Filomat 36:7 (2022), 2153–2170
https://doi.org/10.2298/FIL2207153L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Global Solutions for a General Predator-Prey Model with Prey-Stage Structure and Cross-Diffusion

Xiaojuan Li[a,b], Shenghu Xu[a,c]

[a] *School of Economics, North Minzu University, Yinchuan, Ningxia 750021,, P. R. China*
[b] *College of Mathematics and Information Science, Neijiang Normal University, Neijiang, Sichuan 641112, P.R. China*
[c] *School of Mathematics and Statistics, Longdong University, Qingyang, Gansu 745000, P. R. China*

Abstract. In this paper, a cross-diffusion predator-prey model with general functional response and stage-structure for the prey is analyzed. The global existence of classical solutions to the system of strong coupled reaction-diffusion type is proved when the space dimension less than ten by the energy estimates and the bootstrap arguments. The crucial point of the proof is to deal with the cross-diffusion term and the nonlinear predation term .

1. Introduction and the mathematical model

The dynamic relationship between predator and prey has long been, and will continue to be, one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Many kinds of predator-prey models have been studied extensively (see, [1, 2]). In the natural world, there are many species whose individual members have a life history that takes them through two stages: immature and mature. In particular, we have in mind mammalian populations and some amphibious animals, which exhibit these two stages. Due to the above realistic evidences, the stage-structured models have received much attention in recent years, see, [3–14, 31, 32, 34, 45–47] and the references therein. In the model of Aiello and Freedman [2], the population has a life history and is divided into two stages: immature and mature. They built and studied a time delay model of single species growth with stage structure. Then, in [3], Zhang *et al*. proposed the following of a Lotka-Volterra predator-prey model with prey-stage structure

$$\begin{aligned}
&\tfrac{dx_1}{dt} = Bx_2 - Cx_1 - D_1x_1 - \gamma x_1^2 - kx_1y, \quad t > 0,\\
&\tfrac{dx_2}{dt} = Cx_1 - D_2x_2, \quad t > 0, \qquad\qquad (A)\\
&\tfrac{dy}{dt} = y(-D_3 + \delta_1 kx_1 - \eta y), \quad t > 0.
\end{aligned}$$

2020 *Mathematics Subject Classification*. 35J60, 35B32, 92D25
Keywords. predator-prey model; stage structure; cross-diffusion; Global solution
Received: 04 January 2020; Revised: 20 January 2020; Accepted: 21 January 2020
Communicated by Maria Alessandra Ragusa
The first author is partially supported by the Doctor Start-up Funding of North Minzu University(Grants nos.2022QHPY16), Scientific research fund of Sichuan province provincial education department(Grants nos.18ZB0319), and the second author is partially supported by the Doctor Start-up Funding of North Minzu University(Grants nos.2022QHPY16), National Natural Science Foundation of China (Grants nos.11661051), Sichuan province science and technology plan project(Grants nos.2017JY0195), Research and innovation team of Neijiang Normal University(Grants nos.17TD04)
Email addresses: lixiaojuan114@126.com. (Xiaojuan Li), xuluck2001@163.com (Shenghu Xu)

On the other hand, in order to understand the dynamics of a predator-prey model involves not only the size and structure of the population, but also the ability to capture prey and renew itself. One significant component of the predator-prey relationship is the predator's functional response, i.e., the rate of prey consumption by an average predator. Generally, the functional response can be classified into two types: *prey-dependent* and *predator-dependent*. Prey-dependent indicates that the functional response is only a function the preys density, while predator-dependent means that the functional response is a function of both the preys and the predator's densities. The classical Holling types I-III [37, 37], the Holling type IV(or Monod-Haldane type)[39], the Ivlev type[38] and Rosenzweig type [44] are strictly prey-dependent functional response; Ratio-dependent type [35] Hassell-Varley type [43], Beddington-DeAngelis type by Beddington [40] and DeAngelis et al. [41] as well as Crowley-Martin type [42] are predator-dependent functional response.

We note that an important factor in modelling of predator-prey is the choice of functional responses governing the prey-predator interactions. The above system (A) assume the predator with Holling tyepe I funcational response kx, which is linear and prey-dependent. However, this assumption seems not to be so reliable all the time. Motivated by the above papers, it is realistic and interesting for us to construct a stage-structured predator-prey model with general functional response function which depend on the numbers of immature prey and predator. We also assume the predator only preys on immature prey. Under the above assumptions, we establish the ODE prey-predator model with general functional response and stage-structure for the prey as follows

$$\begin{aligned} \frac{dx_1}{dt} &= Bx_2 - Cx_1 - D_1x_1 - \gamma x_1^2 - \phi(x_1, y)y, \quad t > 0, \\ \frac{dx_2}{dt} &= Cx_1 - D_2x_2, \quad t > 0, \\ \frac{dy}{dt} &= y(-D_3 + \delta_1\phi(x_1, y) - \eta y), \quad t > 0, \end{aligned} \tag{1}$$

where x_1, x_2 are the population densities of immature and mature prey species, respectively. y denotes the density of predator population. η is *nonnegative* constant. $B, C, D_1, D_2, D_3, \gamma, \delta_1$ are positive constants. B represents the birth rate of the immature prey, C denotes the transmission rate from immature prey individuals, γ and η is the intra-specific competition rate of the immature prey and predator, respectively; D_1 and D_2 represent the death rates of immature and mature prey, respectively. D_3 is the death rate of predator, δ_1 is the conversion rate. Furthermore, we assume that the functional response function $\phi(x_1, y)$ satisfies:

$(H_1)'$: $\phi(0, y) = 0$, for all $y \geq 0$.

$(H_2)'$: $\frac{\partial\phi(x_1,y)}{\partial y} \leq 0$, for all $x_1 \geq 0$ and $y \geq 0$.

From the biological point of view, the functional response function $\phi(x_1, y)$ satisfies $(H_1)'$ and $(H_2)'$. The condition $(H_1)'$ implies that, as the prey population extinction, the capture rate of the predator is identical to zero. The condition $(H_2)'$ implies that, as the predator population increases, the consumption rate of prey per predator decreases. Some explicit forms for the predator functional response that have been used are

$\phi(x) = L_1(1 - e^{-px})$ [Ivlev type (1961)[38]],

$\phi(x) = L_1x^q (q < 1)$ [Rosenzweig (1971)[44]];

$\phi(x) = L_1x, \frac{L_1x}{a+x}, \frac{L_1x^2}{a+x^2}$ [Holling types I-III (1959)[36, 37]];

$\phi(x) = \frac{L_1x}{1+ay+bx^2}$ [Holling type IV type (1968) [39]];

$\phi(x, y) = \frac{L_1x}{ay^\delta+x}(\delta \in (0, 1))$ [Hassell-Varley type (1969)[43]];

$\phi(x, y) = \frac{L_1x}{ay+x}$ [ratio-dependent type (1989)[35]];

$\phi(x, y) = \frac{L_1x}{1+ax+by}$ [Beddington-DeAngelis type (1975)[40, 41]];

$\phi(x, y) = \frac{L_1x}{(1+ax)(1+by)}$ [Crowley-Martin type (1989)[42]];

Using the scaling $u_1 = \frac{1}{D_2}x_1, u_2 = \frac{1}{C}x_2, u_3 = \frac{1}{D_2}y, d\tau = D_2 dt$, and denoting τ by t again, the system (1.1) becomes

$$\begin{aligned} &\tfrac{du_1}{dt} = au_2 - bu_1 - \gamma u_1^2 - g(u_1,u_3)u_3, \quad t>0,\\ &\tfrac{du_2}{dt} = u_1 - u_2, \quad t>0,\\ &\tfrac{du_3}{dt} = u_3(-r + \delta g(u_1,u_3) - \eta u_3), \quad t>0, \end{aligned} \tag{2}$$

where $a = \frac{BC}{D_2^2}, b = \frac{C+D_1}{D_2}, r = \frac{D_3}{D_2}, \delta = \frac{\delta_1}{D_2}$, and $g(u_1,u_3) = \phi(D_2u_1, D_2u_3)$, so the conditions $(H_1)' - (H_2)'$ become:

(H_1) : $g(0,u_3) = 0$, for all $u_3 \geq 0$.

(H_2) : $\frac{\partial g(u_1,u_3)}{\partial u_3} \leq 0$, for all $u_1 \geq 0$ and $u_3 \geq 0$.

Note that the above ten functional responses satisfy the hypotheses $(H_1) - (H_2)$. We also remark that while there have been many results about prey-predator models with stage-structure for the predator, such as [6, 11–13].

In the last decades, there has been a great interest in using cross diffusion to model physical and biological phenomena, such as chemotaxis phenomenon in biomathematics, generalized drift diffusion and energy transport model in semiconductor science, separation of granular material, etc[6, 12, 15–23, 30–33]. In real applications, such kinds of cross diffusion models describe the phenomena in consideration more clearly than the classical weakly coupled diffusion systems, we shall include the cross diffusion term in the third equation, as follows:

$$\begin{aligned} &u_{1t} - \Delta[(d_1 + \alpha_{11}u_1)u_1] = au_2 - bu_1 - \gamma u_1^2 - g(u_1,u_3)u_3, x \in \Omega, t>0,\\ &u_{2t} - \Delta[(d_2 + \alpha_{22}u_2)u_2] = u_1 - u_2, x \in \Omega, t>0,\\ &u_{3t} - \Delta[(d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + \alpha_{33}u_3)u_3] = u_3(-r + \delta g(u_1,u_3) - \eta u_3), x \in \Omega, t>0,\\ &\tfrac{\partial u_1}{\partial \nu} = \tfrac{\partial u_2}{\partial \nu} = \tfrac{\partial u_3}{\partial \nu} = 0, x \in \partial\Omega, t>0,\\ &u_i(x,0) = u_{i0}(x) \geq 0, i = 1,2,3, x \in \Omega, \end{aligned} \tag{3}$$

where $\alpha_{11}, \alpha_{22} \geq 0$ and $\alpha_{31}, \alpha_{32}, \alpha_{33} > 0$. $d_i (i = 1,2,3)$ are positive constants. $d_i (i = 1,2,3)$ are the random diffusion rates of the three species, respectively. $\alpha_{ii} (i = 1,2,3)$ are self-diffusion rates, and α_{31} and α_{32} are cross-diffusion rates. For more details on the biological background, see [17].

