i.i.d. random variables $Z_j, j \geq 1$, and $N_n = Y_1 + Y_2 + \cdots + Y_n$.

Further, if the random variables $Z_j, j \geq 1$ are i.i.d. copies of Z, then notion of compound $\varphi-$ stable random variable will be formulated as follows.

Definition 2.3. *A $\varphi-$ ID random variable Z (and its probability distribution) is said to be compound $\varphi-$ stable random variable if Z can be represented in form*

$$Z \stackrel{D}{=} \varphi(N_n) \sum_{j=1}^{N_n} Z_j, \tag{11}$$

where $Z_1, Z_2, \cdots$ are i.i.d copies of Z ($Z \stackrel{D}{=} Z_j, j \geq 1$). Here $Y_1, Y_2, \cdots, Y_n$ are i.i.d positive integer-valued random variables, independent of all $Z_j, j \geq 1$, and $N_n = Y_1 + Y_2 + \cdots + Y_n$.

Remark 2.4.
1. *The difference between the concepts of compound $\varphi-$ ID random variable (Definition 2.2) and compound $\varphi-$ stable random variable (Definition 2.3) depend upon that the $Z_1, Z_2, \cdots$ are i.i.d. random variables (in formula (10)), or they are i.i.d. copies of the original random variable Z (in formula (11)).*
2. *The notions of compound $\varphi-$decomposibility, compound $\varphi-$infinitely divisible and compound $\varphi-$ stability are extensions of concepts of infinite divisibility and geometric-infinite divisibility, mentioned by Butzer et al [3], [4], [5], [6]; Klebanov et al [24] and Kotz et al [27].*

The next propositions will show the examples of compound $\varphi-$stable random variables (Definition 2.3) depending upon various normalizing functions $\varphi(N_n)$, which will be used in subsequent sections.

Proposition 2.5. *The standard normally distributed random variable X^* with characteristic function $\mathbb{E}\Big(\exp(iX^*t)\Big) = e^{-\frac{1}{2}t^2}$, is compound $\varphi-$ stable with $\varphi(N_n) \stackrel{a.s.}{=} A_{N_n}^{-1} = \left(\sum\limits_{j=1}^{N_n} a_j^2\right)^{-1/2}$. Namely,*

$$X^* \stackrel{D}{=} A_{N_n}^{-1} \sum_{j=1}^{N_n} a_j X_j^*,$$

where $\{a_j, j \geq 1\}$ is a non-decreasing sequence of positive real numbers. Here $X^*_j (j \geq 1)$ are i.i.d copies of X^*, and $Y_1, Y_2, \cdots, Y_n$ are i.i.d. positive integer-valued random variables, independent of $X^*_1, X^*_2, \cdots$ and $N_n = Y_1 + Y_2 + \cdots + Y_n$.

Proof. According to the law of total probability ([14], page 18), the characteristic function of compound random sum $A^{-1}_{N_n} \sum_{j=1}^{N_n} a_j X^*_j$ is given by

$$\mathbb{E}\Big(\exp\Big\{itA^{-1}_{N_n}\sum_{j=1}^{N_n} a_j X^*_j\Big\}\Big) = \sum_{m=1}^{\infty} \mathbb{P}\Big(N_n = m\Big)\mathbb{E}\Big(\exp\Big\{itA^{-1}_{m}\sum_{j=1}^{m} a_j X^*_j\Big\}\Big)$$
$$= \sum_{m=1}^{\infty} \mathbb{P}\Big(N_n = m\Big)\prod_{j=1}^{m}\mathbb{E}\Big(\exp\Big\{itA^{-1}_{m} a_j X^*_j\Big\}\Big)$$
$$= \sum_{m=1}^{\infty} \mathbb{P}\Big(N_n = m\Big)\mathbb{E}\Big(\exp\Big\{-\frac{1}{2}t^2 A^{-2}_{m}\sum_{j=1}^{m} a^2_j\Big\}\Big)$$
$$= \sum_{m=1}^{\infty} \mathbb{P}\Big(N_n = m\Big)\mathbb{E}\Big(\exp\Big\{-\frac{1}{2}t^2\Big\}\Big) = \exp\Big\{-\frac{1}{2}t^2\Big\}$$
$$= \mathbb{E}(\exp\{itX^*\}).$$

On account of the uniqueness theorem ([14], Theorem 1.2, page 160), we have

$$X^* \stackrel{D}{=} A^{-1}_{N_n}\sum_{j=1}^{N_n} a_j X^*_j.$$

The proof is immediate. □

Proposition 2.6. *The standard normally distributed random variable X^* is compound $\varphi-$ stable with $\varphi(N_n) \stackrel{a.s.}{=} N^{-1/2}$, that is*

$$X^* \stackrel{D}{=} N^{-1/2}_n \sum_{j=1}^{N_n} X^*_j,$$

where $X^*_j, j \geq 1$ are i.i.d copies of X^*, and $Y_1, Y_2, \cdots, Y_n$ are i.i.d. positive integer-valued random variables, independent of all $X^*_j, j \geq 1$, with notation $N_n = Y_1 + Y_2 + \cdots + Y_n$.

Proof. The proof follows directly from Proposition 2.5 when $a_j \equiv 1$ for $j \geq 1$. □

Proposition 2.7. *The random variable X_μ degenerated at point $\mu \in \mathbb{R}$ with characteristic function $\mathbb{E}\Big(\exp(iX_\mu t)\Big) = e^{i\mu t}$, is compound $\varphi-$ stable with $\varphi(N_n) \stackrel{a.s.}{=} N^{-1}_n$, that is*

$$X_\mu \stackrel{D}{=} N^{-1}_n \sum_{j=1}^{N_n} X_\mu(j),$$

where $X_\mu(j), j \geq 1$ are i.i.d copies of X_μ, $N_n = Y_1 + Y_2 + \cdots + Y_n$, and $Y_1, Y_2, \cdots, Y_n$ are i.i.d. positive integer-valued random variables, independent of all $X_\mu(j)$ for $j \geq 1$.

Proof. On account of law of total probability ([14], page 18), the characteristic function of compound random sum $N_n^{-1}\sum_{j=1}^{N_n} X_\mu(j)$ is given by

$$\mathbb{E}\Big(\exp\Big\{iN_n^{-1}\sum_{j=1}^{N_n} X_\mu(j)t\Big\}\Big) = \sum_{m=1}^{\infty}\mathbb{P}\Big(N_n = m\Big)\mathbb{E}\Big(\exp\Big\{im^{-1}\sum_{j=1}^{m} X_\mu(j)t\Big\}\Big)$$
$$= \sum_{m=1}^{\infty}\mathbb{P}\Big(N_n = m\Big)\mathbb{E}\Big(e^{i\mu t}\Big) = e^{i\mu t}$$
$$= \mathbb{E}(e^{iX_\mu t}).$$

According to the uniqueness theorem ([14], Theorem 1.2, page 160), it follows that

$$X_\mu \stackrel{D}{=} N_n^{-1}\sum_{j=1}^{N_n} X_\mu(j).$$

This finishes the proof. □

Proposition 2.8. *The standard Cauchy distributed random variable* $C_{0,1}$ *with characteristic function* $\mathbb{E}(\exp(iC_{0,1}t)) = e^{-|t|}$*, denoted by* $C_{0,1} \sim Cauchy(0,1)$*, is a compound* $\varphi-$*stable with* $\varphi(N_n) = N_n^{-1}$*, that is*

$$C_{0,1} \stackrel{D}{=} N_n^{-1}\sum_{j=1}^{N_n} C_{0,1}(j),$$

where $C_{0,1}(j), j \geq 1$ *are i.i.d. copies of* $C_{0,1}, N_n = Y_1 + \cdots + Y_n$*, and* $Y_1, Y_2, \cdots, Y_n$ *are i.i.d. positive integer-valued random variables, independent of all* $C_{0,1}(j)$ *for* $j \geq 1$*.*

Proof. It is easily seen that the characteristic function of compound random sum $N_n^{-1}\sum_{j=1}^{N_n} C_{0,1}(j)$ is given by

$$\mathbb{E}\Big(\exp\Big\{iN_n^{-1}\sum_{j=1}^{N_n} C_{0,1}(j)t\Big\}\Big) = \sum_{m=1}^{\infty}\mathbb{P}\Big(N_n = m\Big)\mathbb{E}\Big(\exp\Big\{im^{-1}\sum_{j=1}^{m} C_{0,1}(j)t\Big\}\Big)$$
$$= \sum_{m=1}^{\infty}\mathbb{P}\Big(N_n = m\Big)\mathbb{E}\Big(e^{-|t|}\Big) = e^{-|t|}$$
$$= \mathbb{E}\Big(\exp(iC_{0,1}t)\Big).$$

