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Soft Outer Measure and Soft Premeasure

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Abstract. Molodtsov introduced the concept of soft set as a new mathematical tool for dealing with problems treating uncertainties, and, later, Riaz at al. developed various concepts of soft sets. Here, we study the concept of soft semiring and soft premeasure and explore their basic properties. We define a $[0, \infty]$–valued function on some soft set $\mathcal{E}$, that is not a soft σ–algebra, and then we extend this function to a measure on the soft σ–algebra generated by $\mathcal{E}$. We also introduce various set functions like soft content and soft premeasures, and give their basic properties.

1. Introduction

Probability theory is not always sufficient and is often not appropriate for studying vague, imprecise, uncertain, incomplete real-life information. Apart from Zadeh's fuzzy systems theory based on fuzzy sets (see [34]), other significant alternatives dealing with the above problems and information include Shafer's evidence theory (see [31] or [14]), Pawlak's rough set theory (see [23] or [35]), and Molodtsov's soft set theory (see [20] or [21]). Our focus will be on some aspects of Molodtsov's soft set theory.

The soft set theory was proposed not so long ago in 1999 by Molodtsov [20], as a new mathematical tool for dealing with various types of uncertain, fuzziness, or inaccuracy. The soft set theory has many applications in both the natural and many social sciences. According to Molodtsov (see [20] and [21]), the soft set theory is very successful in many mathematical fields, such as operations research, Riemann integration, game theory, measure theory, and many others. Since 1999, research on soft sets has become more frequent and is advancing rapidly.

Many papers define different types of operations with soft sets and their applications. Well-known operations with soft sets and the properties of such operations have been defined and proven by Maji et al. [17]. Ali et al. [4] and Yang [32] pointed out some gaps and shortcomings in such operations. To fill in the gaps of some operations, Ali et al. [4], Cagman and Enginoglu [7], Pai and Miao [24], and Sezgin and Atagun [30] have contributed to working soft sets. In 2011, Sezgin and Atagun [30] proved the basic theorems on soft set operation and the connection between the union and the intersection of soft sets. In that paper, they defined the union and intersection of soft sets with restricted and extended conditions, but defined difference operation only as restricted. In 2019, Sezgin, Ahmad and Mehmood [29] defined an

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extended difference operation. Maji et al. [16],[17] used the soft set theory in problems requiring decision-making and defined some more of the operations. Ali et al. [4] suggested some other operations on soft sets that have become very useful in soft set theory. Chen et al. [9] studied some basic problems where soft set theory would be successfully applied.

Kharal and Ahmad [15] defined mappings on soft classes, and images and inverse images of soft sets. Samanta and Majumdar [19] proposed the concept of soft groups and studied soft mappings on soft sets. Riaz and Naeem [26] studied measurable soft mappings and some applications of soft set theory. Many authors studied the topology of soft sets in papers [28], [6] and [12]. In paper [33], Yüksel et al. studied soft filters and some of their basic properties, and in paper [18], Mukherjee et al. studied measurable soft sets. Khameneh and Kilicman [13] studied soft σ-algebras and soft probability spaces.

In Paper [27], Riaz et al. defined a soft measure and a soft outer measure, and we opt to continue to study such structures.

This paper aims to define a suitable $[0,\infty]$–valued function on a soft set other than a soft algebra, and then extend the observed function to a soft measure on a soft σ–algebra generated by $\mathcal{E}$. We will also extend our functions from so-called soft semirings to soft σ–algebras. We also investigate the different properties of soft ring of sets and soft σ–algebras. We also define special functions on soft sets, such as soft content and soft premeasure, and then give and prove their basic and very important properties.

2. Preliminaries

This section provides basic definitions and basic properties in soft set theory that were introduced and proven earlier by many authors.

Let X be an initial universe set and E_X be the set of all possible parameters under consideration with respect to X. The power set of X is denoted by $\mathcal{P}(X)$ and A is a subset of E. Usually, parameters are attributes, characteristics, or properties of objects in X. In what follows, E_X (simply denoted by E) always means the universe set of parameters with respect to X, unless otherwise specified.

Definition 2.1. *[20] A pair (F, A) is called a soft set over X where $A \subseteq X$ and $F : A \to \mathcal{P}(X)$ is a set valued mapping. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\forall e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). It is worth noting that $F(e)$ may be arbitrary. Some of them may be empty, and some may have nonempty intersection.*

Definition 2.2. *[18] A soft set F_A on the universe X is defined by the set of ordered pairs $F_A = \{(e, f_A(e)) \mid e \in E, f_A(e) \in \mathcal{P}(X)\}$, where $f_A : E \to \mathcal{P}(X)$, such that $f_A(e) \neq \emptyset$ if $e \in A \subseteq E$ and $f_A(e) = \emptyset$, if $e \notin A$. Here, F_A is called an approximate function of the soft set F_A. The value of $f_A(e)$ may be arbitrary.*

Note that the set of all soft sets over X will be denoted by $\mathcal{S}(X, E)$.