Ever since the fundamental work by Amann (see, [24, 25]), the question of local existence of solutions to (3) was settled by Amann's work but global existence results seem to be answered in only very few cases. However, Mathematically, one of the most important problem for (3) is to establish the existence of global solutions. In particular, the global existence of classical solutions for (3) is open and interesting question to understand the problem in the high-dimensional space. This question is on the list of open problems (for two species predator-prey model with cross-diffusion) made by Y. Yamada in [48]. The main purpose of this paper is to understand the global existence of classical solutions of (3) for higher-dimensional space.

The fundamental characteristics of this model are:

(C_1) : The functional response function $g(u_1,u_3)$ is dependent on the densities of the immature prey and predator.

(C_2) : The intra-specific competition rate of the predator, η is *nonnegative* constant. At this point, it becomes important whether $\eta = 0$ or $\eta > 0$ in estimating the term $\int_{Q_t} u_3^q(-r + \delta g(u_1,u_3) - \eta u_3)dxds$.

(C_3) : The system (3) is a strongly coupled parabolic systems. In particular, in the case $\alpha_{22}, \alpha_{32}, \alpha_{32} > 0$. Recently, Fu et.al, in [32] showed the existence of global solutions for the system (A) with cross-diffusion, However, they only consider the system (3) the case when $\alpha_{22} = \alpha_{32} = 0$, $G(u_1,u_3) = du$ and $\eta > 0$. First, global existence results for (3) are stated in a different style according as $\alpha_{11} = \alpha_{22} = 0$ or $\alpha_{11}, \alpha_{22} > 0$. If $\alpha_{11} = \alpha_{22} = 0$, then (3) possesses a unique global solution for any initial functions, and any space dimension N, while if $\alpha_{11}, \alpha_{22} > 0$ some restriction on N or the nonlinear diffusion coefficients is necessary to ensure global existence.

Main results. The purpose of this paper is to establish the global existence of classical solutions to (3). Precisely, we prove the following results:

(a) In case $\eta > 0$:

Theorem 1.1. *Let* $(H_1) - (H_2)$ *hold. Assume* $\alpha_{11}, \alpha_{22}, \alpha_{33} > 0$ *and* $1 \leq N \leq 9$. *Assume also that initial data* $u_{01}, u_{20}, u_{30} \geq 0$ *satisfy the zero Neumann boundary condition and belong to* $C^{2+\lambda}(\overline{\Omega})$ *for some* $0 < \lambda < 1$. *Then* (3) *possesses a unique non-negative solution* $u_1, u_2, u_3 \in C^{2+\lambda, 1+\frac{\lambda}{2}}(\overline{\Omega} \times [0, \infty))$.

Theorem 1.2. *Let* $(H_1) - (H_2)$ *hold. Assume* $\alpha_{11} = \alpha_{22} = 0, \alpha_{33} > 0$. *If initial data* $u_{01}, u_{20}, u_{30} \geq 0$ *satisfy the zero Neumann boundary condition and belong to* $C^{2+\lambda}(\overline{\Omega})$ *for some* $0 < \lambda < 1$. *Then* (3) *possesses a unique non-negative solution* $u_1, u_2, u_3 \in C^{2+\lambda, 1+\frac{\lambda}{2}}(\overline{\Omega} \times [0, \infty))$.

(b) In case $\eta = 0$. In this case, we assume that $f(u_1, u_3) \equiv g(u_1, u_3)u_3$ satisfy:

(H_3) : For all $u_1, u_3 \geq 0, 0 \leq f(u_1, u_3) \leq Ch(u_1)$ for some positive constant C and continuous function $h(u_1)$.

Theorem 1.3. *Let* $(H_1) - (H_3)$ *hold. Assume* $\alpha_{11}, \alpha_{22} \geq 0, \alpha_{33} > 0$. *Assume also that initial data* $u_{01}, u_{20}, u_{30} \geq 0$ *satisfy the zero Neumann boundary condition and belong to* $C^{2+\lambda}(\overline{\Omega})$ *for some* $0 < \lambda < 1$. *Then* (3) *possesses a unique non-negative solution* $u_1, u_2, u_3 \in C^{2+\lambda, 1+\frac{\lambda}{2}}(\overline{\Omega} \times [0, \infty))$.

Remark 1.4. *Theorem1.1-Theorem1.3 also hold for* (3) *but with homogeneous Dirichlet boundary condition.*

Remark 1.5. *Although we have stated the existence of global solutions(Theorem1.1 and Theorem1.2), we do not have enough information about the uniform boundedness of solutions and their asymptotic behaviors as* $t \to \infty$. *In order to study the asymptotic behavior of* u_1, u_2, u_3 *as* $t \to \infty$, *we have to establish the uniform boundedness of global solutions. However, we have to leave an open question here that our above results whether can establish the uniform boundedness of global solutions? This may be more challenging from mathematical point of view.*

Remark 1.6. *In the proof of Theorem1.1 -Theorem1.3, the positivity of self-diffusion coefficient* α_{33} *has played an important role. However, in case of* $\alpha_{31}, \alpha_{32} > 0$ *and* $\alpha_{33} = 0$ *in* (3), *it is difficult to show the existence of global solutions. Unfortunately, we have to leave an open question here.*

Remark 1.7. *We believe that the condition* $n < 10$ *of Theorem1.1 and the condition* (H_3) *of Theorem1.3 are just the technical conditions. To drop these conditions, more new ideas and techniques must be developed.*

2. Local existence and A priori estimate

2.1. Local existence

For the time-dependent solutions of (3), the local existence of non-negative solutions is established by Amann in the seminal papers [24, 25]. The results can be summarized as follows:

Theorem 2.1. *Suppose that* u_{10}, u_{20}, u_{30} *are in* $W^1_p(\Omega)$ *for some* $p > n$. *Then* (3) *has a unique non-negative smooth solution* $u_1, u_2, u_3 \in [C([0, T), W^1_p(\Omega)) \bigcap C^\infty((0, T), C^\infty(\Omega))]$ *with maximal existence time* T. *Moreover, if the solution* (u, v) *satisfies the estimate*

$$\sup\{\|u_i(\cdot, t)\|_{W^1_p(\Omega)} : t \in (0, T)\} < \infty, i = 1, 2, 3.$$

then $T = \infty$.

We denote

$$
\begin{aligned}
&Q_T = \Omega \times [0, T),\\
&\|u\|_{L^{p,q}(Q_T)} = \Big(\int_0^T (\int_\Omega |u(x,t)|^p dx)^{\frac{q}{p}} dt \Big)^{1/q}, L^p(Q_T) := L^{p,p}(Q_T),\\
&\|u\|_{W_p^{2,1}(Q_T)} := \|u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|\nabla^2 u\|_{L^p(Q_T)},\\
&\|u\|_{V_2(Q_T)} := \sup_{0\le t\le T} \|u(.,t)\|_{L^2(\Omega)} + \|\nabla u(x,t)\|_{L^2(Q_T)},
\end{aligned}
$$

T be the maximal existence time for the solution (u_1, u_2, u_3) of (3). In order to the proof of Theorem 1.1-Theorem 1.3 , we need the following Lemmas.

2.2. *A priori estimate*

Lemma 2.2. **(i)** *Let (u_1, u_2, u_3) be a nonnegative solution of* (3) *in $[0, T)$. Then there exists positive M_0 such that*

$$0 < u_1, u_2 < M_0, \quad and \quad u_3 > 0 \quad in \quad Q_T. \tag{4}$$

(ii) *For any $T > 0$. Then there exist positive $C(T)$ such that*

$$\sup_{0\le t\le T} \|u_3(.,t)\|_{L^1(\Omega)} < C_1. \tag{5}$$

Proof. (**i**) Applying the maximum principle to (3), it is not hard to verify that $u_i > 0, i = 1, 2, 3$. Now we prove that $u_i \le M_0, i = 1, 2$. To this end, we consider the auxiliary problem

$$
\begin{cases}
u_{1t} - \Delta[(d_1 + \alpha_{11} u_1)u_1] = f_1(\mathbf{u}), \quad x \in \Omega, t > 0,\\
u_{2t} - \Delta[(d_2 + \alpha_{22} u_2)u_2] = f_2(\mathbf{u}), \quad x \in \Omega, t > 0,\\
\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0,\\
u_i(x, 0) = u_{i0}(x) \ge 0, i = 1, 2, \quad x \in \Omega,
\end{cases} \tag{6}
$$

where $f_1(\mathbf{u}) = au_2 - bu_1 - \gamma u_1^2 - g(u_1, u_3)u_3$, $f_2(\mathbf{u}) = u_1 - u_2$. Notice that the functions f_1 and f_2 are sufficiently smooth in R^2 and quasimonotone in R_+^2. Let $(0, 0)$ and (N_1, N_2) are a pair of upper-lower solutions for (6), where $N_i, i = 1, 2$ are positive constants. Then we have

$$
\begin{aligned}
&aN_2 - bN_1 - \gamma N_1^2 \le 0,\\
&N_1 - N_2 \le 0,\\
&u_{10} \le N_1, \quad u_{20} \le N_2,
\end{aligned} \tag{7}
$$

yields

$$
\begin{aligned}
N_1 &= \max\{\tfrac{|a-b|}{\gamma}, \|u_{10}|_{L^\infty(\Omega)}\},\\
N_2 &= \max\{N_1, \|u_{20}|_{L^\infty(\Omega)}\}.
\end{aligned}
$$

It follows that there exists $M_0 = K \max\{N_1, N_2\}$, for any $t > 0$ such that $u_1, u_2 < M_0$, where K is a sufficiently large positive constant.