On account of the uniqueness theorem ([14], Theorem 1.2, page 160), it may be concluded that

$$C_{0,1} \stackrel{D}{=} N_n^{-1}\sum_{j=1}^{N_n} C_{0,1}(j).$$

The proof is complete. □

3. Trotter distance

For the purpose of the present paper, a more limited definition of probability metrics will be used. For a deeper discussion of probability metrics we refer the reader to [21], [23], [16], [17] and [19]. From now on, we denote by $\mathfrak{X}$ the set of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Definition 3.1. *([21]) Let $X, Y \in \mathfrak{X}$. The mapping $d : \mathfrak{X} \times \mathfrak{X} \longrightarrow [0, +\infty)$ is called the probability metric of X and Y, denoted by $d(X, Y)$, if it possesses the following properties:*

1. *Identity property. Let $X \overset{a.s.}{=} Y$ for any $X, Y \in \mathfrak{X}$. Then, $d(X, Y) = 0$.*
2. *Symmetry. $d(X, Y) = d(Y, X)$ for any $X, Y \in \mathfrak{X}$.*
3. *Triangle inequality. $d(X, Z) \leq d(Z, Y) + d(Y, Z)$ for any $X, Y, Z \in \mathfrak{X}$.*

It is an immediate corollary of the identity property through the triangle inequality property that the probability metric $d \geq 0$.

We need to recall the definition and properties of Trotter distance based on Trotter operator [37] (see [16] and [19] for more details). Let us denote by $C_B(\mathbb{R})$ the set of all bounded, uniformly continuous functions f on $\mathbb{R} = (-\infty, +\infty)$, with norm $\| f \| = \sup_{x\in\mathbb{R}} | f(x) |$, and

$$C_B^k(\mathbb{R}) = \left\{ f \in C_B(\mathbb{R}) : f^{(j)} \in C_B(\mathbb{R}), j = 1, 2, \cdots, k; k \in \mathbb{N} \right\}.$$

Definition 3.2. *([37]) The Trotter operator associated with a random variable $X \in \mathfrak{X}$, is the mapping $T_X : C_B(\mathbb{R}) \mapsto C_B(\mathbb{R})$, such that*

$$T_X f(y) := \mathbb{E} f(X + y) = \int_{\mathbb{R}} f(x + y) dF_X(x), \quad \text{for every} \quad f \in C_B(\mathbb{R}),$$

where $y \in \mathbb{R}$ and $F_X(x) := \mathbb{P}(X \leq x)$ is distribution function of a random variable X.

We refer the reader to [37], [11], [2], [3], [4], [5], [6], [9], [29], [23], [33], [34], [35], [16], [17], [19] and [20] for a more general and detailed discussion of Trotter operator method and its applications in limit theorems for partial sums and for random sums of independent random variables.

We will consider the definition of Trotter distance based on the Trotter operator. It is worth mentioning that the name of Trotter distance firstly recommended by Kirschfink [23].

Definition 3.3. *([16]) Let $X, Y \in \mathfrak{X}$. Trotter distance of X and Y associated to a function $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$, denoted by $d_T(X, Y; f)$, is defined as follows*

$$d_T(X, Y; f) = \sup_{y\in\mathbb{R}} \left| \mathbb{E} f(X + y) - \mathbb{E} f(Y + y) \right|.$$

Based on properties of Trotter operator [37], several important properties of Trotter distance are summarized as follows (see [16] and [19] for more details).

1. Trotter distance $d_T(X, Y; f)$ is a probability metric, i.e., for $X, Y, Z \in \mathfrak{X}$, the following properties hold:
   (a) For every $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$, the distance $d_T(X, Y; f) = 0$ if $X \overset{a.s.}{=} Y$;
   (b) $d_T(X, Y; f) = d_T(Y, X; f)$ for every $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$;
   (c) $d_T(X, Y; f) \leq d_T(X, Z; f) + d_T(Z, Y; f)$ for every $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$.
2. Let $d_T(X, Y; f) = 0$ for all $f \in C_B^k(\mathbb{R}), k \geq 1$. Then $X \overset{D}{=} Y$.
3. Let $d_T(X_n, X; f) = o(1)$ as $n \to \infty$, for $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$. Then $X_n \overset{D}{\longrightarrow} X$ as $n \to \infty$.
4. Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two sequences of independent random variables (in each sequence). Then, for $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$ and for $m \in \mathbb{N}$,

$$d_T\left(\sum_{j=1}^{m} X_j, \sum_{j=1}^{m} Y_j; f \right) \leq \sum_{j=1}^{m} d_T(X_j, Y_j; f). \tag{12}$$

   Particularly, if $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ are two sequences of i.i.d. (in each sequence) random variables, then for $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$ and for $m \in \mathbb{N}$,

$$d_T\left(\sum_{j=1}^{m} X_j, \sum_{j=1}^{m} Y_j; f \right) \leq m \times d_T(X_1, Y_1; f). \tag{13}$$

5. Let $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ be two sequences of independent random variables (in each sequence). Let $\{N_n, n \geq 1\}$ be a sequence of positive, integer-valued random variables, independent of all X_j and Y_j for $j \geq 1$. Then, for $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$,

$$d_T\left(\sum_{j=1}^{N_n} X_j, \sum_{j=1}^{N_n} Y_j; f\right) \leq \sum_{m=1}^{\infty} P(N_n = m) \sum_{j=1}^{m} d_T(X_j, Y_j; f). \tag{14}$$

In particular, assume that $\{X_j, j \geq 1\}$ and $\{Y_j, j \geq 1\}$ are two sequences of i.i.d. random variables (in each sequence). Then, for $f \in C_B^k(\mathbb{R}), k \in \mathbb{N}$,

$$d_T\left(\sum_{j=1}^{N_n} X_j, \sum_{j=1}^{N_n} Y_j; f\right) \leq \sum_{m=1}^{\infty} P(N_n = m) \times m \times d_T(X_1, Y_1; f). \tag{15}$$

Note that the relationship between the Trotter distance and various well-known probability metrics like Kolmogorov's metric, Levy's metric, Zolotarev's metric, etc. have been discussed by Hung (2007) in [16].

4. Modulus of continuity and Lipschitz class

We introduce the notions of modulus of continuity and Lipschitz class for a function $f \in C_B(\mathbb{R})$, following [2] and [22].

Definition 4.1. *([2]) For $f \in C_B(\mathbb{R})$ and for $\delta > 0$, the modulus of continuity is defined in the form*

$$\omega(f; \delta) = \sup_{|h| \leq \delta} \sup_{x \in \mathbb{R}} \left| f(x+h) - f(x) \right|.$$

From [22] (Chapter 9, page 407), it follows that

1. If $0 < \delta_1 < \delta_2$, then $\omega(f; \delta_1) \leq \omega(f; \delta_2)$.
2. $\omega(f; \delta) = o(1)$ as $\delta \to 0^+$.
3. $\omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta)$ for all $\lambda \in (0, +\infty)$.

Definition 4.2. *([2]) A function $f \in C_B(\mathbb{R})$ is said to satisfy a Lipschitz condition of order $\alpha, 0 < \alpha \leq 1$, in symbols $f \in Lip(\alpha)$, if $\omega(f; \delta) = O(\delta^\alpha)$.*

It is obvious that $f' \in C_B(\mathbb{R})$ implies $f \in Lip(1)$.

Next proposition states one of the most important properties of moments of $X \in \mathfrak{X}$ that will be used in this paper.

Proposition 4.3. *([2]) Let $X \in \mathfrak{X}$ with $\mathbb{E}|X|^k < +\infty$. Then, $\mathbb{E}|X|^j < +\infty$, for any $1 \leq j \leq k$, and*

$$\mathbb{E}|X|^j \leq 1 + \mathbb{E}|X|^k. \tag{16}$$

Proof. The proof is elementary and this can be found in [20] (Appendix, page 230). □

5. General Approximation Theorems and consequences

Recall that $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ and $\{Y_j, j \geq 1\}$ are two sequences introduced in Section 1 with notations of the partial sum N_n and compound random sum S_{n,N_n}. Let Z be a compound $\varphi-$ decomposable random variable, defined in (9), that is

$$Z \stackrel{D}{=} \varphi(N_n) \sum_{j=1}^{N_n} Z_j,$$

where Z_j, $j \geq 1$ are independent random variables with the probability distributions $H_j(x) := \mathbb{P}(Z_j \leq x)$. Note that the random variables $Z_1, Z_2, \cdots$ are independent of $Y_1, Y_2, \cdots$. Moreover, assume that the condition $\varphi(N_n) \xrightarrow{a.s.} 0$ holds as $n \to \infty$.
The following theorem deals with the large $O-$ rate of convergence for distribution of normalized compound random sum $\varphi(N_n)S_{n,N_n} := \varphi(N_n)\sum_{j=1}^{N_n} X_j$ to distribution of a compound $\varphi-$ decomposable random variable Z, in term of Trotter distance.

