Filomat 36:7 (2022), 2143–2152
https://doi.org/10.2298/FIL2207143M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Properties of the Inverse Degree Index and Coindex of Trees

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Abstract. Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple graph without isolated vertices, with the sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(v_i)$. If vertices v_i and v_j are adjacent in G, we write $i \sim j$, otherwise we write $i \nsim j$. The inverse degree topological index of G is defined to be $ID(G) = \sum_{i=1}^{n} \frac{1}{d_i} = \sum_{i\sim j} \left(\frac{1}{d_i^2} + \frac{1}{d_j^2} \right)$, and the inverse degree coindex $\overline{ID}(G) = \sum_{i\nsim j} \left(\frac{1}{d_i^2} + \frac{1}{d_j^2} \right)$. We obtain a number of inequalities which determine bounds for the $ID(G)$ and $\overline{ID}(G)$ when G is a tree.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple graph with n vertices, m edges, and a sequence of vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. Denote by $\overline{G}$ a complement of G. If vertices v_i and v_j are adjacent in G we write $i \sim j$, whereas $i \nsim j$ denotes that v_i and v_j are adjacent in $\overline{G}$.

A topological index of graph is a numerical quantity which is invariant under automorphisms of the graph. Topological indices are important and useful tools in mathematical chemistry, nanomaterials, pharmaceutical engineering, etc. used for quantifying information on molecules.

The first Zagreb index is a vertex–degree–based graph invariant defined as

$$M_1(G) = \sum_{i=1}^{n} d_i^2 .$$

The quantity M_1 was first time considered in 1972 [1]. It was recognized to be a measure of the extent of branching of the carbon–atom skeleton of the underlying molecule. The first Zagreb index became one of the most popular and most extensively studied graph-based molecular structure descriptors. Details of its theory and applications can be found in surveys [2–6] and the references cited therein.

Various generalizations of the first Zagreb index have been proposed. In [7] a so called general zeroth–order Randić index was introduced. It was conceived as

$$^0R_\alpha(G) = \sum_{i=1}^{n} d_i^\alpha ,$$

2020 *Mathematics Subject Classification*. Primary 05C12; Secondary 05C50
Keywords. Topological indices; vertex degree; general zeroth order Randić index.
Received: 26 January 2020; Accepted: 28 June 2020
Communicated by Dragan S. Djordjević
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where α is an arbitrary real number. It is also met under the names the first general Zagreb index [8] and variable first Zagreb index [9].

Besides the first Zagreb index, in this paper we are interested in the following special cases of ${}^0R_\alpha(G)$. For $\alpha = -1$, the inverse degree index, $ID(G)$, is obtained [11]

$$ID(G) = \sum_{i=1}^{n} \frac{1}{d_i} \,.$$

More on $ID(G)$ and its properties can be found in [12, 13].

For $\alpha = -2$, a so called modified first Zagreb index ${}^mM_1(G)$ is obtained. It was defined in [6] (see also [9]) as

$${}^mM_1(G) = \sum_{i=1}^{n} \frac{1}{d_i^2} \,.$$

A family of 148 disrete Adriatic indices was introduced and analyzed in [18]. Especially interesting subclass of these indices consists of 20 indices which are useful for predicting certain physicochemical properties of chemical compounds. One of them is the symmetric division deg index, $SDD(G)$, defined as

$$SDD(G) = \sum_{i\sim j} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right) .$$

More on its applications and mathematical properties can be found in [19–21].

Let

$$TI(G) = \sum_{i\sim j} F(d_i, d_j) \,, \tag{1}$$

be an arbitrary vertex–degree-based topological index, where $F(x, y)$ is a real valued function such that $F(x, y) = F(y, x)$. In [14] a concept of topological coindices was introduced. In this case the sum runs over the edges of the complement of G. In a view of (1) the corresponding coindex of G can be defined as

$$\overline{TI}(G) = \sum_{i\nsim j} F(d_i, d_j) \,.$$

In this article we prove a number of inequalities that determine upper and lower bounds for the inverse degree index and coindex of trees as well as relationships with some of the aforementioned indices.

2. Preliminaries

In this section we recall some discrete analytical inequalities that will be used frequently in the proofs of theorems, as well as one result for $ID(G)$, when G is a tree, which is specifically interesting for us.

Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be positive real number sequences. In [23] it was proven that for arbitrary $r \geq 0$, holds

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r} \,. \tag{2}$$

Equality holds if and only if $r = 0$ or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n$, be sequences of nonnegative and positive real numbers, respectively. In [15] (see also [16]), it was proven that for any real r, $r \leq 0$ or $r \geq 1$, holds

$$\left(\sum_{i=1}^{n} p_i \right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left(\sum_{i=1}^{n} p_i a_i \right)^r \,. \tag{3}$$

When $0 \leq r \leq 1$, the sense of (3) reverses. Equality is valid if and only either if $r = 0$, or $r = 1$, or $a_1 = a_2 = \cdots = a_n$, or $p_1 = p_2 = \cdots = p_t = 0$ and $a_{t+1} = \cdots = a_n$, for some t, $1 \leq t \leq n-1$.

In [24] the following bounds for the $ID(G)$ index when G is a tree T, that is $G \cong T$, were obtained

$$\frac{n+2}{2} \leq ID(T) \leq n-1+\frac{1}{n-1}, \tag{4}$$

with equality on the left-hand side if and only if $T \cong P_n$, and on the right-hand side if and only if $T \cong K_{1,n-1}$.

3. Main results

In the next theorem we set up upper and lower bounds for $ID(T)$ for arbitrary tree, in terms of n and Δ.

Theorem 3.1. *Let T be a tree with n vertices. If $n \geq 4$, then*

$$ID(T) \geq \frac{1}{\Delta}+2+\frac{(n-3)^2}{2n-\Delta-4}. \tag{5}$$

If $n \geq 2$, then

$$ID(T) \leq n-\frac{n-2}{\Delta}. \tag{6}$$

Equality in (5) holds if and only if T is a tree with the property $\Delta = d_1 \geq d_2 = \cdots = d_{n-2} \geq d_{n-1} = d_n = \delta = 1$, whereas in (6) if and only if T is a tree with the property $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-2$.

Proof. Let G be a simple graph of order $n \geq 4$ without isolated vertices. Based on the inequality between the arithmetic and harmonic means, the AM–HM inequality, (see e.g. [16]), we have that

$$\sum_{i=2}^{n-2}\frac{1}{d_i}\sum_{i=2}^{n-2}d_i \geq (n-3)^2,$$

that is

$$\left(ID(G)-\frac{1}{\Delta}-\frac{1}{d_{n-1}}-\frac{1}{d_n}\right)(2m-\Delta-d_{n-1}-d_n) \geq (n-3)^2. \tag{7}$$

Suppose G is an arbitrary tree, i.e. $G \cong T$. Then, at least two of its vertices are of degree 1. If G is a tree then $m = n-1$ and $d_{n-1} = d_n = \delta = 1$. Then the inequality (7) becomes

$$\left(ID(T)-\frac{1}{\Delta}-2\right)(2(n-1)-\Delta-2) \geq (n-3)^2, \tag{8}$$

from which (5) is obtained.

Equality in (8), and thus in (5), holds if and only if $\Delta = d_1 \geq d_2 = \cdots = d_{n-2} \geq d_{n-1} = d_n = \delta = 1$.

Let G be a simple graph of order $n \geq 2$ without isolated vertices. Then for the degree d_i of vertex v_i holds

$$(d_i-\delta)(\Delta-d_i) \geq 0,$$

that is

$$d_i+\frac{\Delta\delta}{d_i} \leq \Delta+\delta. \tag{9}$$

Summing the above inequality over i, $i = 1, 2, \ldots, n$, we obtain

$$\sum_{i=1}^{n}d_i+\Delta\delta\sum_{i=1}^{n}\frac{1}{d_i} \leq (\Delta+\delta)\sum_{i=1}^{n}1,$$

that is

$$2m + \Delta\delta\, ID(G) \le n(\Delta + \delta)\,.$$

If G is a tree, then $m = n-1$, $\delta = 1$, $n \ge 2$, and the above inequality turns into

$$\Delta\, ID(T) \le n(\Delta + 1) - 2(n-1)\,, \tag{10}$$

from which the inequality (6) is obtained.

Equality in (10), and hence in (6), holds if T is a tree with the property $\Delta = d_1 = \cdots = d_t \ge d_{t-1} = \cdots = d_n = \delta = 1$, for some t, $1 \le t \le n-2$. □

Remark 3.2. *The function*

$$f(x) = \frac{1}{x} + 2 + \frac{(n-3)^2}{2n-4-x}$$

is monotone increasing for $x \ge 2$. Therefore for $x = \Delta \ge 2$ we have

$$\frac{1}{\Delta} + 2 + \frac{(n-3)^2}{2n-4-\Delta} \ge \frac{1}{2} + 2 + \frac{(n-3)^2}{2n-6} = \frac{n+2}{2}\,,$$

which means that the inequality (5) *is stronger than the left-hand part of inequality* (4).