(**ii**) Integrating the third equation of (3) over the domain Ω and by the assumption (H_2), (4) and Hölder inequality, we have

$$\frac{d}{dt} \int_\Omega u_3 dx \le \int_\Omega u_3(|-r| + \delta g(u_1, 0) - \eta u_3) dx \le \rho \int_\Omega u_3 dx - \frac{\eta}{|\Omega|} \Big(\int_\Omega u_3 dx \Big)^2, \tag{8}$$

where $\rho = r + \delta M, M = \max g(u_1, 0)$.

We note that

$$\int_\Omega u_3 dx \leq \begin{cases} \int_\Omega u_{30} dx, & \text{if } \rho = 0, \eta = 0; \\ \frac{\int_\Omega u_{30} dx}{1+\eta|\Omega|^{-1}(\int_\Omega u_{30} dx)t}, & \text{if } \rho = 0, \eta \neq 0; \\ e^{\rho t}(\int_\Omega u_{30} dx), & \text{if } \rho \neq 0, \eta = 0. \end{cases}$$

Now, we assume that $\rho \neq 0$ and $\eta \neq 0$. From (8), we have

$$\int_\Omega u_3 dx \leq \frac{e^{\rho t} Y}{1 + YZ(e^{\rho t} - 1)} \equiv L(t), \quad t \geq 0,$$

where $Y = \int_\Omega u_{30} dx, Z = \eta|\Omega|^{-1}\rho^{-1}$. Then

$$\frac{dL(t)}{dt} = \frac{\rho(1 - YZe^{\rho t} Y)}{((1 - YZ) + YZe^{\rho t})^2}.$$

Thus, when $YZ \geq 1$, we have

$$L(t) \leq L(0) = Y.$$

Then, we have

$$\int_\Omega u_3 dx \leq \int_\Omega u_{30} dx, \quad t \geq 0, \quad \text{if} \quad \int_\Omega u_{30} dx \geq \frac{\rho}{\eta}|\Omega|.$$

On the other hand, when $YZ < 1$, we have

$$L(t) = \frac{e^{\rho t} Y}{(1 - YZ) + YZe^{\rho t}} < \frac{e^{\rho t} Y}{e^{\rho t} YZ} = Z^{-1}.$$

Then, we have

$$\int_\Omega u_3 dx \leq Z^{-1} = \frac{\rho}{\eta}|\Omega|, \quad t \geq 0, \quad \text{if} \quad \int_\Omega u_{30} dx < \frac{\rho}{\eta}|\Omega|.$$

Hence, when $\rho \neq 0$ and $\eta \neq 0$, we have

$$\int_\Omega u_3 dx \leq \max\left\{\int_\Omega u_{30} dx, \frac{\rho}{\eta}|\Omega|\right\}, \quad t \geq 0.$$

Combing above results, it follows that

$$\sup_{0 \leq t \leq T} ||u_3(., t)||_{L^1(\Omega)} < C_1(T).$$

□

We shall establish L^p-estimates and $V_2(Q_T)-$ estimates for u_3.

Lemma 2.3. **(i)** *When $\eta > 0$, then there exists a constant $C_2(T)$, such that*

$$||\nabla u_1||_{L^4(Q_T)} \leq C_2(T).$$

(ii) *When $\eta = 0$, and we assume that $(\mathbf{H_3})$ hold. Then there exists a constant $C_3(T)$, such that*

$$||\nabla u_1||_{L^p(Q_T)} \leq C_3(T) \quad \text{for any } p > 1.$$

(iii) *When $\eta = 0$, and we assume that $(\mathbf{H_3})$ hold. There exists a constant $C_4(T)$, such that*

$$||\nabla u_2||_{L^p(Q_T)} \leq C_4(T) \quad \text{for any } p > 1.$$

Proof. (**i**) Let $w_1 = (d_1 + \alpha_{11}u_1)u_1$. In case $\eta > 0$. First of all, integrating the inequality (8) from 0 to t, $t \in [0, T]$, we have

$$\|u_3\|_{L^2(Q_T)} \leq C_5(T), \tag{9}$$

where $C_5(T) > 0$ is a constant which depends only on T, the initial data u_{10}, u_{30} and the coefficients of (3). On the other hand, multiplying the first equation of (3) by u_1 and integrating the result over Q_T and using the Gronwall inequality, we have

$$\sup_{0\leq t\leq T}\int_\Omega u_1^2dx + d_1\int_{Q_T}|\nabla u_1|^2dxds + 2\alpha_{11}\int_{Q_T}u_1|\nabla u_1|^2dxds \leq C_6(T),$$

which implies that

$$\|u_1\|_{V^2(Q_T)} < C_6(T), \tag{10}$$

with a constant $C_6(T) > 0$ depending on T, the initial data u_{10} and the coefficients of (3). Next, we note that w_1 satisfies the equation

$$w_{1t} = (d_1 + 2\alpha_{11}u_1)\Delta w_1 + h_1 + h_2u_3, \tag{11}$$

where $h_1 = (d_1 + 2\alpha_{11}u_1)(au_2 - bu_1 - \gamma u_1^2)$, $h_2 = -(d_1 + 2\alpha_{11}u_1)g(u_1, u_3)u_3$. From Lemma 2.2 and ($H_2$), we know that h_1 and h_2 are bounded. Then multiplying the equation of (11) by $-\Delta w_1$, integrating the resulting expression over Q_T and using (9), (10) and Young's inequality, we have

$$\|\Delta w_1\|_{L^2(Q_T)} \leq C_7(T).$$

From this and the elliptic regularity estimates, we get $(w_1)_{x_ix_j} \in L^2(Q_T)$ for all $i, j = 1, 2, \ldots, n$. From this, (9) and (11), we have $\|w_1\|_{W_2^{2,1}(Q_T)} \leq C_2(T)$.

Moreover, it is easy to see that w_1 satisfies

$$w_{1t} \leq \sqrt{d_1^2 + 4\alpha_{11}d_1w_1}\Delta w_1 + (d_1 + 2\alpha_{11}u_1)au_2.$$

Applying [20, Proposition 2.1] to the above equation with $p = 2$ and deduce that

$$\|\nabla u_1\|_{L^4(Q_T)} \leq C_2(T).$$

(**ii**) In case $\eta = 0$. The equation of u_1 can be written in the divergence form as

$$u_{1t} = \nabla \cdot [(d_1 + 2\alpha_{11}u_1)\nabla u_1] + au_2 - bu_1 - \gamma u_1^2 - g(u_1, u_3)u_3. \tag{12}$$

Since $d_1 + 2\alpha_{11}u_1$ and $au_2 - bu_1 - \gamma u_1^2 - g(u_1, u_3)u_3$ are bounded on Q_T by the assumption ($\mathbf{H_3}$) and Lemma 2.2, by applying the Hölder continuity result to (12), we have

$$u_1 \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T), \alpha > 0. \tag{13}$$

In (11), $d_1 + 2\alpha_{11}u_1 \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$ by (13), $(d_1 + 2\alpha_{11}u_1)(au_2 - bu_1 - \gamma u_1^2 - g(u_1, u_3)u_3) \in L^\infty(Q_T)$ by lemma 2.2 and the assumption ($\mathbf{H_3}$). The parabolic regularity theorem can be applied to (11) so that

$$\|w_1\|_{W_p^{2,1}(Q_T)} \leq C_3(T) \quad \text{for any } p > 1.$$

This implies

$$\nabla u_1 = \frac{1}{d_1 + 2\alpha_{11}u_1}\nabla w_1 \in L^p(Q_T) \quad \text{for any } p > 1.$$

(**iii**) Using the similar arguments as in the preceding of lemma 2.3 (ii), it can be also obtains the desired result. □

Lemma 2.4. **(i)** *Let $\alpha_{11} = \alpha_{22} = 0, \alpha_{33} > 0$. Then, for each $q > 1$, there is a constant $C(q, T)$ such that for every $T_1 \in (0, T]$*

$$\sup \|u_3\|^q_{L^q(\Omega)} + \|\nabla u_3^{\frac{q+1}{2}}\|^2_{L^2(Q_{T_1})} \leq C(1 + \|u_3\|^{q+1}_{L^{q+1}(Q_{T_1})}). \tag{14}$$

(ii) *Let $p > 2$ and $\alpha_{ii} > 0, i = 1, 2, 3$. Assume that there is a positive constant $M_1 < \infty$ such that*

$$\|\nabla u_i\|_{L^p(Q_T)} \leq M_1 (i = 1, 2).$$

Then, for each $q > 1$, there exists positive constant $C(q, T, M_1)$ such that for every $T_1 \in (0, T]$

$$\begin{aligned} &\|u_3(., t)\|^q_{L^q(\Omega)} + \frac{4(q-1)d_3}{q}\|\nabla(u_3^{\frac{q}{2}})\|^2_{L^2(Q_{T_1})} + \|\nabla(u_3^{\frac{q+1}{2}})\|^2_{L^2(Q_{T_1})} \\ &\leq C\left(1 + \|u_3\|^{q-1}_{L^{\frac{p(q-1)}{p-2}}(Q_{T_1})}\right). \end{aligned} \tag{15}$$