Theorem 5.1. *(General Approximation Theorem for independent (not necessarily identically distributed) random variables) Let $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ be an array of row-wise of independent (not necessarily identically distributed) random variables with $\mathbb{E}(|X_{n,j}|^k) < +\infty$ for $1 \leq k \leq r-1, r \in \mathbb{N}$. Let Z be a compound $\varphi-$composable random variable with independent components Z_j, having the probability distributions $H_j(x)$, $j \geq 1$. Let $Y_1, Y2, \cdots, Y_n$ be a sequence of i.i.d positive integer-valued random variables, independent of all $X_{n,1}, X_{n,2}, \cdots$ and $Z_1, Z_2, \cdots$ for $n \geq 1$. Write $N_n = Y_1 + Y_2 + \cdots + Y_n$. Assume that the following condition*

$$\mathbb{E}\left(X_{n,j}^k\right) = \mathbb{E}\left(Z_j^k\right) \quad for \quad j \in \mathbb{N}, 1 \leq k \leq r-1, r \in \mathbb{N} \tag{17}$$

holds. Then, for each $f \in C_B^{r-1}(\mathbb{R}), r \in \mathbb{N}$,

$$\begin{aligned} & d_T\Big(\varphi(N_n)S_{n,N_n}, Z; f\Big) \\ & \leq 2\mathbb{E}\Big\{\frac{[\varphi(N_n)]^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)}; \varphi(N_n)\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{nj}|^r + \mathbb{E}|Z_j|^r + 1\Big]\Big\}. \end{aligned} \tag{18}$$

In particular, if $f \in Lip(\alpha), 0 < \alpha \leq 1$, then the bound in (18) takes the form

$$d_T\Big(\varphi(N_n)S_{n,N_n}, Z; f\Big) = O\Big(\mathbb{E}\Big\{\frac{[\varphi(N_n)]^{r-1+\alpha}}{(r-1)!}\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{n,j}|^r + \mathbb{E}|Z_j|^r + 1\Big]\Big\}\Big). \tag{19}$$

Proof. Since $f \in C_B^{r-1}(\mathbb{R}), r \in \mathbb{N}$ one has by Taylor series expansion (see [22], Theorem 4.3.1, Pages 108–109),

$$f(x+y) = f(y) + \sum_{k=1}^{r-1}\frac{f^{(k)}(y)}{k!}x^k + \frac{x^{r-1}}{(r-1)!}\Big(f^{(r-1)}(\eta) - f^{(r-1)}(y)\Big),$$

where η is some number between y and $x+y$. This yields, for $f \in C_B^{r-1}(\mathbb{R})$,

$$\begin{aligned} \mathbb{E}f\Big(\varphi(m_n)X_{n,j} + y\Big) &= \int_{\mathbb{R}} f\Big(\varphi(m_n)x + y\Big)dF_{n,j}(x) \\ &= f(y) + \sum_{k=1}^{r-1}\frac{f^{(k)}(y)}{k!}[\varphi(m_n)]^k\mathbb{E}(X_{n,j}^k) + \frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\int_{\mathbb{R}}\Big(f^{(r-1)}(\eta_1) - f^{(r-1)}(y)\Big)x^{r-1}dF_{n,j}(x), \end{aligned} \tag{20}$$

where η_1 is now between y and $y + \varphi(m_n)x$, hence $|\eta_1 - y| < \varphi(m_n)|x|$.
Analogously, for $f \in C_B^{r-1}(\mathbb{R})$,

$$\begin{aligned} \mathbb{E}f\Big(\varphi(m_n)Z_j + y\Big) &= \int_{\mathbb{R}} f\Big(\varphi(m_n)x + y\Big)dH_j(x) \\ &= f(y) + \sum_{k=1}^{r-1}\frac{f^{(k)}(y)}{k!}[\varphi(m_n)]^k\mathbb{E}(Z_j^k) + \frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\int_{\mathbb{R}}\Big(f^{(r-1)}(\eta_2) - f^{(r-1)}(y)\Big)x^{r-1}dH_j(x), \end{aligned} \tag{21}$$

where η_2 is now between y and $y+\varphi(m_n)x$, hence $|\eta_2 - y| < \varphi(m_n)|x|$.
Combining (21) and (20), from (17), for each $j = 1, 2, \cdots, m_n$, it follows that

$$\begin{aligned}&\left|\mathbb{E}f\Big(\varphi(m_n)X_{n,j}+y\Big)-\mathbb{E}f\Big(\varphi(m_n)Z_j+y\Big)\right|\\ &\le \frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\int\limits_{\mathbb{R}}\left|f^{(r-1)}(\eta_1)-f^{(r-1)}(y)\right||x|^{r-1}dF_{n,j}(x)\\ &+\frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\int\limits_{\mathbb{R}}\left|f^{(r-1)}(\eta_2)-f^{(r-1)}(y)\right||x|^{r-1}dH_j(x),\end{aligned} \tag{22}$$

where η_1 and η_2 are some numbers between y and $y+\varphi(m_n)x$, hence $|\eta_j - y| < \varphi(m_n)|x|$ for $j = 1, 2$.
According to Definition 4.1 and properties of the modulus of continuity (Section 4), from Proposition 4.3, for the first term on the right-hand side of inequality (22), we have the following estimates

$$\begin{aligned}&\frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\int\limits_{\mathbb{R}}\left|f^{(r-1)}(\eta_1)-f^{(r-1)}(y)\right||x|^{r-1}dF_{n,j}(x)\\ &\le \frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\int\limits_{\mathbb{R}}|x|^{r-1}\omega\Big(f^{(r-1)};|\eta_1-y|\Big)dF_{n,j}(x)\\ &\le \frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)};\varphi(m_n)\Big)\int\limits_{\mathbb{R}}|x|^{r-1}(1+|x|)dF_{nj}(x)\\ &\le \frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)};\varphi(m_n)\Big)\Big(2\mathbb{E}|X_{nj}|^r+1\Big).\end{aligned} \tag{23}$$

Analogously, for the second term on the right-hand side of inequality (22), it follows that

$$\begin{aligned}&\frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\int\limits_{\mathbb{R}}\left|f^{(r-1)}(\eta_1)-f^{(r-1)}(y)\right||x|^{r-1}dH_j(x)\\ &\le \frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)};\varphi(m_n)\Big)\Big(2\mathbb{E}|Z_j|^r+1\Big).\end{aligned} \tag{24}$$

Combining (22), (23) and (24), from Definition 4.2, for $f \in C_B^{r-1}(\mathbb{R})$, we conclude that

$$\begin{aligned}&d_T(\varphi(m_n)X_{nj},\varphi(m_m)Z_j;f)\\ &\le 2\frac{[\varphi(m_n)]^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)};\varphi(m_n)\Big)\Big[\mathbb{E}|X_{n,j}|^r+\mathbb{E}|Z_j|^r+1\Big].\end{aligned} \tag{25}$$

According to inequality (14), from (25), for $f \in C_B^{r-1}(\mathbb{R})$, the desired estimate in (18) is proved as follows

$$d_T\Big(\varphi(N_n)S_{n,N_n},Z;f\Big)\le \mathbb{E}\Big\{\sum_{j=1}^{N_n}d_T\Big(\varphi(N_n)X_{n,j},\varphi(N_n)Z_j;f\Big)\Big\}$$

$$\le 2\mathbb{E}\Big\{\frac{[\varphi(N_n)]^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)};\varphi(N_n)\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{n,j}|^r+\mathbb{E}|Z_j|^r+1\Big]\Big\}.$$

It is easy to check that if $f \in Lip(\alpha), 0 < \alpha \le 1$, then the bound in (19) will be followed.
The proof is straightforward. □

Remark 5.2. *Note that the equality of moments in (17) have been used in Butzer et al ([2], condition (3.1)), Butzer and Hahn ([3], Theorem 1, condition (3.1); [4], Theorem 9, conditions (6.2) and (6.5)). The equality of moments have been also used in [4] for weak limit theorems for random sum of martingale difference sequences (MDS) by Butzer and Schulz ([4], Theorem 1, condition (3.3); Theorem 3, condition (5.1)).*

As a particular case of Theorem 5.1, the following approximation theorem deals with the case of i.i.d random variables $X_{n,j}$ $(1 \leq j \leq m_n, n \geq 1)$ and the limiting random variable Z belongs to a class of compound $\varphi-$ infinitely divisible ($\varphi-$ ID) random variables.