Since $\Delta \le n-1$, we have that

$$n - \frac{n-2}{\Delta} \le n - \frac{n-2}{n-1} = n - 1 + \frac{1}{n-1}\,,$$

which implies that (6) *is stronger than the right-hand side of inequality* (4).

In analogy with the bounds (4) for $ID(G)$, in the next theorem we determine bounds for the $\overline{ID}(G)$.

Theorem 3.3. *Let T be a tree with $n \ge 2$ vertices. Then*

$$\frac{(n-2)(n+5)}{4} \le \overline{ID}(T) \le (n-1)(n-2)\,, \tag{11}$$

with equality on the left-hand side if and only if $T \cong P_n$, and on the right-hand side if and only if $T \cong K_{1,n-1}$.

Proof. For $r = 2$ the inequality (3) can be considered as

$$\sum_{i=1}^{n-2} p_i \sum_{i=1}^{n-2} p_i a_i^2 \ge \left(\sum_{i=1}^{n-2} p_i a_i\right)^2 .$$

Let G be a simple graph of order $n \ge 3$ and size m, without isolated vertices. For $p_i = \frac{n-1-d_i}{d_i^2}$, $a_i = d_i$, $i = 1, 2, \ldots, n-2$, the above inequality becomes

$$\sum_{i=1}^{n-2} \frac{n-1-d_i}{d_i^2} \sum_{i=1}^{n-2} (n-1-d_i) \ge \left(\sum_{i=1}^{n-2} \frac{n-1-d_i}{d_i}\right)^2 ,$$

that is

$$\left(\sum_{i=1}^{n} \frac{n-1-d_i}{d_i^2} - \frac{n-1-d_{n-1}}{d_{n-1}^2} - \frac{n-1-d_n}{d_n^2}\right)\left(\sum_{i=1}^{n}(n-1-d_i) - (n-1-d_{n-1}) - (n-1-d_n)\right) \ge$$

$$\ge \left(\sum_{i=1}^{n} \frac{n-1-d_i}{d_i} - \frac{n-1-d_{n-1}}{d_{n-1}} - \frac{n-1-d_n}{d_n}\right)^2 ,$$

which gives

$$\left(\overline{ID}(G)-\frac{n-1-d_{n-1}}{d_{n-1}^2}-\frac{n-1-d_n}{d_n^2}\right)\left((n-1)(n-2)-2m+d_{n-1}+d_n\right)\geq\left((n-1)\left(ID(G)-\frac{1}{d_{n-1}}-\frac{1}{d_n}\right)-n+2\right)^2 .$$

If G is a tree, $G\cong T$, with n vertices, the above inequality becomes

$$(n-2)(n-3)(\overline{ID}(T)-2(n-2))\geq((n-1)(ID(T)-2)-n+2)^2 .$$

Combining the above and the left-hand part of inequality (4), we obtain the left-hand part of (11).

Equality in the left-hand part of inequality (4) holds if and only if $T\cong P_n$, which implies that the equality in the left-hand part of (11) if and only if $T\cong P_n$.

Let T be a tree with $n\geq 2$ vertices. Then, for the degree d_i of any vertex v_i, holds

$$(n-1-d_i)(d_i-1)\geq 0 ,$$

that is

$$d_i^2+(n-1)\leq nd_i .$$

After multiplying the above inequality by $\frac{n-1-d_i}{d_i^2}$ and summung over i, $i=1,2,\ldots,n$, we obtain

$$(n-1)\sum_{i=1}^{n}\frac{n-1-d_i}{d_i^2}\leq n\sum_{i=1}^{n}\frac{n-1-d_i}{d_i}-\sum_{i=1}^{n}(n-1-d_i) ,$$

that is

$$(n-1)\overline{ID}(T)\leq n((n-1)ID(T)-n)-(n(n-1)-2(n-1)) .$$

From the above inequality and the right-hand part of (4) we arrive at right-hand part of (11).

Equality in the right-hand part of (4) holds if and only if $T\cong K_{1,n-1}$, which means that equality in the right-hand part of (11) holds if and only if $T\cong K_{1,n-1}$. □

From (4) and (11) we obtain the following result.