Proof. For any constant $q > 1$, multiplying the third equation of (3) by qu_3^{q-1} and using the integration by parts, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_3^q dx &= -q(q-1)\int_\Omega u_3^{q-2}(d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + 2\alpha_{33}u_3)|\nabla u_3|^2 dx \\ &\quad -\alpha_{32}(q-1)\int_\Omega \nabla(u_3^q)\cdot\nabla u_2 dx - \alpha_{31}(q-1)\int_\Omega \nabla(u_3^q)\cdot\nabla u_1 dx \\ &\quad +q\int_\Omega u_3^q(-r + \delta g(u_1, u_3) - \eta u_3)dx \\ &\leq -q(q-1)d_3\int_\Omega u_3^{q-2}|\nabla u_3|^2 dx - 2\alpha_{33}q(q-1)\int_\Omega u_3^{q-1}|\nabla u_3|^2 dx \\ &\quad -\alpha_{32}(q-1)\int_\Omega \nabla(u_3^q)\cdot\nabla u_2 dx - \alpha_{31}(q-1)\int_\Omega \nabla(u_3^q)\cdot\nabla u_1 dx \\ &\quad +q\int_\Omega u_3^q(-r + \delta g(u_1, u_3) - \eta u_3)dx \\ &= -\frac{4(q-1)d_3}{q}\int_\Omega |\nabla(u_3^{\frac{q}{2}})|^2 dx - \frac{8\alpha_{33}q(q-1)}{(q+1)^2}\int_\Omega |\nabla(u_3^{\frac{q+1}{2}})|^2 dx \\ &\quad -\alpha_{32}(q-1)\int_\Omega \nabla(u_3^q)\cdot\nabla u_2 dx - \alpha_{31}(q-1)\int_\Omega \nabla(u_3^q)\cdot\nabla u_1 dx \\ &\quad +q\int_\Omega u_3^q(-r + \delta g(u_1, u_3) - \eta u_3)dx. \end{aligned}$$

Integrating the above inequality from 0 to t, we have

$$\begin{aligned} &\int_\Omega u_3^q(x, t)dx + \frac{4(q-1)d_3}{q}\int_{Q_t} |\nabla(u_3^{\frac{q}{2}})|^2 dxds + \frac{8\alpha_{33}q(q-1)}{(q+1)^2}\int_{Q_t} |\nabla(u_3^{\frac{q+1}{2}})|^2 dxds \\ &\leq \int_\Omega u_3^q(x, 0)dx - \alpha_{31}(q-1)\int_{Q_t} \nabla(u_3^q)\cdot\nabla u_1 dxds - \alpha_{32}(q-1)\int_{Q_t} \nabla(u_3^q)\cdot\nabla u_2 dxds \\ &+ q\int_{Q_t} u_3^q(-r + \delta g(u_1, u_3) - \eta u_3)dxds. \end{aligned} \tag{16}$$

Now, we will divide the proof of Lemma 2.4 into two cases according to the different values of α_{11} and α_{22}.

Case (i). $\alpha_{11} = \alpha_{22} = 0$.

When $\eta > 0$, the last term in (16) may be estimated by

$$\begin{aligned} & q \int_{Q_t} u_3^q(-r + \delta g(u_1, u_3) - \eta u_3)\, dx\, dt \\ & \le -\eta q \|u_3\|^{q+1}_{L^{q+1}(Q_t)} + \delta q M \|u_3\|^q_{L^q(Q_t)} \\ & \le -\eta q \|u_3\|^{q+1}_{L^{q+1}(Q_t)} + \delta q M |Q_T|^{\frac{1}{q+1}} \|u_3\|^q_{L^{q+1}(Q_t)} \\ & \le -\eta q \|u_3\|^{q+1}_{L^{q+1}(Q_t)} + \delta q M \Big[\varepsilon \|u_3\|^{q+1}_{L^{q+1}(Q_t)} + \varepsilon^{-q} |Q_T|^{\frac{q}{q+1}} \Big] \\ & \le C_8. \end{aligned} \tag{17}$$

Note that when $\eta = 0$, this becomes

$$q \int_{Q_t} u_3^q(-r + \delta g(u_1, u_3))\, dx\, dt \le C_9(1 + \|u_3\|^{q+1}_{L^{q+1}(Q_t)}), \forall t \in [0, T). \tag{18}$$

We also note

$$\Big| - \int_{Q_t} \nabla(u_3^q) \cdot \nabla u_1 \; dx\, dt \Big| = \Big| \int_{Q_t} u_3^q \Delta u_1 \; dx\, dt \Big| \le \|u_3\|^q_{L^{q+1}(Q_T)} \cdot \|\Delta u_1\|_{L^{q+1}(Q_T)}.$$

We will make use of the maximal regularity theory for parabolic equations(see, e.g., [26]) to estimate $\|\Delta u_1\|_{L^{q+1}(Q_T)}$. It follows from the first equation in (3) that

$$\begin{aligned} & \|\Delta u_1\|_{L^{q+1}(Q_T)} + \|u_{1t}\|_{L^{q+1}(Q_T)} \\ & \le C_{10}\Big(\|a u_2 - b u_1 - \gamma u_1^2 - g(u_1, u_3) u_3\|_{L^{q+1}(Q_T)} + \|u_{10}\|_{W^2_{q+1}(\Omega)} \Big) \\ & \le C_{11}\Big(1 + \|u_3\|^{q+1}_{L^{q+1}(Q_T)}\Big), \end{aligned}$$

with some positive numbers C_{10} and C_{11}. Here we have used the assumption (H_2) and lemma 2.2(**i**). Note that when $\eta = 0$ and the assumption (H_3), this becomes

$$\|\Delta u_1\|_{L^{q+1}(Q_T)} + \|u_{1t}\|_{L^{q+1}(Q_T)} \le C_{12}.$$

Combining above inequalities, we see that

$$\Big| - \int_{Q_t} \nabla(u_3^q) \cdot \nabla u_1 \; dx\, dt \Big| \le C_{13}\Big(1 + \|u_3\|^{q+1}_{L^{q+1}(Q_T)}\Big) \tag{19}$$

with a positive constant C_{13}. In a similar fashion, we get

$$\Big| - \int_{Q_t} \nabla(u_3^q) \cdot \nabla u_2 \; dx\, dt \Big| \le C_{14}\Big(1 + \|u_3\|^{q+1}_{L^{q+1}(Q_T)}\Big), \tag{20}$$

where C_{14} is a positive constant.

Substituting (17)-(20) into (16) enables us to derive (14).

Case (**ii**). $\alpha_{ii} > 0, i = 1, 2, 3$. In this case, (16) and (17) are also valid, but it is difficult to estimate $\int_{Q_t} \nabla(u_3^q) \cdot \nabla u_i \; dx\, dt, i = 1, 2$ and $q \int_{Q_t} u_3^q(-r + \delta g(u_1, u_3)) dxdt$.

Since that $\frac{1}{p}+\frac{1}{2}+\frac{p-2}{2p}=1$ and $\nabla u_i, i=1,2$ is in $L^p(Q_T)$, by the Hölder's inequality, we have

$$\begin{aligned}\left|-\int_{Q_t}\nabla(u_3^q)\cdot\nabla u_1dxds\right| &= \frac{2q}{q+1}\left|\int_{Q_t}u_3^{\frac{q-1}{2}}\cdot\nabla(u_3^{\frac{(q+1)}{2}})\cdot\nabla u_1dxds\right| \\ &\le \frac{2q}{q+1}\|u_3^{\frac{q-1}{2}}\|_{L^{\frac{2p}{p-2}}(Q_t)}\cdot\|\nabla(u_3^{\frac{q+1}{2}})\|_{L^2(Q_t)}\cdot\|\nabla u_1\|_{L^p(Q_t)} \\ &\le \frac{2q}{q+1}\|u_3\|^{\frac{q-1}{2}}_{L^{\frac{p(q-1)}{p-2}}(Q_t)}\cdot\|\nabla(u_3^{\frac{q+1}{2}})\|_{L^2(Q_t)}\cdot\|\nabla u_1\|_{L^p(Q_t)} \\ &\le \frac{2q}{q+1}M_1\|u_3\|^{\frac{q-1}{2}}_{L^{\frac{p(q-1)}{p-2}}(Q_t)}\cdot\|\nabla(u_3^{\frac{q+1}{2}})\|_{L^2(Q_t)}. \end{aligned} \tag{21}$$

Similarly,

$$\left|-\int_{Q_t}\nabla(u_3^q)\cdot\nabla u_2dxds\right| \le \frac{2q}{q+1}M_1\|u_3\|^{\frac{q-1}{2}}_{L^{\frac{p(q-1)}{p-2}}(Q_t)}\cdot\|\nabla(u_3^{\frac{q+1}{2}})\|_{L^2(Q_t)}. \tag{22}$$

On the other hand, using Hölder's inequality and Poincaré inequality, we can easily arrive at the following estimate

$$\begin{aligned} q\int_{Q_t}u_3^q(-r+\delta g(u_1,u_3))dxds &\le q\int_{Q_t}(\delta g(u_1,0))u_3^qdxds \\ &\le \int_{Q_t}(q\delta M)u_3^qdxds \\ &= \int_{Q_t}(q\delta M)\cdot u_3^{\frac{q-1}{2}}\cdot u_3^{\frac{q+1}{2}}dxds \\ &\le \|u_3^{\frac{q-1}{2}}\|_{L^{\frac{2p}{p-2}}(Q_t)}\cdot\|u_3^{\frac{q+1}{2}}\|_{L^2(Q_t)}\cdot\|q\delta M\|_{L^p(Q_t)} \\ &\le C_{15}\|u_3\|^{(q-1)/2}_{L^{\frac{p(q-1)}{p-2}}(Q_t)}\cdot\|\nabla(u_3^{\frac{q+1}{2}})\|_{L^2(Q_t)}. \end{aligned} \tag{23}$$