Theorem 5.3. *(General Approximation Theorem for i.i.d. random variables) Let $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ be an array of row-wise of i.i.d. random variables with $\mathbb{E}(|X_{n,1}|^k < +\infty, 1 \leq k \leq r-1, r \in \mathbb{N}$. Let Z be a compound $\varphi-$ ID random variable with the i.i.d. components $Z_j, j \geq 1$, such that $\mathbb{E}(|Z_1|^k < +\infty, 1 \leq k \leq r-1, r \in \mathbb{N}$. Assume that the following condition*

$$\mathbb{E}\left(X_{n,1}^k\right) = \mathbb{E}\left(Z_1^k\right) \quad for \quad 1 \leq k \leq r, r \in \mathbb{N} \tag{26}$$

holds. Suppose that $Y_1, Y_2, \cdots, Y_n$ are i.i.d. positive integer-valued random variables, independent of both sequences $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ and $\{Z_j, j \geq 1\}$. Write $N_n = Y_1 + Y_2 + \cdots + Y_n$.
Then, for each $f \in C_B^{r-1}(\mathbb{R}), r \in \mathbb{N}$,

$$\begin{aligned} & d_T\Big(\varphi(N_n)S_{n,N_n}, Z; f\Big) \\ & \leq 2\mathbb{E}\Big(N_n \frac{[\varphi(N_n)]^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)}; \varphi(N_n)\Big)\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|Z_1|^r + 1\Big]\Big). \end{aligned} \tag{27}$$

In particular, if $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, then the bound in (27) takes the form

$$d_T\Big(\varphi(N_n)S_{n,N_n}, Z; f\Big) = O\Big(\mathbb{E}\Big(N_n[\varphi(N_n)]^{r+\alpha-1}\Big)\Big). \tag{28}$$

Proof. By an argument analogous to that used for the proof of Theorem 5.1, using inequalities (13) and (15), from Definition 4.2 with Proposition 4.3, the proof of the Theorem 5.3 follows immediately. □

Note that the limiting random variable Z in Theorem 5.1 was chosen as a compound $\varphi-$ decomposable random variable (Section 2) and in Theorem 5.3 it was taken as a compound $\varphi-$ ID random variable (Definition 2.2). Further, the standard normal distributed random variable X^* and the random variable X_μ degenerated at point μ, are both compound $\varphi-$ stable (see Definition 2.3, Proposition 2.5, Proposition 2.6 and Proposition 2.7). Therefore, the large $O-$rates of convergence in next compound random–sum central limit theorem and compound random–sum weak law of large numbers will be estimated as follows.

Theorem 5.4. *(Large $O-$rate of convergence in compound random-sum central limit theorem) Let $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ be an array of row-wise independent (not necessarily identically distributed) random variables with $\mathbb{E}|X_{n,j}|^k < +\infty$, for $1 \leq k \leq r-1, r \in \mathbb{N}, j \geq 1$. Let X^* be a standard normally distributed random variable with probability distribution $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp\{-0.5y^2\}dy$. Assume that for $1 \leq k \leq r-1, r \in \mathbb{N}, j \geq 1$, the following condition*

$$\mathbb{E}\left(X_{n,j}^k\right) = a_j^k \mathbb{E}\left(X_j^{*k}\right) \tag{29}$$

holds, where $X_1^*, X_2^*, \cdots$ are i.i.d. copies of X^* and $\{a_j, j \geq 1\}$ is a sequence of positive real numbers. Assume that $Y_1, Y_2, \cdots, Y_n$ are i.i.d. positive integer-valued random variables, independent of two sequences $\{X_{n,j}, j \geq 1\}$ and

$\{X_j^*, j \geq 1\}$. Write $N_n = Y_1 + Y_2 + \cdots + Y_n$.
Then, for $f \in C_B^{r-1}(R), r \in \mathbb{N}$

$$d_T\Big(A_{N_n}^{-1} S_{n,N_n}, X^*; f\Big) \leq 2\mathbb{E}\Big(\frac{A_{N_n}^{-(r-1)}}{(r-1)!}\omega\Big(f^{(r-1)}; A_{N_n}^{-1}\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{nj}|^r + a_j^r \mathbb{E}|X_j^*|^r + 1\Big]\Big). \tag{30}$$

where $A_{N_n} := \Big(\sum_{j=1}^{N_n} a_j^2\Big)^{1/2}$.
In particular, if $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, then the bound in (30) takes on the form

$$d_T\Big(A_{N_n}^{-1}\sum_{j=1}^{N_n} X_{nj}, X^*; f\Big) = O\Big\{\mathbb{E}\Big[\frac{A_{N_n}^{-(r-1+\alpha)}}{(r-1)!}\sum_{j=1}^{N_n}\Big(\mathbb{E}|X_{nj}|^r + a_j^r \mathbb{E}|X_j^*|^r + 1\Big)\Big]\Big\}.$$

Proof. On account of Proposition 2.5, the X^* is a compound $\varphi-$ stable random variable. It means that for each $n \in \mathbb{N}$ there exist independent, normally distributed random variables $X_j^*, 1 \leq j \leq m_n, n \geq 1$, such that

$$X^* \stackrel{D}{=} \Big(\sum_{j=1}^{m_n} a_j\Big)^{-1/2}\sum_{j=1}^{m_n} a_j X_j^*. \tag{31}$$

Further, from (29), it follow that $E(|a_j^k X^{*k}|) < +\infty$, for $1 \leq k \leq r-1, r \in \mathbb{N}, j \geq 1$. For the sake of convenience we shall adopt the following notation

$$A_{m_n} := \Big(\sum_{j=1}^{m_n} a_j^2\Big)^{1/2}.$$

By an argument analogous to the previous proof of Theorem 5.1, we get for $f \in C_B^{r-1}(\mathbb{R})$,

$$\begin{aligned} \mathbb{E}f\Big(A_{m_n}^{-1}X_{n,j} + y\Big) &= \int_{\mathbb{R}} f\Big(A_{m_n}^{-1}x + y\Big)dF_{n,j}(x) \\ &= f(y) + \sum_{k=1}^{r-1}\frac{f^{(k)}(y)}{k!}(A_{m_n}^{-k})\mathbb{E}(X_{n,j}^k) + \frac{A_{m_n}^{-(r-1)}}{(r-1)!}\int_{\mathbb{R}}\Big(f^{(r-1)}(\eta_3) - f^{(r-1)}(y)\Big)x^{r-1}dF_{n,j}(x), \end{aligned} \tag{32}$$

where η_3 is some number between y and $y + A_{m_n}^{-1}x$, hence $|\eta_3 - y| \leq A_{m_n}^{-1}|x|$.
Analogously,

$$\begin{aligned} \mathbb{E}f\Big(A_{m_n}^{-1}a_j X_j^* + y\Big) &= \int_{\mathbb{R}} f\Big(A_{m_n}^{-1}a_j x + y\Big)d\Phi(x) \\ &= f(y) + \sum_{k=1}^{r-1}\frac{f^{(k)}(y)}{k!}(A_{m_n}^{-k})\mathbb{E}([a_j X_j^*]^k) \\ &+ \frac{A_{m_n}^{-(r-1)}a_j^{r-1}}{(r-1)!}\int_{\mathbb{R}}\Big(f^{(r-1)}(\eta_4) - f^{(r-1)}(y)\Big)x^{r-1}d\Phi(x), \end{aligned} \tag{33}$$

where η_4 is some number between y and $y + A_{m_n}^{-1}a_j x$, hence $|\eta_4 - y| \le A_{m_n}^{-1}|x|$.
Combining (32) and (33), based on properties of the modulus of continuity and Proposition 4.3, from (29), it may be concluded that

$$\begin{aligned} d_T\Big(A_{m_n}^{-1}X_{nj}, A_{m_n}^{-1}a_jX_j^*; f\Big) &= \sup_{y\in\mathbb{R}}\Big|\mathbb{E}f\Big(A_{m_n}^{-1}X_{nj}+y\Big) - \mathbb{E}f\Big(A_{m_n}^{-1}a_jX_j^*+y\Big)\Big| \\ &\le \frac{(A_{m_n}^{1-r})}{(r-1)!}\int\limits_{\mathbb{R}}\Big|f^{(r-1)}(\eta_3) - f^{(r-1)}(y)\Big||x|^{r-1}dF_{nj}(x) \\ &+ \frac{(A_{m_n}^{1-r})}{(r-1)!}\int\limits_{\mathbb{R}}\Big|f^{(r-1)}(\eta_4) - f^{(r-1)}(y)\Big|a_j^{r-1}|x|^{r-1}d\Phi(x) \\ &\le 2\frac{(A_{m_n}^{1-r})}{(r-1)!}\omega\Big(f^{(r-1)}; A_{m_n}^{-1}\Big)\Big[\mathbb{E}|X_{n,j}|^r + a_j^r\mathbb{E}|X_j^*|^r + 1\Big], \end{aligned} \tag{34}$$

for $f \in C_B^{r-1}(\mathbb{R})$. According to inequalities (12) and (14), from (34) it follows that

$$\begin{aligned} & d_T\Big(A_{N_n}^{-1}S_{n,N_n}, X^*; f\Big) \\ &\le 2\mathbb{E}\Big(\frac{A_{N_n}^{1-r}}{(r-1)!}\omega\Big(f^{(r-1)}; A_{N_n}^{-1}\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{nj}|^r + a_j^r\mathbb{E}|X_j^*|^r + 1\Big]\Big). \end{aligned}$$