Corollary 3.4. *Let T be a tree with $n\geq 2$. Then*

$$\frac{(n-1)(n+6)}{4}\leq ID(T)+\overline{ID}(T)\leq (n-1)^2+\frac{1}{n-1} .$$

Equality in the left-hand side of the above inequality holds if and only if $T\cong P_n$, and in the right-hand side if and only if $T\cong K_{1,n-1}$.

Since

$$ID(T)+\overline{ID}(T)=\sum_{i=1}^{n}\frac{1}{d_i}+\sum_{i=1}^{n}\frac{n-1-d_i}{d_i^2}=(n-1)\sum_{i=1}^{n}\frac{1}{d_i^2}=(n-1)\,{}^{m}M_1(T) ,$$

we have the following result.

Corollary 3.5. *Let T be a tree with $n\geq 2$ vertices. Then*

$$\frac{n+6}{4}\leq {}^{m}M_1(T)\leq (n-1)+\frac{1}{(n-1)^2} .$$

Equality on the left-hand side holds if and only if $T\cong P_n$, whereas on the right-hand side if and only if $T\cong K_{1,n-1}$.

In [12] (see also [17]) the following inequality was proved for the connected graph of order $n \geq 2$

$$\sum_{1\leq i<j\leq n}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)=2mID(G)-n\,.$$

Since

$$\sum_{1\leq i<j\leq n}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)=\sum_{i\sim j}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)+\sum_{i\nsim j}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right),$$

we have that

$$SDD(G)+\overline{SDD}(G)=2mID(G)-n\,,$$

and

$$SDD(T)+\overline{SDD}(T)=2(n-1)ID(T)-n\,,$$

which implies that the following result is valid.

Corollary 3.6. *Let T be a tree with $n \geq 2$ vertices. Then*

$$n^2-2\leq SDD(T)+\overline{SDD}(T)\leq 2n^2-5n+4\,.$$

Equality on the left-hand side holds if and only if $T \cong P_n$, whereas in the right-hand side if and only if $T \cong K_{1,n-1}$.

Corollary 3.7. *Let T be a tree with $n \geq 4$ vertices. Then*

$$SDD(T)+\overline{SDD}(T)\geq 2(n-1)\left(\frac{1}{\Delta}+2+\frac{(n-3)^2}{2n-\Delta-4}\right)-n\,. \tag{12}$$

When $n \geq 2$ then

$$SDD(T)+\overline{SDD}(T)\leq 2(n-1)\left(n-\frac{n-2}{\Delta}\right)-n\,. \tag{13}$$

Equality in (12) holds if and only if T is a tree with the property $\Delta = d_1 \geq d_2 \geq \cdots = d_{n-2} \geq d_{n-1} = d_n = \delta = 1$. Equality in (13) holds if and only if T is a tree with the property $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-2$.

In the next theorem we establish a relationship between $ID(T)$ and $M_1(T)$.

Theorem 3.8. *Let T be a tree with $n \geq 2$ vertices. Then*

$$\Big(\Delta(ID(T)-1)-n+1\Big)\Big((2n-3)\Delta-M_1(T)+1\Big)\geq\Big((n-1)(\Delta-2)+1\Big)^2, \tag{14}$$

and

$$\Big((n-1)\Delta+1-\Delta ID(T)\Big)\Big(M_1(T)-2(n-1)-\Delta(\Delta-1)\Big)\geq\Delta(n-1-\Delta)^2\,. \tag{15}$$

Equality in (14) holds if and only if T is a tree with the property $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-2$, and in (15) if and only if $\Delta = d_1 \geq d_2 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-2$.

Proof. For $r = 2$ the inequality (3) can be considered as

$$\sum_{i=1}^{n-1}p_i\sum_{i=1}^{n-1}p_ia_i^2\geq\left(\sum_{i=1}^{n-1}p_ia_i\right)^2\,.$$