Therefore, from (17), (21), (22), (23) and (16), it follows that

$$\begin{aligned} &\int_\Omega u_3^q(x,t)dx+\frac{4(q-1)d_3}{q}\int_{Q_t}|\nabla(u_3^{\frac{q}{2}})|^2dxdt+\frac{8\alpha_{33}q(q-1)}{(q+1)^2}\int_{Q_t}|\nabla(u_3^{\frac{q+1}{2}})|^2dxdt \\ \le\ & C_{16}+C_{17}\|u_3\|^{\frac{q-1}{2}}_{L^{\frac{p(q-1)}{p-2}}(Q_t)}\cdot\|\nabla(u_3^{\frac{q+1}{2}})\|_{L^2(Q_t)} \\ \le\ & C_{16}+\frac{C_{17}}{4\varepsilon}\|u_3\|^{q-1}_{L^{\frac{p(q-1)}{p-2}}(Q_t)}+C_{17}\varepsilon\|\nabla(u_3^{\frac{q+1}{2}})\|^2_{L^2(Q_t)}. \end{aligned}$$

For any $\varepsilon>0$, from above expression and by choosing a sufficiently small ε, such that $C_{17}\varepsilon<\frac{8\alpha_{33}q(q-1)}{(q+1)^2}$, we get (15). This completes the proof of the lemma. □

Combining lemma 2.3 and lemma 2.4 of [19], we can prove the following Lemma.

Lemma 2.5. *Let $q>1, \widetilde{q}=2+\frac{4q}{N(q+1)}$, $\widetilde{\beta}$ in $(0,1)$ and let $C_T>0$ be any number which may depend on T. Then there is a constant M' depending on $q, n, \Omega, \widetilde{\beta}$ and C_T such that for any g in $C([0,T), W_2^1(\Omega))$ with $(\int_\Omega|g(.,t)|^{\widetilde{\beta}}dx)^{\frac{1}{\widetilde{\beta}}}\le C_T$ for all $t\in[0,T]$, we have the following inequality*

$$\|g\|_{L^{\widetilde{q}}(Q_T)}\le M'\left\{1+\left(\sup_{0\le t\le T}\|g(.,t)\|_{L^{2q/q+1}(\Omega)}\right)^{4q/N(q+1)\widetilde{q}}\|\nabla g\|^{2/\widetilde{q}}_{L^2(Q_T)}\right\}.$$

The proof of the above lemma can be found in [19, Lemmas 2.3, 2.4].

Now, we establish L^p-estimates of u_3 for any $p > 1$. For any number a, we denote $a_+ = \max\{a, 0\}$.

Lemma 2.6. *Let $\eta > 0$ and $\alpha_{33} > 0$.*

(i) *When $\alpha_{11}, \alpha_{22} > 0$, then there is a constant $C_{18} > 0$ such that*

$$\|u_3\|_{V_2(Q_T)} \le C_{18}.$$

Moreover, for any constant $p < \frac{4(N+1)}{(N-2)_+}$, there exists a positive constant C_{19} such that

$$\|u_3\|_{L^p(Q_T)} \le C_{19}.$$

(ii) *When $\alpha_{11} = \alpha_{22} = 0$, then there exist positive constants C_{20} and C_{21}, such that*

$$\|u_3\|_{L^p(Q_T)} \le C_{20} \quad \text{for any} \quad p > 1,$$

and

$$\|u_3\|_{V_2(Q_T)} \le C_{21}.$$

Proof. (**i**) Set $v = u_3^{\frac{q+1}{2}}$, and

$$\begin{aligned} E &\equiv \sup_{0\le t\le T} \int_\Omega u_3^q(x,t)dx + \int_{Q_T} |\nabla(u_3^{(q+1)/2})|^2 dxds \\ &= \sup_{0\le t\le T} \int_\Omega v^{2q/q+1} dx + \int_{Q_T} |\nabla v|^2 dxds. \end{aligned}$$

Let $p_0 = 4, \overline{p} = \frac{2p_0}{p_0 - 2}$. It follows from lemma 2.3 **(i)**, **(ii)** and lemma 2.4 **(ii)** that

$$E + \frac{4(q-1)d_3}{q}\|\nabla(u_3^{\frac{q}{2}})\|^2_{L^2(Q_T)} \le C_{22}\left(1 + \|v\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{\overline{p}(q-1)}{q+1}}(Q_T)}\right). \tag{24}$$

For any $q > 1$, if

$$(N\overline{p} - 2N - 4)q \le 2N + N\overline{p}, \tag{25}$$

then, $\frac{\overline{p}(q-1)}{q+1} \le \widetilde{q} = 2 + \frac{4q}{N(q+1)}$. By Hölder's inequality, we have

$$\|v\|_{L^{\frac{\overline{p}(q-1)}{q+1}}(Q_T)} \le C_{23}\|v\|_{L^{\widetilde{q}}(Q_T)}, \tag{26}$$

where $C_{23} = |Q_T|^{\frac{q+1}{\overline{p}(q-1)} - \frac{1}{\widetilde{q}}}$. Setting $\widetilde{\beta} = 2/(q+1) \in (0,1)$, by (5) we get

$$\|v(.,t)\|_{L^{\widetilde{\beta}}(\Omega)} = \|u_3(.,t)\|^{\frac{1}{\widetilde{\beta}}}_{L^1(\Omega)} \le (C_1)^{\frac{1}{\widetilde{\beta}}}, \forall t \in [0,T). \tag{27}$$

Therefore, by (27), Lemma 2.5 and the definition of E, from (26) we have

$$\|v\|_{L^{\overline{p}(q-1)/q+1}(Q_T)} \le C_{23}\|v\|_{L^{\widetilde{q}}(Q_T)} \le C_{23}M_1\left\{1 + E^{2/n\widetilde{q}}E^{\frac{1}{\widetilde{q}}}\right\}. \tag{28}$$

From (24) and (28), we get

$$E \le C_{24}(1 + E^\mu E^\nu) \tag{29}$$

with

$$\mu = \frac{4(q-1)}{n\widetilde{q}(q+1)}, \quad \nu = \frac{2(q-1)}{\widetilde{q}(q+1)}.$$

As

$$\mu + \nu = \frac{2(q-1)}{\widetilde{q}(q+1)}\left[\frac{2}{N}+1\right] < \frac{1}{\widetilde{q}}\left[\frac{4q}{N(q+2)}+2\right] = 1.$$

Hence, there exists a positive constant C_{25} such that $E \le C_{25}$. By (28) and (29) we get $v \in L^{\widetilde{q}}(Q_T)$, this implies that $u_3 \in L^p(Q_T)$ with $p = \frac{\widetilde{q}(q+1)}{2}$ for any q satisfying (25). Looking at (25), when $N \le 2$,

$$N\overline{p} - 2N - 4 = 2(N-2) \le 0,$$

then (25) holds for all q. Hence, for $N \le 2$, $u_3 \in L^p(Q_T)$ for all $p > 1$. when $n > 2$, then (25) is equivalent to

$$1 < q < q_0 := \frac{2N + N\overline{p}}{(N\overline{p} - 2N - 4)} = \frac{3N}{N-2}.$$

By

$$\frac{\widetilde{q}(q+1)}{2} = q + 1 + \frac{2q}{N} \le \overline{p}_1 := q_0 + 1 + \frac{2q_0}{n} = \frac{4(N+1)}{N-2}.$$

We have that u_3 is in $L^p(Q_T)$ for all$1 < p \le \overline{p}_1$. Namely, there exist positive constant C_{19} such that $\|u_3\|_{L^p(Q_T)} \le C_{19}$ for $p < \frac{4(N+1)}{(N-2)_+}$. Since (25) holds true for $q = 2$. Hence E is finite for $q = 2$. Therefore, $u_3 \in V_2(Q_T)$ for any n by (24).

(**ii**) From the definition of E and (14), we have

$$E \equiv \sup_{0\le t\le T} \|v(t)\|^{\frac{2q}{q+1}}_{L^{\frac{2q}{q+1}}(\Omega)} + \|\nabla v\|^2_{L^2(Q_T)} \le C_{26}\Big(1 + \|v\|^2_{L^2(Q_T)}\Big). \tag{30}$$

By (9), this implies $u_3 \in L^2(Q_T)$, so $\|v\|_{L^{\frac{4}{q+1}}(Q_T)} \le C_{27}$. Since $\frac{4}{q+1} < 2 \le \widetilde{q}$. Then we see from Hölder's inequality

$$\|v\|^2_{L^2(Q_T)} \le \|v\|^{2(1-\lambda)}_{L^{\widetilde{q}}(Q_T)} \|v\|^{2\lambda}_{L^{\frac{4}{q+1}}(Q_T)} \le C_{27}^{2\lambda}\|v\|^{2(1-\lambda)}_{L^{\widetilde{q}}(Q_T)}, \tag{31}$$

where $\lambda = (\frac{1}{2} - \frac{1}{\widetilde{q}})/(\frac{q+1}{4} - \frac{1}{\widetilde{q}})$. Setting $\widetilde{\beta} = 2/(q+1) \in (0,1)$, we have $\|v(.,t)\|_{L^{\widetilde{\beta}}(\Omega)} = \|u_3(.,t)\|^{\frac{1}{\widetilde{\beta}}}_{L^1(\Omega)} \le C_1(T)^{\frac{1}{\widetilde{\beta}}}$ for all $t \in [0,T)$ by Lemma 2.2. Then it follow from (30), (31) and Lemma 2.5 that

$$E \le C_{28}(1 + E^{\theta}) \tag{32}$$

with

$$\theta = \frac{2(1-\lambda)}{\widetilde{q}}\Big(\frac{2}{n}+1\Big).$$

A simple calculation show $0 < \theta < 1$. It follows from (32) that

$$\sup_{0\le t\le T} \|v(t)\|^{\frac{2q}{q+1}}_{L^{\frac{2q}{q+1}}(\Omega)} \le E \le C_{29}.$$

with some $C_{29} > 0$, let $p = q > 1$, so that $\sup_{0\le t\le T} \|u_3(t)\|_{L^p(\Omega)} \le C_{20}$.