It is clear that if $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \le 1$, then

$$d_T\Big(A_{N_n}^{-1}S_{n,N_n}, X^*; f\Big) = O\Big\{\mathbb{E}\Big[\frac{A_{N_n}^{-(r-1+\alpha)}}{(r-1)!}\sum_{j=1}^{N_n}\Big(\mathbb{E}|X_{nj}|^r + a_j^r\mathbb{E}|X_j^*|^r + 1\Big)\Big]\Big\}.$$

The proof is complete. □

Theorem 5.5. *(Large $O-$rate of convergence in compound random-sum central limit theorem for i.i.d. random variables) Let $\{X_{n,j}; 1 \le j \le m_n, n \ge 1\}$ be an array of row-wise i.i.d. random variables with $\mathbb{E}|X_{n,1}|^k < +\infty$ for $1 \le k \le r-1, r \in \mathbb{N}$. Let X^* be a standard normally distributed random variables with distribution function $\Phi(x)$. Assume that the following condition*

$$\mathbb{E}\left(X_{n,1}^k\right) = \mathbb{E}\left(X^{*k}\right) \quad for \quad 1 \le k \le r-1, r \in \mathbb{N}, \tag{35}$$

holds. Assume that $Y_1, Y_2, \cdots, Y_n$ are i.i.d. positive integer-valued random variables, independent of random variables X^* and $X_{n,j}$ for $j \ge 1$ and for $n \ge 1$. Write $N_n := Y_1 + Y_2 + \cdots + Y_n$.
Then, for $f \in C_B^{r-1}(\mathbb{R})$,

$$d_T\Big(N_n^{-1/2}S_{n,N_n}, X^*; f\Big) \le 2\mathbb{E}\Big(\frac{N_n^{1-(r-1)/2}}{(r-1)!}\omega(f^{(r-1)}; N_n^{-1/2})\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|X_j^*|^r + 1\Big]\Big). \tag{36}$$

If, in addition, $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \le 1$, then (36) takes the form

$$d_T\Big(N_n^{-1/2}S_{n,N_n}, X^*; f\Big) = O\Big\{\mathbb{E}\Big[\frac{N_n^{1-(r-1+\alpha)/2}}{(r-1)!}\Big(\mathbb{E}|X_{n,1}|^r + \mathbb{E}|X_j^*|^r + 1\Big)\Big]\Big\}. \tag{37}$$

Proof. The proof parallels that of Theorem 5.4 (with $a_j = 1$ for $j \ge 1$) and will be omitted. □

The following theorem is a particular case of Theorem 5.4 and it extends a result due to Chen et al (2011) in [8] (Theorem 10.6, page 271).

Theorem 5.6. *Let $\{X_{n,j}; 1 \le j \le m_n, n \ge 1\}$ be an array of row-wise independent (not necessarily identically distributed) random variables with distribution functions $F_{n,j}(x) = \mathbb{P}(X_{n,j} \le x)$, expected values $\mu_{n,j} = \mathbb{E}(X_{n,j})$ and finite variances $\sigma^2_{n,j} = \mathbb{V}ar(X_{n,j}) \in (0, +\infty)$ for $1 \le k \le r-1, r \in \mathbb{N}, j \ge 1$. Let X^* be a standard normally distributed random variable with distribution function $\Phi(x)$. Assume that $\mathbb{E}|X_{n,j} - \mu_{n,j}|^k < +\infty$ for $2 \le k \le r-1, n \ge 1$ and the following condition*

$$\mathbb{E}(X_{n,j} - \mu_{n,j})^k = \sigma^k_{n,j}\mathbb{E}(X^*_j)^k, \quad for \quad j \ge 1 \quad and \quad 1 \le k \le r-1, r \in \mathbb{R}, n \ge 1 \tag{38}$$

holds, where $X^*_1, X^*_2, \cdots$ are i.i.d. copies of X^*. Assume that $Y_1, Y_2, \cdots$, are i.i.d. positive integer-valued random variables, independent of both sequences $\{X_{n,j}; 1 \le j \le m_n, n \ge 1\}$ and $\{X^*_j, j \ge 1\}$. Set $N_n = Y_1 + Y_2 + \cdots + Y_n$. Then, for $f \in C^{r-1}_B(\mathbb{R})$,

$$d_T\Big(\frac{S_{n,N_n} - \mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) \le \mathbb{E}\Big\{2\frac{N_n^{-(r-1)/2}}{(r-1)!}\omega\Big(f^{(r-1)}, N_n^{-1/2}\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|W_{n,j}|^r + \mathbb{E}|X^*_j|^r + 1\Big]\Big\}, \tag{39}$$

where $W_{n,j} = \frac{X_{n,j} - \mu_{n,j}}{\sigma_{n,j}}, j \ge 1, n \ge 1$.
In particular, if, $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \le 1$, then

$$d_T\Big(\frac{S_{n,N_n} - \mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) = O\Big\{\mathbb{E}\Big(\frac{N_n^{-(r-1+\alpha)/2}}{(r-1)!}\sum_{j=1}^{N_n}\Big[\mathbb{E}|W_{n,j}|^r + \mathbb{E}|X^*_j|^r + 1\Big]\Big)\Big\}. \tag{40}$$

Proof. We shall begin with showing that

$$\begin{aligned} d_T\Big(\frac{S_{n,N_n} - \mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) &= \sum_{m_n=1}^{\infty}\mathbb{P}\Big(N_n = m_n\Big)d_T\Big(\frac{S_{n,m_n} - \mathbb{E}\big(S_{n,m_n}\big)}{[\mathbb{V}ar(S_{n,m_n})]^{1/2}}, X^*; f\Big) \\ &\le \sum_{m_n=1}^{\infty}\mathbb{P}\Big(N_n = m_n\Big)\sum_{j=1}^{m_n} d_T\Big(m_n^{-1/2}\frac{X_{nj} - \mu_{nj}}{\sigma_{nj}}, m_n^{-1/2}X^*_j; f\Big), \end{aligned} \tag{41}$$

where, from Proposition 2.6, $X^*_j, j \ge 1$ are i.i.d. copies of X^* such that

$$X^* \overset{D}{=} N_n^{-1/2}\sum_{j=1}^{N_n} X^*_j.$$

For $n \ge 1, j \ge 1$ set $W_{n,j} = (X_{n,j} - \mu_{n,j})/\sigma_{n,j}$. It is clear that

$$\mathbb{E}(W_{n,j}) = 0, \mathbb{V}ar(W_{n,j}) = 1 \quad \text{and} \quad \mathbb{E}|W_{n,j}|^k < +\infty$$

for $j \ge 1, 1 \le k \le r-1, r \in \mathbb{N}, n \ge 1$. Further,

$$F_{W_{n,j}}(x) = \mathbb{P}(W_{n,j} \le x) = \mathbb{P}(X_{n,j} \le \sigma_{n,j}x + \mu_{n,j}) = F_{n,j}(\sigma_{n,j}x + \mu_{n,j}).$$

Moreover, it is obvious that $W_{n,j} (j \ge 1)$ are independent of $N_n := Y_1 + Y_2 + \cdots + Y_n$ for $n \ge 1$ and

$$\mathbb{E}(W_{n,j})^k = \mathbb{E}(X^*_j)^k \quad \text{for} \quad 1 \le k \le r-1, r \in \mathbb{N}. \tag{42}$$