Suppose G is a simple graph without isolated vertices. For $p_i = \frac{\Delta - d_i}{d_i}$, $a_i = d_i$, $i = 1, 2, \ldots, n-1$, the above inequality becomes

$$\sum_{i=1}^{n-1} \frac{\Delta - d_i}{d_i} \sum_{i=1}^{n-1} (\Delta - d_i) d_i \geq \left(\sum_{i=1}^{n-1} (\Delta - d_i) \right)^2 ,$$

that is

$$\left(\sum_{i=1}^{n} \frac{\Delta - d_i}{d_i} - \frac{\Delta - \delta}{\delta} \right) \left(\sum_{i=1}^{n} (\Delta - d_i) d_i - \delta(\Delta - \delta) \right) \geq \left(\sum_{i=1}^{n} (\Delta - d_i) - (\Delta - \delta) \right)^2 ,$$

which is equivalent to

$$\left(\Delta\, ID(G) - n - \frac{\Delta - \delta}{\delta} \right) (2m\Delta - M_1(G) - \delta(\Delta - \delta))(n\Delta - 2m - (\Delta - \delta))^2 .$$

If G is a tree, $G \cong T$, with n vertices, then $m = n-1$, $\delta = 1$, and from the above inequality we get

$$(\Delta ID(T) - n - \Delta + 1)(2(n-1)\Delta - M_1(T) - \Delta + 1) \geq (n\Delta - 2(n-1) - \Delta + 1)^2 , \tag{16}$$

from which we obtain (14).

Similarly, for $r = 2$ the inequality (3) can be viewed as

$$\sum_{i=2}^{n} p_i \sum_{i=2}^{n} p_i a_i^2 \geq \left(\sum_{i=2}^{n} p_i a_i \right)^2 .$$

Now, for $p_i = \frac{d_i - \delta}{d_i}$, $a_i = d_i$, $i = 2, 3, \ldots, n$ the above inequality becomes

$$\sum_{i=2}^{n} \frac{d_i - \delta}{d_i} \sum_{i=2}^{n} d_i(d_i - \delta) \geq \left(\sum_{i=2}^{n} (d_i - \delta) \right)^2 ,$$

that is

$$\left(\sum_{i=1}^{n} \frac{d_i - \delta}{d_i} - \frac{\Delta - \delta}{\Delta} \right) \left(\sum_{i=1}^{n} d_i(d_i - \delta) - \Delta(\Delta - \delta) \right) \geq \left(\sum_{i=1}^{n} (d_i - \delta) - (\Delta - \delta) \right)^2 ,$$

which is equivalent to

$$\left(n - \delta\, ID(G) - \frac{\Delta - \delta}{\Delta} \right) \left(M_1(G) - 2m\delta - \Delta(\Delta - \delta) \right) \geq \left(2m - n\delta - (\Delta - \delta) \right)^2 .$$

If G is a tree with n vertices, then $m = n-1$ and $\delta = 1$, and the above inequality becomes

$$\left(n - ID(T) - 1 + \frac{1}{\Delta} \right) \left(M_1(T) - 2(n-1) - \Delta(\Delta - 1) \right) \geq \left(2(n-1) - n - \Delta + 1 \right)^2 , \tag{17}$$

from which we arrive at (15).

Equality in (16), and consequently in (14), occurs if and only if T is a tree with the property $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-2$. Equality in (17), and thus in (15) holds if and only if $\Delta = d_1 \geq d_2 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-2$. □

In the next theorem we set up a relation between $\overline{M}_1(T)$, $\overline{ID}(T)$ and $ID(T)$.

Theorem 3.9. *Let T be a tree with $n \geq 3$ vertices. Then*

$$\left(\overline{M}_1(T) - 2(n-2) - \Delta(n-1-\Delta) \right) \left(\overline{ID}(T) - 2(n-2) - \frac{n-1-\Delta}{\Delta^2} \right)^2 \geq \left((n-1)ID(T) - 3n + 5 - \frac{n-1}{\Delta} \right)^3 . \tag{18}$$

Equality holds if and only if $T \cong P_n$ or $T \cong K_{1,n-1}$.

Proof. Let T be an arbitrary tree. In that case $d_{n-1} = d_n = \delta = 1$, and we have that

$$\overline{M}_1(T) = \sum_{i=1}^{n}(n-1-d_i)d_i = \sum_{i=2}^{n-2}(n-1-d_i)d_i + (n-1-\Delta)\Delta + 2(n-2)\,. \tag{19}$$

On the other hand, for $r = 2$ the inequality (2) can be viewed as

$$\sum_{i=2}^{n-2}\frac{x_i^3}{a_i^2} \geq \frac{\left(\sum_{i=2}^{n-2} x_i\right)^3}{\left(\sum_{i=2}^{n-2} a_i\right)^2}\,.$$