By(17), (18), (19), (20) and (16), we have

$$\int_\Omega u_3^q(x,t)dx + \frac{4(q-1)d_3}{q}\int_{Q_t} |\nabla(u_3^{\frac{q}{2}})|^2 dxdt + \frac{8\alpha_{33}q(q-1)}{(q+1)^2}\int_{Q_t} |\nabla(u_3^{\frac{q+1}{2}})|^2 dxdt$$
$$\le C_{30}(1 + \|u_3\|^{q+1}_{L^{q+1}(\Omega)}). \tag{33}$$

Since $\widetilde{q} > 2$, by Lemma2.5 and the definition of v, we have for any $q > 1$,

$$\|u_3\|_{L^{q+1}(Q_T)} = \|v\|_{L^2(Q_T)}^{\frac{2}{q+1}} \le C_{30}\|v\|_{L^{\widetilde{q}}(Q_T)}^{\frac{2}{q+1}} \le C_{31}.$$

Letting $q = 2$ in (33) and use the above inequality, we have

$$\sup \|u_3\|_{L^2(\Omega)}^2 + \|\nabla u_3\|_{L^2(Q_{T_1})}^2 \le C_{32}$$

with $C_{32} > 0$, and the proof is complete. □

Lemma 2.7. *Let $\eta > 0$ and $\alpha_{ii} > 0, i = 1, 2, 3$, and suppose that there are $p_1 > \max\{\frac{N+2}{2}, 3\}$ and a positive constant Cp_1, T such that*

$$\|u_3\|_{L^{p_1}(Q_T)} \le Cp_1, T.$$

Then, there exists a positive M_2 such that

$$\|u_3\|_{L^p(Q_T)} \le M_2 \quad \text{for any } p > 1$$

Proof. The proof of Lemma 2.7 is similar to the proof of Lemma 3.2 in [31](or Lemma 3.7 in [32]), we omit it. □

Lemma 2.8. *Let $\eta = 0$, $\alpha_{11}, \alpha_{22} \ge 0$, $\alpha_{33} > 0$, and we assume that* $(\mathbf{H_3})$ *hold. Then there exist constant $C_{33} > 0$ and $C_{34} > 0$, such that*

$$\|u_3\|_{V_2(Q_T)} \le C_{33},$$

and

$$\|u_3\|_{L^p(Q_T)} \le C_{34} \quad \text{for any} \quad p > 1.$$

Proof. First of all, multiplying the third equation of (3) by u_3 and integrating the result on Q_t, $t \in [0, T]$, we obtain

$$\begin{aligned} &\int_\Omega u_3^2 dx + 2d_3 \int_{Q_t} |\nabla u_3|^2 dxds + 4\alpha_{33} \int_{Q_t} u_3 |\nabla u_3|^2 dxds \\ &\le -\alpha_{31} \int_{Q_t} \nabla(u_3^2) \cdot \nabla u_1 dxds - \alpha_{32} \int_{Q_t} \nabla(u_3^2) \cdot \nabla u_2 dxds + 2\delta M \int_{Q_t} u_3^2 dxds + \int_\Omega u_{30}^2 dx. \end{aligned} \tag{34}$$

Let $w_i = (d_i + \alpha_{ii} u_i) u_i, i = 1, 2$. Then by Lemma 2.3 (ii) and (iii), we have

$$\|w_i\|_{W_p^{2,1}(Q_T)} \le C_{35}, (i = 1, 2) \quad \text{for any } p > 1.$$

Thus, the Sobolev inequality yields

$$w_i \in C^{\beta+1, \frac{\beta+1}{2}}(\overline{Q}_T) \quad \text{for any } \beta \in (0, 1). \tag{35}$$

Taking into account that $u_i = \frac{-d_i + \sqrt{d_i^2 + 4w_i\alpha_{ii}}}{2\alpha_{ii}}, i = 1, 2$ (note that when $\alpha_{ii} = 0$, this becomes $u_i = \frac{w_i}{d_i}, i = 1, 2$). By (35), we have

$$u_i \in C^{\beta+1, \frac{\beta+1}{2}}(\overline{Q}_T) \quad \beta \in (0, 1). \tag{36}$$

By (36) and using the Young inequality, we have

$$\begin{aligned} \left| \int_{Q_t} u_3 \nabla u_3 \cdot \nabla u_1 dxds \right| &\le C_{36} \int_{Q_t} |u_3||\nabla u_3| dxds \\ &\le C_{36}\varepsilon \int_{Q_t} |\nabla u_3|^2 dxds + \frac{C_{36}}{4\varepsilon} \int_{Q_t} u_3^2 dxds. \end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|\int_{Q_t} u_3\nabla u_3\cdot\nabla u_2 dxds\right| &\le C_{37}\int_{Q_t}|u_3||\nabla u_3|dxds\\ &\le C_{37}\varepsilon\int_{Q_t}|\nabla u_3|^2dxds+\frac{C_{37}}{4\varepsilon}\int_{Q_t}u_3^2dxds.\end{aligned}$$

It follows from the above two inequalities and (34), we obtain

$$\sup_{0\le t\le T}\int_\Omega u_3^2dx+\int_{Q_T}|\nabla u_3|^2dxds+\int_{Q_t}|\nabla(u_3^2)|^2dxds\le C_{38}.$$

The above inequality implies that

$$u_3\in L^2(Q_T),\quad u_3\in V^2(Q_T). \tag{37}$$

Now, We will divide the proof of Lemma 2.8 into two cases according to the different values of α_{11},α_{22}.

Case (a). $\alpha_{11},\alpha_{22}>0$. From Lemma 2.3 (ii) and (iii), which implies $\|\nabla u_i\|_{L^p(Q_T)}\le M_1(i=1,2)$ hold. From Lemma 2.4 (ii), we have (15) holds for any $q>1$ and $p>2$. Hence (15) holds true for $p=q+1$. Letting $p=q+1$ in (15) and using the inequality $a^qb\le qa^{q+1}/(q+1)+b^{q+1}/(q+1)$, we have

$$\begin{aligned}&\|u_3(.,t)\|^q_{L^q(\Omega)}+\frac{4(q-1)d_3}{q}\|\nabla(u_3^{\frac{q}{2}})\|^2_{L^2(Q_{T_1})}+\|\nabla(u_3^{\frac{q+1}{2}})\|^2_{L^2(Q_{T_1})}\\ &\le C\left(1+\|u_3\|^{q+1}_{L^{q+1}(Q_{T_1})}\right).\end{aligned} \tag{38}$$

Note that (37), (38) and the definition of E. By the similar arguments in the proof of Lemma2.6 (ii), we can show $\|u_3\|_{L^p(Q_T)}\le C_{34}$ for any $p>1$.

Case (b). $\alpha_{11}=\alpha_{22}=0$. From the definition of E and (14), we have (30) hold. By (37), we find (31) holds true. The rest of the proof is the same as in the proof of Lemma2.6 (ii), so we omit it. □

3. Proof of Theorem 1.1 and Theorem 1.2

Now we begin with the proof of Theorem 1.1 and Theorem 1.2. We divide the proof into the following two steps. The first step of the proof is to show that $u_3\in L^\infty(Q_T)$. In order to show that $u_3\in L^\infty(Q_T)$, we need the the following maximum principle is a modification of [29, Theorem 7.1, p.181].

Lemma 3.1. *Suppose that $w\in V_2^{1,0}(Q_T)$ satisfies*

$$\begin{aligned}&w_t-\tfrac{\partial}{\partial x_i}(a_{ij}w_{x_j}+a_iw)+b_iw_{x_i}+aw\le f,\quad in\ Q_T,\\ &w(.,0)=w_0\quad in\ \Omega,\end{aligned} \tag{39}$$

and the boundary condition

$$\nu_ia_{ij}w_{x_i}\le 0\quad on\ \partial\Omega\times[0,T), \tag{40}$$

where $\nu=(\nu_1,\nu_2,\dots,\nu_n)$ is the outward normal vector on $\partial\Omega$. Suppose also that the coefficients a_{ij},a_i,b_i,a and f satisfy the following conditions

$$\lambda_1|\xi|^2\le a_{ij}(x,t)\xi_i\xi_j\quad for\ all\quad \xi\in\mathbb{R}^N,\quad for\ some\quad \lambda_1>0, \tag{41}$$

and

$$\left\|\sum_{i=1}^N a_i^2;\sum_{i=1}^N b_i^2;a;f\right\|_{L^{q,r}(Q_T)}\le\mu_1<\infty, \tag{42}$$

for

$$\frac{1}{r} + \frac{N}{2q} = 1 - \kappa_1, \tag{43}$$

with

$$q \in \left[\frac{N}{2(1-\kappa_1)}, \infty\right], \quad r \in \left[\frac{1}{1-k_1}, \infty\right], \quad 0 < \kappa_1 < 1, \quad \text{for } N \geq 2, \tag{44}$$

$$q \in [1, \infty], \quad r \in \left[\frac{1}{1-k_1}, \frac{2}{1-2k_1}\right], \quad 0 < \kappa_1 < \frac{1}{2}, \quad \text{for } N = 1. \tag{45}$$

Assume also that w_0 is bounded above and $a_i \nu_i \leq 0$ on $\partial\Omega \times [0, T)$. Then,

$$ess \sup_{Q_T} w$$

is finite.