Since $f \in C^{r-1}_B(\mathbb{R})$, using Taylor series expansion, we have

$$\begin{aligned} \mathbb{E}f\Big(m_n^{-1/2}W_{n,j} + y\Big) &= f(y) + \mathbb{E}\Big\{\sum_{k=1}^{r-1}\frac{f^{(k)}(y)}{k!}(m_n^{-1/2})^k W^k_{n,j}\Big\} \\ &\quad + \frac{(m_n^{-1/2})^{r-1}}{(r-1)!}\int_{\mathbb{R}}\Big(f^{(r-1)}(\eta_5) - f^{(r-1)}(y)\Big)x^{r-1}dF_{n,j}(\sigma_{n,j}x + \mu_{n,j}), \end{aligned} \tag{43}$$

where η_5 is some number between y and $y + m_n^{-1/2}x$, hence $|\eta_5 - y| \le m_n^{-1/2}|x|$.
Analogously,

$$\mathbb{E}f\Big(m_n^{-1/2}X_j^* + y\Big) = f(y) + \mathbb{E}\Big\{\sum_{k=1}^{r-1} \frac{f^{(k)}(y)}{k!}(m_n^{-1/2})^k (X_j^*)^k\Big\} + \frac{(m_n^{-1/2})^{r-1}}{(r-1)!}\int_{\mathbb{R}} \Big(f^{(r-1)}(\eta_6) - f^{(r-1)}(y)\Big)x^{r-1}d\Phi(x), \tag{44}$$

where η_6 is some number between y and $y + m_n^{-1/2}x$, hence $|\eta_6 - y| \le m_n^{-1/2}|x|$.
Combining (43) and (44), from (42), for $f \in C_B^{r-1}(\mathbb{R})$, we conclude that

$$d_T\Big(m_n^{-1/2}W_{n,j}, m_n^{-1/2}X_j^*; f\Big) \le 2\frac{(m_n^{-1/2})^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)}, m_n^{-1/2}\Big)\Big[\mathbb{E}|W_{n,j}|^r + \mathbb{E}|X_j^*|^r + 1\Big]. \tag{45}$$

According to inequalities (12) and (14), from (44) it may be conclude that

$$\begin{aligned} & d_T\Big(\frac{S_{n,N_n} - \mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) \\ & \le \sum_{m_n=1}^{\infty} \mathbb{P}(N_n = m_n) \sum_{j=1}^{m_n} \Big\{2\frac{(m_n^{-1/2})^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)}, m_n^{-1/2}\Big)\Big[\mathbb{E}|W_{n,j}|^r + \mathbb{E}|X_j^*|^r + 1\Big]\Big\} \\ & \le \mathbb{E}\Big\{2\frac{N_n^{-(r-1)/2}}{(r-1)!}\omega\Big(f^{(r-1)}, N_n^{-1/2}\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|W_{n,j}|^r + \mathbb{E}|X_j^*|^r + 1\Big]\Big\}. \end{aligned}$$

If $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \le 1$, from previous inequality, it follows that

$$d_T\Big(\frac{S_{n,N_n} - \mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) = O\Big\{\mathbb{E}\Big(\frac{N_n^{-(r-1+\alpha)/2}}{(r-1)!}\sum_{j=1}^{N_n}\Big[\mathbb{E}|W_{n,j}|^r + \mathbb{E}|X_j^*|^r + 1\Big]\Big)\Big\}.$$

This completes the proof. □

The following corollary is an immediate consequence of Theorem 5.6.

Corollary 5.7. *Let $\{X_{n,j}; 1 \le j \le m_n, n \ge 1\}$ be an array of row-wise i.i.d. random variables with a distribution F, common expected value μ and finite variance $\sigma^2 \in (0, +\infty)$. Assume that $\mathbb{E}|X_{n,1} - \mu|^k < +\infty$ for $2 \le k \le r-1, n \ge 1$ and the following condition*

$$\mathbb{E}(X_{n,1} - \mu)^k = \sigma^k \mathbb{E}(X^*)^k, \quad for \quad and \quad 1 \le k \le r-1, r \in \mathbb{R}, n \ge 1 \tag{46}$$

holds. Then, for $f \in C_B^{r-1}(\mathbb{R})$,

$$d_T\Big(\frac{S_{n,N_n} - \mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) \le \mathbb{E}\Big\{2\frac{N_n^{1-(r-1)/2}}{(r-1)!}\omega\Big(f^{(r-1)}, N_n^{-1/2}\Big)\Big[\mathbb{E}|W_{n,1}|^r + \mathbb{E}|X^*|^r + 1\Big]\Big\}, \tag{47}$$

where $W_{n,1} = \frac{X_{n,1}-\mu}{\sigma}$.
In particular, if $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \le 1$, then

$$d_T\Big(\frac{S_{n,N_n} - \mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) = O\Big\{\mathbb{E}\Big(\frac{N_n^{1-(r-1+\alpha)/2}}{(r-1)!}\Big[\mathbb{E}|W_{n,1}|^r + \mathbb{E}|X^*|^r + 1\Big]\Big)\Big\}. \tag{48}$$

Proof. By an argument analogous to that used for the proof of Theorem 5.6, we get

$$
\begin{aligned}
& d_T\Big(\frac{S_{n,N_n}-\mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) \\
& \le \sum_{m_n=1}^{\infty}\mathbb{P}(N_n=m_n)m_n\Big\{2\frac{(m_n^{-1/2})^{r-1}}{(r-1)!}\omega\Big(f^{(r-1)}, m_n^{-1/2}\Big)\Big[\mathbb{E}|W_{n,1}|^r+\mathbb{E}|X^*|^r+1\Big]\Big\} \\
& \le \mathbb{E}\Big\{2\frac{N_n^{1-(r-1)/2}}{(r-1)!}\omega\Big(f^{(r-1)}, N_n^{-1/2}\Big)\Big[\mathbb{E}|W_{n,1}|^r+\mathbb{E}|X^*|^r+1\Big]\Big\}.
\end{aligned}
$$

If, $f^{(r-1)} \in Lip(\alpha), 0<\alpha\le 1$, then from above estimate, we have

$$
d_T\Big(\frac{S_{n,N_n}-\mathbb{E}\big(S_{n,N_n}\big)}{[\mathbb{V}ar(S_{n,N_n})]^{1/2}}, X^*; f\Big) = O\Big\{\mathbb{E}\Big(\frac{N_n^{1-(r-1+\alpha)/2}}{(r-1)!}\Big[\mathbb{E}|W_{n,1}|^r+\mathbb{E}|X^*|^r+1\Big]\Big)\Big\}.
$$

The proof is complete. □

The following theorem gives the large $O-$rate of convergence in Markov-type weak law of large numbers introduced by Petrov (1995) ([28], Theorem 4.16, page 134) and the Khinchin-type weak law of large numbers presented by Giang and Hung (2018) ([12], Theorem 3.3, page 612), in term of Trotter distance. This result is also an extension of Hung (2007) ([16], Theorem 5.3, page 30).

Theorem 5.8. *(Large $O-$rate of convergence in compound $\varphi-$ random weak law of large numbers) Let $\{X_{n,j}; 1\le j\le m_n, n\ge 1\}$ be an array of row-wise independent (not necessarily identically distributed) random variables with distribution functions $F_{n,j}(x)=\mathbb{P}(X_{n,j}\le x)$, expected values $\mathbb{E}(X_{n,j})=\mu_{n,j}$ and finite absolute moments $\mathbb{E}|X_{n,j}|^k<+\infty$ for $1\le k\le r-1, r\in\mathbb{N}, n\ge 1$. Let $\mu=m_n^{-1}\sum_{j=1}^{m_n}\mu_{n,j}$ denote the average of expected values $\mu_{n,j}, 1\le j\le m_n, n\ge 1$, and X_μ the random variable degenerated at point μ. Assume that for two sequences $\{X_{n,j}: 1\le j\le m_n, n\ge 1\}$ and $\{X_\mu(j), j\ge 1\}$ the following condition*

$$
\mathbb{E}\left(X_{n,j}^k\right)=\mathbb{E}\left(X_\mu^k(j)\right) \quad \text{for} \quad 1\le k\le r-1, r\in\mathbb{N}, j\ge 1, n\ge 1 \tag{49}
$$

holds, where $X_\mu(1), X_\mu(2),\cdots$ are i.i.d. copies of X_μ, such that

$$
X_\mu \overset{D}{=} \varphi(N_n)\sum_{j=1}^{N_n}X_\mu(j).
$$

Suppose that $Y_1, Y_2,\cdots$ are i.i.d positive integer-valued random variables, independent of both sequences $\{X_{n,j}; 1\le j\le m_n, n\ge 1\}$ and $\{X_\mu(j), j\ge 1\}$. Set $N_n=Y_1+Y_2+\cdots+Y_n$. Moreover, assume that $\varphi(N_n)\overset{a.s.}{=}o(1)$ as $n\to\infty$. Then, for $f\in C_B^{r-1}(\mathbb{R})$,

$$
\begin{aligned}
& d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) \\
& \le 2\mathbb{E}\Big(\frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1}\omega\Big(f^{(r-1)};\varphi(N_n)\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{n,j}|^r+\mathbb{E}|X_\mu(j)|^r+1\Big]\Big).
\end{aligned} \tag{50}
$$