For $x_i = \frac{n-1-d_i}{d_i}$, $a_i = \frac{n-1-d_i}{d_i^2}$, $i = 2, 3, \dots, n-2$, the above inequality transforms into

$$\sum_{i=2}^{n-2}(n-1-d_i)d_i \geq \frac{\left(\sum_{i=2}^{n-2}\frac{n-1-d_i}{d_i}\right)^3}{\left(\sum_{i=2}^{n-2}\frac{n-1-d_i}{d_i^2}\right)^2}\,. \tag{20}$$

Now, from (19) and (20) we get

$$\overline{M}_1(T) - (n-1-\Delta)\Delta - 2(n-2) \geq \frac{\left((n-1)ID(T) - n - \frac{n-1-\Delta}{\Delta} - 2(n-2)\right)^3}{\left(\overline{ID}(T) - \frac{n-1-\Delta}{\Delta^2} - 2(n-2)\right)^2}\,,$$

from which we obtain (18).

Equality in (20) holds if and only if $d_2 = d_3 = \cdots = d_{n-2}$, which implies that equality in (18) holds if and only if $\Delta = d_1 \geq d_2 = d_3 = \cdots = d_{n-2} \geq d_{n-1} = d_n = \delta = 1$, that is if and only if $T \cong P_n$ or $T \cong K_{1,n-1}$. □

In the next theorem we prove the inequality of Nordhaus-Gaddum type (see e.g. [25]) for the inverse degree index.

Theorem 3.10. *Let T be a tree, $T \not\cong K_{1,n-1}$, with $n \geq 4$ vertices. Then we have*

$$ID(T) + ID(\overline{T}) \geq (n-1)\left(\frac{2}{n-2} + \frac{n-2}{2(n-3)}\right)\,, \tag{21}$$

with equality if and only if $T \cong P_n$.

Proof. Let G be a simple graph of order $n \geq 4$ without isolated vertices for which vertex degrees hold $d_i \neq n-1$, $i = 1, 2, \dots, n$. Then we have

$$\sum_{i=1}^{n-2}\frac{1}{d_i} + \sum_{i=1}^{n-2}\frac{1}{n-1-d_i} = (n-1)\sum_{i=1}^{n-2}\frac{1}{d_i(n-1-d_i)}\,. \tag{22}$$

On the other hand, for $r = 1$ the inequality (2) can be considered as

$$\sum_{i=1}^{n-2}\frac{x_i^2}{a_i} \geq \frac{\left(\sum_{i=1}^{n-2} x_i\right)^2}{\sum_{i=1}^{n-2} a_i}\,.$$

Now, for $x_i = 1$, $a_i = d_i(n-1-d_i)$, $i = 1, 2, \dots, n-2$, the above inequality becomes

$$\sum_{i=1}^{n-2}\frac{1}{d_i(n-1-d_i)} \geq \frac{\left(\sum_{i=1}^{n-2} 1\right)^2}{\sum_{i=1}^{n-2} d_i(n-1-d_i)}\,,$$

that is

$$\sum_{i=1}^{n-2} \frac{1}{d_i(n-1-d_i)} \ge \frac{(n-2)^2}{2m(n-1) - M_1(G) - d_{n-1}(n-1-d_{n-1}) - d_n(n-1-d_n)} \, .$$

From the above inequality and identity (22) we have that

$$ID(G)+ID(\overline{G}) \ge \frac{n-1}{d_{n-1}(n-1-d_{n-1})} + \frac{n-1}{d_n(n-1-d_n)} + \frac{(n-1)(n-2)^2}{2m(n-1) - M_1(G) - d_{n-1}(n-1-d_{n-1}) - d_n(n-1-d_n)} \, .$$

When G is a tree T, and $T \not\cong K_{1,n-1}$, with $n \ge 4$ vertices, the above inequality becomes

$$ID(T) + ID(\overline{T}) \ge \frac{2(n-1)}{n-2} + \frac{(n-1)(n-2)^2}{2(n-1)^2 - M_1(T) - 2(n-2)} \, . \tag{23}$$

In [4] it was proven that

$$M_1(T) \ge 4n - 6 \, . \tag{24}$$

Now from (23) and (24) we arrive at (21).

Equality in (24) holds if and only if $T \cong P_n$, which implies that equality in (21) holds if and only if $T \cong P_n$. □

Acknowledgement

This work was supported by the Serbian Ministry for Education, Science and Technological development, grant No. 451-03-68/2022-14/200102.

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