Proof. [Proof of Theorem 1.1 and Theorem 1.2] For clarity, the proof will be divided into two steps.

Step 1. L^∞ estimate.

Lemma 3.2. *(L^∞ estimates for u_3) Let $\eta > 0$ and $\alpha_{33} > 0$. Suppose* (i) *$\alpha_{11} = \alpha_{22} = 0$ or* (ii) *$\alpha_{11}, \alpha_{22} > 0$ and $n < 10$. Then there exists M_2 such that*

$$\|u_3\|_{L^\infty(Q_T)} \leq M_2.$$

Proof. The third equation of (3) can be written as the linear equation

$$u_{3t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u_3}{\partial x_j}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i u_3) - a u_3 \tag{46}$$

where

$$a_{ij}(x,t) = (d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + \alpha_{33}u_3)\delta_{ij}, a_i = \alpha_{31}\frac{\partial u_1}{\partial x_i} + \alpha_{32}\frac{\partial u_2}{\partial x_i}, a = -(-r + \delta g(u_1, u_3) - \eta u_3)$$

with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

(i) Fix any $p > \frac{n+2}{2}$. Then it follows from Lemma 2.6 (ii) and [29, Theorem 9.1, p.341-342] that $\|u_i\|_{W_p^{2,1}}(Q_T)(i = 1, 2)$ is bounded. By [29, Lemma3.3, p.80], $\nabla u_1, \nabla u_2 \in L^{\frac{(n+2)p}{n+2-p}}(Q_T)$. Since $\|u_3\|_{V_2(Q_T)}$ is bounded by Lemma 2.6 (ii). By Lemma 3.1 and (46), we see that u_3 is bounded in $\overline{Q}_T$.

(ii) It follows from Lemma 2.6 and Lemma 2.7 that $\frac{N+2}{2} < \frac{4(N+1)}{N-2}$ for $N \leq 9$. By Lemma 2.6 and Lemma 2.7, we have $u_3 \in L^p(Q_T) \cap V_2(Q_T)$ for any $p > 1$.

The equations of u_1 and u_2 can be written in the divergence form as

$$u_{1t} = \nabla \cdot [(d_1 + 2\alpha_{11}u_1)\nabla u_1] + au_2 - bu_1 - \gamma u_1^2 - g(u_1, u_3)u_3, \tag{47}$$

and

$$u_{2t} = \nabla \cdot [(d_2 + 2\alpha_{22}u_2)\nabla u_2] + u_1 - u_2, \tag{48}$$

Since $d_i + 2\alpha_{ii}u_i(i = 1, 2)$ and $u_1 - u_2$ are bounded in $\overline{Q}_T$ by Lemma 2.2 and $au_2 - bu_1 - \gamma u_1^2 - g(u_1, u_3)u_3$ is in $L^p(Q_T)$ for $p > 1$. Application of the Hölder continuity result [29, Theorem 10.1]to (47) and (48), we have

$$u_1, u_2 \in C^{\beta, \frac{\beta}{2}}(\overline{Q}_T) \ \text{ with some } \ \beta > 0. \tag{49}$$

Let $w_i = (d_i + \alpha_{ii}u_i)u_i, i = 1,2$. Then w_i satisfies

$$w_{it} = (d_i + 2\alpha_{ii}u_i)\Delta w_i + f_i, i = 1,2,$$

where $f_1 = (d_1 + 2\alpha_{11}u_1)(au_2 - bu_1 - \gamma u_1^2 - g(u_1,u_3)u_3)$, $f_2 = (d_2 + 2\alpha_{22}u_2)(u_1 - u_2)$ are bounded in $\overline{Q}_T$ by Lemma 2.2 and Lemma 2.6, $(d_i + 2\alpha_{ii}u_i) \in C^{\beta,\frac{\beta}{2}}(Q_T)$ (i=1,2) by (49). By Theorem 9.1 in [29], we have

$$\|w_i\|_{W_r^{2,1}}(Q_T) < M_3, i = 1,2 \quad \text{for any} \quad r > 1.$$

By [29, Lemma3.3, p.80],

$$w_i \in C^{1+\beta^*,\frac{(1+\beta^*)}{2}}(\overline{Q}_T), \quad \forall\, 0 < \beta^* < 1.$$

And direct calculation $w_i = (d_i + \alpha_{ii}u_i)u_i, i = 1,2$ yields $u_i = \frac{-d_i + \sqrt{d_i^2 + 4w_i\alpha_{ii}}}{2\alpha_{ii}}, i = 1,2$. Therefore,

$$u_i \in C^{1+\beta^*,\frac{(1+\beta^*)}{2}}(\overline{Q}_T), \quad \forall\, 0 < \beta^* < 1. \tag{50}$$

Application of maximum principle(Lemma 3.1) to (46) yields $u_3 \in L^\infty(Q_T)$. □

Step 2. Schauder estimate.

We give the proof only in case $\alpha_{11}, \alpha_{22} > 0$ because the proof for $\alpha_{11} = \alpha_{22} = 0$ is essentially the same.

Note that the equation of u_3 can be rewritten as

$$u_{3t} = \nabla \cdot [(d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + 2\alpha_{33}u_3)\nabla u_3 + (\alpha_{31}\nabla u_1 + \alpha_{32}\nabla u_2)u_3] + f_3(x,t),$$

where $f_3(x,t) = u_3(-r + \delta g(u_1,u_3) - -\eta u_3)$, u_1, u_2, u_3, ∇u_1 and ∇u_2 are all bounded functions because of Lemma 2.2, Lemma 3.2 and (50). By Theorem 10.1 in [29], we have

$$u_3 \in C^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T) \quad \text{with some} \quad 0 < \sigma < 1. \tag{51}$$

We now turn to the equations for u_1, u_2 and rewrite its as

$$\begin{aligned} u_{1t} &= (d_1 + 2\alpha_{11}u_1)\Delta u_1 + f_1^*(x,t), \\ u_{2t} &= (d_2 + 2\alpha_{22}u_2)\Delta u_2 + f_2^*(x,t), \end{aligned} \tag{52}$$

where $f_1^*(x,t) = 2\alpha_{11}|\nabla u_1|^2 + (au_2 - bu_1 - \gamma u_1^2 - g(u_1,u_3)u_3), f_2^*(x,t) = 2\alpha_{22}|\nabla u_2|^2 + (u_1 - u_2) \in C^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T)$ by (50) and (51). Then the Schuader estimate in [29] applied to (52) yields

$$u_1, u_2 \in C^{2+\sigma^*,\frac{2+\sigma^*}{2}}(\overline{Q}_T) \quad \text{with } \sigma^* = \min\{\lambda, \sigma\}. \tag{53}$$

Let $w_3 = (d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + \alpha_{33}u_3)u_3$, which satisfies

$$w_{3t} = (d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + 2\alpha_{33}u_3)\Delta w_3 + f_3^*(x,t), \tag{54}$$

where $f_3^*(x,t) = (d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + 2\alpha_{33}u_3)u_3(-r + \delta g(u_1,u_3) - \eta u_3) + (\alpha_{31}u_{1t} + \alpha_{32}u_{2t})u_3 \in C^{\sigma^*,\frac{\sigma^*}{2}}(\overline{Q}_T)$ by (51) and (53), $d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + 2\alpha_{33}u_3 \in C^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T)$ by (50) and (51), by applying the Schuader estimate to the equation (54), we have

$$w_3 \in C^{2+\sigma^*,\frac{2+\sigma^*}{2}}(\overline{Q}_T). \tag{55}$$

Then

$$u_3 = \frac{-(d_3 + \alpha_{31}u_1 + \alpha_{32}u_2) + \sqrt{(d_3 + \alpha_{31}u_1 + \alpha_{32}u_2)^2 + 4w_3\alpha_{33}}}{2\alpha_{33}} \in C^{2+\sigma^*,\frac{2+\sigma^*}{2}}(\overline{Q}_T). \tag{56}$$

Now repeat the procedure by making use of (53) and (56) in place of (50) and (51), we have

$$u_1, u_2, u_3 \in C^{2+\lambda,\frac{2+\lambda}{2}}(\overline{Q}_T). \tag{57}$$

Finally, by Theorem 2.1 we have (u_1, u_2, u_3) exists globally in time. The proof of Theorem 1.1 and Theorem 1.2 is now complete. □

4. Proof of Theorem 1.3

Proof. [Proof of Theorem 1.3] The third equation of (3) can be written as the linear equation

$$u_{3t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u_3}{\partial x_j}\right) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i u_3) - a u_3 \tag{58}$$

where

$$a_{ij}(x,t) = (d_3 + \alpha_{31}u_1 + \alpha_{32}u_2 + \alpha_{33}u_3)\delta_{ij}, a_i = \alpha_{31}\frac{\partial u_1}{\partial x_i} + \alpha_{32}\frac{\partial u_2}{\partial x_i}, a = -(-r + \delta g(u_1, u_3))$$

with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Since $0 < u_1, u_2 \leq M_0$ by Lemma 2.2, $\nabla u_1, \nabla u_2 \in C^{\beta,\frac{\beta}{2}}(\overline{Q}_T)$ by(36), $\|u_3\|_{L^p(Q_t)}$, $\|u_3\|_{V_2(Q_t)}$ are finite by Lemma 2.8 and $u_3 g(u_1, u_3)$ is bounded by the assumption (H_3) and Lemma 2.2. By applying the maximum principle of Lemma 3.1 to the equation (58)ensures that u_3 is bounded in Q_T . The rest of the proof is same as in the case $\eta > 0$. Therefore, $u_1, u_2, u_3 \in C^{2+\lambda,1+\frac{\lambda}{2}}(\overline{\Omega} \times [0, \infty))$. This concludes the proof of Theorem 1.3. □

Acknowledgments

The authors are very grateful to the anonymous referees for their valuable comments and suggestions, which greatly improved the presentation of this work. The first author is partially supported by the Doctor Start-up Funding of North Minzu University(Grants nos.2022QHPY16), Scientific research fund of Sichuan province provincial education department(Grants nos.18ZB0319), and the second author is partially supported by the Doctor Start-up Funding of North Minzu University(Grants nos.2022QHPY16), National Natural Science Foundation of China (Grants nos.11661051), Sichuan province science and technology plan project(Grants nos.2017JY0195), Research and innovation team of Neijiang Normal University(Grants nos.17TD04).