In particular, if $f^{(r-1)}\in Lip(\alpha), 0<\alpha\le 1$, then the bound in (50) takes the form

$$
d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) = O\Big\{\mathbb{E}\Big(\frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1+\alpha}\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{n,j}|^r+\mathbb{E}|X_\mu(j)|^r+1\Big]\Big)\Big\}. \tag{51}
$$

Proof. From Taylor series expansion (see [22], Theorem 4.3.1, Pages 108, 109) for $f \in C_B^{r-1}(\mathbb{R})$, it follows that

$$\begin{aligned} \mathbb{E} f\Big(\varphi(m_n)X_{nj} + y\Big) &= \int_{\mathbb{R}} f\Big(\varphi(m_n x + y)\Big) dF_{n,j}(x) \\ &= f(y) + \sum_{k=1}^{r-1} \frac{f^{(k)}(y)}{(k-1)!}[\varphi(m_n)]^k \mathbb{E}(X_{nj}^k) + \frac{[\varphi(m_n)]^{r-1}}{(r-1)!} \int_{\mathbb{R}} \Big(f^{(r-1)}(\eta_7) - f^{(r-1)}(y)\Big) x^{r-1} dF_{n,j}(x), \end{aligned} \tag{52}$$

where η_7 is some number between y and $y + \varphi(m_n)x$, hence $|\eta_7 - y| \le \varphi(m_n)|x|$.
Analogously,

$$\begin{aligned} \mathbb{E} f\Big(\varphi(m_n)X_\mu(j) + y\Big) &= \int_{\mathbb{R}} f\Big(\varphi(m_n x + y)\Big) dF_\mu(x) \\ &= f(y) + \sum_{k=1}^{r-1} \frac{f^{(k)}(y)}{(k-1)!}[\varphi(m_n)]^k \mathbb{E} X_\mu^k(j) + \frac{[\varphi(m_n)]^{r-1}}{(r-1)!} \int_{\mathbb{R}} \Big(f^{(r-1)}(\eta_8) - f^{(r-1)}(y)\Big) x^{r-1} dF_\mu(x), \end{aligned} \tag{53}$$

where η_8 is some number between y and $\varphi(m_n)x$, hence $|\eta_7 - y| \le \varphi(m_n)|x|$. In (53) the notation $F_\mu(x) = \mathbb{P}\Big(X_\mu \le x\Big)$ stands for the probability distribution of X_μ.
Combining (52) and (53), from condition (49) and Proposition 4.3, we have for $f \in C_B^{r-1}(R)$,

$$\begin{aligned} & d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) \\ & \le 2\mathbb{E}\Big(\frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1}\omega\Big(f^{(r-1)}; \varphi(N_n)\Big) \sum_{j=1}^{N_n} \Big[\mathbb{E}|X_{n,j}|^r + \mathbb{E}|X_\mu(j)|^r + 1\Big]\Big). \end{aligned}$$

On account of $f^{(r-1)} \in Lip(\alpha), (0 < \alpha < 1)$, from the bound (53), it is obviously that

$$d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) = O\Big\{\mathbb{E}\Big(\frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1+\alpha} \sum_{j=1}^{N_n} \Big[\mathbb{E}|X_{n,j}|^r + \mathbb{E}|X_\mu(j)|^r + 1\Big]\Big)\Big\}.$$

The proof is complete. □

The following corollary is an immediate consequence of Theorem 5.8.

Corollary 5.9. *Let $\{X_{n,j}; 1 \le j \le m_n, n \ge 1\}$ be an array of row-wise of i.i.d random variables with a common distribution F, expected value μ and finite absolute moments $\mathbb{E}|X_{n,1}|^k \in (0, +\infty)$ for $n \ge 1, k \in \mathbb{N}$. Let X_μ be a random variable degenerated at point μ. Assume that the following condition*

$$\mathbb{E}\left(X_{n,1}^k\right) = \mathbb{E}\left(X_\mu^k\right) \quad for \quad 1 \le k \le r-1, r \in \mathbb{N}, n \ge 1, \tag{54}$$

holds. Then

$$\begin{aligned} & d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) \\ & \le 2\mathbb{E}\Big(N_n \frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1}\omega\Big(f^{(r-1)}; \varphi(N_n)\Big)\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|X_\mu|^r + 1\Big]\Big). \end{aligned} \tag{55}$$

If, in addition, $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \le 1$, then the inequality (55) takes the form

$$d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) = O\Big\{\mathbb{E}\Big(N_n \frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1+\alpha}\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|X_\mu|^r + 1\Big]\Big)\Big\}.$$

Proof. Evidently, using inequality (15), when the $X_{n,j}$, for $1 \leq j \leq m_n, n \geq 1$, are i.i.d random variables with common expected value μ, from equality of moments (54), it may be concluded that

$$d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) \leq 2\mathbb{E}\Big(N_n \frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1}\omega\Big(f^{(r-1)};\varphi(N_n)\Big)\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|X_\mu|^r + 1\Big]\Big).$$

On account of $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, from above bound, it follows that

$$d_T\Big(\varphi(N_n)S_{n,N_n}, X_\mu; f\Big) = O\Big\{\mathbb{E}\Big(N_n \frac{\|f^{(r-1)}\|}{(r-1)!}[\varphi(N_n)]^{r-1+\alpha}\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|X_\mu|^r + 1\Big]\Big)\Big\}.$$

This finishes the proof. □

Remark 5.10. *The following results are direct consequences of Theorem 5.8.*

1. *When* $\varphi(N_n) \overset{a.s.}{=} N_n^{-1}$, *for i.i.d. random variables* X_{nj} $(1 \leq j \leq m_n, n \geq 1)$, *for* $f \in C_B^2(\mathbb{R})$, *from assumption (54) we have*

$$d_T\Big(N_n^{-1}S_{n,N_n}, X_\mu; f\Big) \leq 2\mathbb{E}\Big(N_n^{-1}\frac{\|f''\|}{2}\omega\Big(f'', N_n^{-1}\Big)\Big[\mathbb{E}|X_{n,j}|^3 + \mathbb{E}|X_\mu|^3 + 1\Big]\Big)$$

2. *From the previous bound, for* $f \in Lip(\alpha), 0 < \alpha \leq 2$, *we get*

$$d_T\Big(N_n^{-1}S_{n,N_n}, X_\mu; f\Big) = O\Big\{\mathbb{E}\Big(N_n^{-(1+\alpha)}\frac{\|f''\|}{2}\Big[\mathbb{E}|X_{n,j}|^3 + \mathbb{E}|X_\mu|^3 + 1\Big]\Big)\Big\}.$$

Following the notation used in [4], let us denote by Z_γ the $\gamma-$ stable random variable with characteristic function $\exp(-c|t|^\gamma), c \geq 0, \gamma \in (0, 2]$.
The next theorem deals with the $\gamma-$ stable random variable $Z_\gamma, 0 < \gamma \leq 2$, which is a compound $\varphi-$ stable random variable with $\varphi(N_n) \overset{a.s.}{=} N_n^{-1/\gamma}$, i. e.,

$$Z_\gamma \overset{D}{=} N_n^{-1/\gamma}\sum_{j=1}^{N_n} Z_\gamma(j), \tag{56}$$

where $Z_\gamma(1), Z_\gamma(2), \cdots$ are i.i.d. copies of Z_γ. Here the random variables $Y_1, Y_2, \cdots$ are independent of $Z_\gamma(1), Z_\gamma(2), \cdots$ and $N_n = Y_1 + Y_2 + \cdots + Y_n$ with assumption that $\varphi(N_n) \overset{a.s.}{=} o(1)$ when $n \to \infty$.