References

[1] J. D. Murray, *Mathematical Biology*, (2nd corrected ed., Springer, Heidelberg, 1993).

[2] W. G. Aiello and H. I. Freedman, A time delay model of single-species growth with stage structure, *Math. Biosci.*, **101** (1990), 139-153.

[3] X. Zhang, L. Chen, A.U. Neumann, The stage-structured predator-prey model and optimal harvesting policy, *Math. Biosci.* **168** (2000), 201-210.

[4] K. L. Cooke, R. H. Elderkin, W. Huang, Predator-prey interactions with delays due to juvenile maturation, *SIAM J. Appl. Math.* **66** (2006), 1050-1079.

[5] J. M. Cushing, *An Introduction to Structured Population Dynamics*, (Society for Industrial and Applied Mathematics, Philadelphia, 1998).

[6] Y. H. Du, P. Y. H. Pang, M. X. Wang, Qualitative analysis of a prey-predator model with stage structure for the predator, *SIAM J. Appl. Math.* **69(2)** (2008), 596-620.

[7] F. D. Chen, Permanence of periodic Holling type predator-prey system with stage structure for prey, *Appl. Math. Comput.* **182** (2006), 1849-1860.

[8] F. D. Chen, M. S. Youa, Permanence, extinction and periodic solution of the predator-prey system with Beddington-DeAngelis functional response and stage structure for prey, *Nonlinear Analysis: RWA*. **9** (2008), 207-221.

[9] W. S. Yang, X. P. Li, Z. J. Bai, Permanence of periodic Holling type-IV predator-prey system with stage structure for prey, *Mathematical and Computer Modelling*. **48** (2008,) 677-684.

[10] Z. K. She, H. Y. Li, Dynamics of a density-dependent stage-structured predator-prey system with Beddington-DeAngelis functional response, *J. Math. Anal. Appl.* **406** (2013) 188-202.

[11] S. Q. Liu, E. Beretta, A stage-structured predator-prey model of Beddington-Deangelis type, *SIAM J. Appl. Math.* **66 (4)** (2006) 1101-1129.

[12] M. X. Wang, Stability and Hopf bifurcation for a prey-predator model with prey-stage structure and diffusion, *Mathematical Biosciences*. **212** (2008), 149-160.

[13] P. Georgescu, Y. H. Hsieh, Global dynamics of predator-prey model with stage structure for the predator, *SIAM J. Appl. Math.* **67 (5)** (2006), 1379-1395.

[14] Z. G. Wang, J. H. Wu, Qualitative analysis for a ratio-dependent predator-prey model with stage structure and diffusion, *Nonlinear Analysis: RWA*. **9** (2008), 2270-2287.

[15] Y. Du, H. Ishii, W. Y. Lin, *Recent Progress in Nonlinear Partial Differential Equations and Viscosity Solutions*, (World Scientific Publishing, 2009).
[16] G. Galiano, A. Jüngle, M. Garzón, Semi-discretization in time and numerical convergence of solutions of a nonlinear cross-diffusion population model, *Numer. Math*. **93** (2003), 655-673.
[17] N. Shigesada, K. Kawasaki, E. Teramoto, Spatial segregation of interacting species, *J. Theor. Biology*. **79** (1979), 83-99.
[18] Y. S. Choi, R. Lui, Y. Yamada, Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion, *Discrete Continuous Dyn. Syst*. **9** (2003), 1193-1200.
[19] Y. S. Choi, R. Lui, Y. Yamada, Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion, *Discrete Continuous Dyn. Syst*. **10** (2004), 719-730.
[20] P. V. Tuoc, On global existence of solutions to a cross-diffusion system, *J. Math. Anal. Appl*. **343** (2008), 826-834.
[21] S. H. Xu, Existence of global solutions for a predator-prey model with cross-diffusion, *Electronic. J. Diff. Eqns*. **06** (2008), 1-14.
[22] S. M. Fu, S. B. Cui, Global existence and stability of solution of a reaction-diffusion model for cancer invasion, *Nonlinear Anal. RWA*. **10** (2009), 1362-1369.
[23] H. L. Li, P. Y. H. Pang, M. X. Wang, Global solutions for a general strongly coupled prey-predator model, *Applied Mathematics Letters*. **22** (2009), 1508-1512.
[24] H. Amann, Dynamic theory of quasilinear parabolic equations: Reaction-diffusion, *Diff. Int. Eqs*. **3**(1990), 13-75.
[25] H. Amann, Dynamic theory of quasilinear parabolic equations: Global existence, *Math. Z*. **202**(1989), 219-250.
[26] H. Amann, *Linear and Quasilinear Parabolic Problems, Vol. 1 Abstract Linear Theory*, Birkhäuser Verlag, Basel-Boston-Berlin, 1995.
[27] C. S. Lin, W. M. Ni, I. Takai, Large amplitude stationary solutions to a chemotaxis systems, *J. Differential Equations*. **72** (1988), 1-27.
[28] Y. Lou, W. M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*. **131** (1996), 79-131.
[29] O. A. Ladyzenskaja and V. A. Solonnikov, N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, (Translations of Mathematical Monographs, 23, AMS, 1968).
[30] B. Andreianov, M. Bendahmane, R. Ruiz Baier, Analysis of a finite volume method for a cross-diffusion model in population dynamics, *Math. Models Methods Appl. Sci.*. **21** (2011) 307-344.
[31] H. W. Xu, S. H. Xu, Existence and stability of global solutions for a cross-diffusion predator-prey model with sex-structure, *Nonlinear Anal. RWA*. **13** (2012) 999-1009.
[32] S. M. Fu, L. Zhang, P. Hu, Global behavior of solutions in a Lotka-Volterra predator-prey model with prey-stage structure, *Nonlinear Anal. RWA*. **14** (2013) 2027-2045.
[33] B. Li, M. Wang, Stationary patterns of the stage-structured predator-prey model with diffusion and cross-diffusion, Math. Comput. Modelling 54 (2011) 1380-1393.
[34] A. M. Atehortúa, L. M. Ladino, J. C. Valverde, Population dynamics of a two-stage migratory species with predation and capture, *Nonlinear Anal. RWA*. **16** (2014) 27-39.
[35] R. Arditi, L. R. Ginzburg, *Coupling in predator-prey dynamics: ratio dependence*, J. Theoret. Biol. 139 (1989), 311-326.
[36] C. Holing. *The functional response of predators to prey density and its role in mimicry and population regulation*. Mem. Ent, Soc. Can, 45(1965), 1-65.
[37] C. S. Holing, The components of predation as revealed by a study of small mammal predation of the European pine sawfly, Canad. Entomologist, 91 (1959), pp. 293-320.
[38] V. S. Ivlev, Experimental Ecology of the Feeding of Fishes, Yale University Press, 1961.
[39] J. F. Andrews, *A mathematical model for the continuous culture of microoganisms utilizing inhibitorysubstrates*, Biotech. Bioeng**10**(1968), 707-723.
[40] Beddington J R. *Mutual interference between parasites or predators and its effect on searhing efficiency*. J. Animal Ecodogy, 1975, 331-340.
[41] DeAngelis D L. Goldstein R A. O'Neill R V A. *A model for tropic interaction*. Ecology, 1975, 56: 881-892.
[42] P. H. Crowley, E. K. Martin, *Functional responses and interference within and between year classes of a dragonfly population*, J. North. Am. Benth. Soc. **8** (1989), 211–221.
[43] M.P. Hassell, G.C. Varley, New inductive population model and its bearing on biological control, Nature 223 (1969) 1133–1137.
[44] M. L. Rosenzweig, Paradox of enrichment: destabilization of exploitation ecosystems in ecological time, Science 171. 3969 (1971) 385-387.
[45] Pulvirenti, S., Pavone, P., Carbonaro, R.A., Costa, R.M.S., 2017. The controversial biography of Paolo Boccone (1633 − 1704) and his "Grand Tour" from the Mediterranean to northern Europe. Plant Biosystems 151(3), 377 − 380
[46] L. Chuan, J. Daqing, W. Ruihua, Periodic Solution of a Stochastic NonAutonomous Lotka-Volterra Cooperative System with Impulsive Perturbations, Filomat 32:4 (2018), 1151–1158.
[47] L. Demou, L., Global stability of solutions in a reaction-diffusion system of predator-prey model, Filomat 32:13 (2018), 4665-4672.
[48] Y. Yamada, Systems of nonlinear diffusion equations with cross-diffusion-systems of reaction-diffusion equations that appear in mathematical ecology. (Japanese) Sugaku 64 (2012), no. 4, 384-406.