Theorem 5.11. *(Large $O-$rate of convergence in compound random–sum $\gamma-$ stable limit theorem) Let $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ be an array of row-wise of independent (not necessarily identically distributed) random variables with $\mathbb{E}|X_{n,j}|^k < +\infty$ for $1 \leq k \leq r-1, r \in \mathbb{N}$. Let Z_γ be a $\gamma-$ stable random variable with representation (56). Assume that for the two sequences $X_{n,1}, X_{n,2}, \cdots$ and $Z_\gamma(1), Z_\gamma(2), \cdots$, the following condition*

$$\mathbb{E}\left(X_{n,j}^k\right) = \mathbb{E}\left(Z_\gamma^k(j)\right) \quad for \quad j \geq 1, 1 \leq k \leq r-1, r \in \mathbb{N}, n \geq 1 \tag{57}$$

holds. Suppose that $Y_1, Y_2, \cdots, Y_n$ are i.i.d. positive integer-valued random variables, independent of both sequences $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ and $\{Z_\gamma(1), Z_\gamma(2), \cdots\}$. Write $N_n = Y_1 + Y_2 + \cdots + Y_n$. Then, for $f \in C_B^{r-1}(\mathbb{R})$,

$$d_T\Big(N_n^{-1/\gamma}S_{n,N_n}, Z_\gamma; f\Big) \leq 2\mathbb{E}\Big\{\frac{\|f^{(r-1)}\|}{(r-1)!}N_n^{-(r-1)/\gamma}\omega\Big(f^{(r-1)}; N_n^{-1/\gamma}\Big)\sum_{j=1}^{N_n}\Big[\mathbb{E}|X_{n,j}|^r + \mathbb{E}|Z_\gamma(j)|^r + 1\Big]\Big\}. \tag{58}$$

In particular, if $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, then the bound in (58) takes the form

$$d_T\Big(N_n^{-1/\gamma} S_{n,N_n}, Z_\gamma; f\Big) = \mathcal{O}\Big\{\mathbb{E}\Big(\frac{\|f^{(r-1)}\|}{(r-1)!} N_n^{-(r-1+\alpha)/\gamma} \sum_{j=1}^{N_n} \Big[\mathbb{E}|X_{n,j}|^r + \mathbb{E}|Z_\gamma(j)|^r + 1\Big]\Big)\Big\}. \tag{59}$$

Proof. Analysis similar to the proofs of previous theorems, using Taylor series expansion for $f \in C_B^{r-1}(\mathbb{R})$, from (57), it follows that

$$\begin{aligned} & d_T\Big(m_n^{-1/\gamma} \sum_{j=1}^{m_n} X_{nj}, m_n^{-1/\gamma} \sum_{j=1}^{m_n} Z_\gamma(j); f\Big) \\ & \leq \frac{\|f^{(r-1)}\|}{(r-1)!} m_n^{-(r-1)/\gamma} \sum_{j=1}^{m_n} \int_{\mathbb{R}} \Big| f^{(r-1)}(\eta_9) - f^{(r-1)}(y)\Big| |x|^{r-1} dF_{n,j}(x) \\ & + \leq \frac{\|f^{(r-1)}\|}{(r-1)!} m_n^{-(r-1)/\gamma} \sum_{j=1}^{m_n} \int_{\mathbb{R}} \Big| f^{(r-1)}(\eta_{10}) - f^{(r-1)}(y)\Big| |x|^{r-1} dF_\gamma(x) \\ & \leq 2\frac{\|f^{(r-1)}\|}{(r-1)!} m_n^{-(r-1)/\gamma} \omega\Big(f^{(r-1)}; m_n^{-1/\gamma}\Big) \sum_{j=1}^{m_n} \Big[\mathbb{E}|X_{n,j}|^r + \mathbb{E}|Z_\gamma(j)|^r + 1\Big], \end{aligned} \tag{60}$$

where η_9 and η_{10} are some numbers between y and $(y + m_n^{-1/\gamma} x)$, hence $|\eta_j - y| \leq m_n^{-1/\gamma}|x|$ for $j = 9, 10$. In (60) the notation $F_\gamma(x)$ stands for the distribution function of $Z_\gamma(j)$, for $1 \leq j \leq m_n$.
According to inequalities (12) and (14), from (60), we have

$$\begin{aligned} & d_T\Big(N_n^{-1/\gamma} S_{n,N_n}, Z_\gamma; f\Big) \\ & \leq 2\mathbb{E}\Big\{\frac{\|f^{(r-1)}\|}{(r-1)!} N_n^{-(r-1)/\gamma} \omega\Big(f^{(r-1)}; N_n^{-1/\gamma}\Big) \sum_{j=1}^{N_n} \Big[\mathbb{E}|X_{nj}|^r + \mathbb{E}|Z_\gamma(j)|^r + 1\Big]\Big\}. \end{aligned}$$

The estimate (58) is proved.
In addition, for $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, from (58), it may be concluded that

$$d_T\Big(N_n^{-1/\gamma} S_{n,N_n}, Z_\gamma; f\Big) = \mathcal{O}\Big(\mathbb{E}\Big\{\frac{\|f^{(r-1)}\|}{(r-1)!} N_n^{-(r-1+\alpha)/\gamma} \sum_{j=1}^{N_n} \Big[\mathbb{E}|X_{n,j}|^r + \mathbb{E}|Z_\gamma(j)|^r + 1\Big]\Big\}\Big).$$

The proof is complete. □

The following corollary is an immediate consequence of Theorem 5.11.

Corollary 5.12. *Let $\{X_{n,j}; 1 \leq j \leq m_n, n \geq 1\}$ be an array of row-wise of i.i.d. random variables with $\mathbb{E}|X_{n,1}|^k < +\infty$ for $1 \leq k \leq r-1, r \in \mathbb{N}$. Let Z_γ be a $\gamma-$ stable random variable presented in (56) with $\gamma \in (0, 2]$ and $\gamma \neq 1$. If, in addition, the condition (57) holds for $j = 1$, then*

$$\begin{aligned} & d_T\Big(N_n^{-1/\gamma} S_{n,N_n}, Z_\gamma; f\Big) \\ & \leq 2\mathbb{E}\Big\{\frac{\|f^{(r-1)}\|}{(r-1)!} N_n^{1-(r-1)/\gamma} \omega\Big(f^{(r-1)}; N_n^{-1/\gamma}\Big)\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|Z_\gamma(1)|^r + 1\Big]\Big\}. \end{aligned} \tag{61}$$

If $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \leq 1$, then the bound (61) takes the form

$$d_T\Big(N_n^{-1/\gamma} S_{n,N_n}, Z_\gamma; f\Big) = \mathcal{O}\Big\{\mathbb{E}\Big(\frac{\|f^{(r-1)}\|}{(r-1)!} N_n^{1-(r-1+\alpha)/\gamma}\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|Z_\gamma(1)|^r + 1\Big]\Big)\Big\}.$$

Proof. For the i.i.d. random variables $X_{n,j}, 1 \le j \le m_n, n \ge 1$, and condition (57) for $j = 1$, we have

$$d_T\Big(N_n^{-1/\gamma} S_{n,N_n}, Z_\gamma; f\Big)$$
$$\le 2\mathbb{E}\Big\{\frac{\|f^{(r-1)}\|}{(r-1)!} N_n^{1-(r-1)/\gamma} \omega\Big(f^{(r-1)}; N_n^{-1/\gamma}\Big)\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|Z_\gamma(1)|^r + 1\Big]\Big\}.$$

Moreover, if $f^{(r-1)} \in Lip(\alpha), 0 < \alpha \le 1$, then the previous bound takes the form

$$d_T\Big(N_n^{-1/\gamma} S_{n,N_n}, Z_\gamma; f\Big) = O\Big\{\mathbb{E}\Big(\frac{\|f^{(r-1)}\|}{(r-1)!} N_n^{1-(r-1+\alpha)/\gamma}\Big[\mathbb{E}|X_{n,1}|^r + \mathbb{E}|Z_\gamma(1)|^r + 1\Big]\Big)\Big\}.$$

This finishes the proof. □

Remark 5.13. *Let $C_{0,1}$ be a standard Cauchy distributed random variable, having the characteristics function $\psi_{C_{0,1}}(t) = e^{-|t|}$ for $t \in \mathbb{R}$. According to Proposition 2.8 when $\gamma = 1$, the $C_{0,1}$ is a compound $N_n^{-1}-$stable with $\varphi(N_n) = N_n^{-1}$, that is*

$$C_{0,1} \overset{D}{=} N_n^{-1} \sum_{j=1}^{N_n} C_{0,1}(j),$$

where $C_{0,1}(j), j \ge 1$ are i.i.d. copies of $C_{0,1}$. However, for the Cauchy distribution , no positive moment of order 1 or more is finite (see, e.g., [27], page 19). Therefore, Theorem 5.11 and Corollary 5.12 can not applied to the standard Cauchy distributed random variables $C_{0,1}$.

6. Concluding remarks

We conclude this paper with the following comments.

1. The obtained results can be extended to various weak limit theorems for compound random sums of different dependent structures of random variables as $m-$ dependent random variables, martingale difference sequences, Markov chain, etc (see for instance DasGupta [10], Gut [14] and Čekanavičius [7]).
2. The matter described in this paper could also be discussed in the case of multidimensional random vectors.

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