Definition 2.3. *[7] Let $F_A \in \mathcal{S}(X, E)$. If $f_A(e) = \emptyset$ for all $e \in E$, then F_A is called an empty soft set, denoted by F_Φ or Φ. $f_A(e) = \emptyset$ means that there is no element in X related to the parameter $e \in E$. Therefore, we do not display such elements in the soft sets, as it is meaningless to consider such parameters.*

Definition 2.4. *[7] Let $F_A \in \mathcal{S}(X, E)$. If $f_A(e) = X$ for all $e \in A$, then F_A is called an A-universal soft set, denoted by $F_{\widetilde{A}} = \widetilde{A}$. If $A = E$, then the A-universal soft set is called a universal soft set, denoted by $F_{\widetilde{E}} = \widetilde{E}$.*

Definition 2.5. *[28] Let Y be a nonempty subset of X, then $\widetilde{Y}$ denotes the soft set Y_E over X for which $Y(e) = Y$, for all $e \in E$. In particular, X_E will be denoted by $\widetilde{X}$.*

Definition 2.6. *[7] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then F_A is a soft subset of G_B, denoted by $F_A \sqsubseteq G_B$ if $f_A(e) \subseteq g_B(e)$, for all $e \in E$.*

Definition 2.7. *[7] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then F_A and G_B are soft equal, denoted by $F_A = G_B$, if and only if $f_A(e) = g_B(e)$, for all $e \in E$.*

Definition 2.8. *[7] Let $F_A, G_B \in \mathcal{S}(X,E)$. Then, the soft union $F_A \sqcup G_B$ of F_A and G_B is defined by the approximate functions $h_{A\cup B}(e) = f_A(e) \cup g_B(e)$, for all $e \in E$.*

Definition 2.9. *[7] Let $F_A, G_B \in \mathcal{S}(X,E)$. Then, the soft intersection $F_A \sqcap G_B$ of F_A and G_B is defined by the approximate functions $h_{A\cap B}(e) = f_A(e) \cap g_B(e)$, for all $e \in E$.*

Definition 2.10. *[7] Let $F_A, G_B \in \mathcal{S}(X,E)$. Then, the soft difference $F_A \backslash G_B$ of F_A and G_B is defined by the approximate functions $h_{A\backslash B}(e) = f_A(e) \backslash g_B(e)$, for all $e \in E$.*

Definition 2.11. *[7] The soft complement F_A^c of $F_A \in \mathcal{S}(X,E)$ is defined by the approximate function $f_{A^c}(e) = f_A^c(e)$, where $f_A^c(e)$ is the complement of the set $f_A(e)$; that is, $f_A^c(e) = X \backslash f_A(e)$, for all $e \in E$.*

Definition 2.12. *[36] Let I be an arbitrary index set and $\{(F_A)_i\}_{i\in I}$ be a subfamily of $\mathcal{S}(X,E)$.*

- *The union of these soft sets is the soft set G_C, where $g_C(e) = \cup_{i\in I}(F_A)_i(e)$ for each $e \in E$. We write $G_C = \sqcup_{i\in I}(F_A)_i$.*
- *The intersection of these soft sets is the soft set H_D, where $h_D(e) = \cap_{i\in I}(F_A)_i(e)$ for each $e \in E$. We write $H_D = \sqcap_{i\in I}(F_A)_i$.*

3. Soft semirings, soft ring and soft algebras

It would be too much to expect that soft measures must be defined on the whole soft power set $\widetilde{\mathcal{P}}(\widetilde{X})$ of a soft set $\widetilde{X}$, so we start by observing subsets $\widetilde{\mathcal{A}} \sqsubseteq \widetilde{\mathcal{P}}(\widetilde{X})$ that are good enough to be sufficient for the domain of a soft measure. It makes sense to require $\widetilde{\mathcal{A}}$ to be closed under countable unions. To be useful, $\widetilde{\mathcal{A}}$ should at least contain Φ and $\widetilde{X}$ etc. Soft measures will be defined on special collections called soft σ-algebras (see Definition 3.1. below). Here a similarity is imposed between the definition of a soft measure and the definition of a soft topology. Despite the differences, we will repeatedly use similar techniques that will be useful for soft measure space as they are useful in topological spaces.

Definition 3.1. *[13] A collection $\widetilde{\mathcal{A}}$ of soft subsets of $\widetilde{X}$ is called a soft σ-algebra on $\widetilde{X}$ if, and only if, it satisfies the following conditions:*

- $\Phi \in \widetilde{\mathcal{A}}$,
- *if $F_A \in \widetilde{\mathcal{A}}$, then $F_A^c = \widetilde{X} \backslash F_A \in \widetilde{\mathcal{A}}$,*
- *if $(F_A)_1, (F_A)_2, (F_A)_3 \ldots$ is a countable collection of soft sets in $\widetilde{\mathcal{A}}$, then $\bigsqcup_{i=1}^{n}(F_A)_i \in \widetilde{\mathcal{A}}$.*

  The pair $(\widetilde{X}, \widetilde{\mathcal{A}})$ is called a soft measurable space and $(F_A)_i \in \widetilde{\mathcal{A}}$ is called a measurable soft set.

Example 3.2. *Let $X = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2\}$. Assume that*
$(F_A)_1 = \Phi$,
$(F_A)_2 = \{(e_1, \{h_1\}), (e_2, \emptyset)\}$,
$(F_A)_3 = \{(e_1, \{h_2\}), (e_2, \{h_1, h_3\})\}$,
$(F_A)_4 = \{(e_1, \{h_3\}), (e_2, \{h_2\})\}$,
$(F_A)_5 = \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_3\})\}$,
$(F_A)_6 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_2, h_3\})\}$,
$(F_A)_7 = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2\})\}$ *and*
$(F_A)_8 = \widetilde{X}$.
Then $\widetilde{\mathcal{A}} = \{(F_A)_i \mid i = 1, 2, 3, \ldots, 8\}$ is a soft σ-algebra over X.

Theorem 3.3. *Let $\widetilde{\mathcal{A}}$ be a soft σ-algebra on $\widetilde{X}$.*

1. *$\widetilde{\mathcal{A}}$ is closed under finite unions: Let $n \in \mathbb{N}$. Then*

$$(F_A)_1, (F_A)_2, (F_A)_3 \ldots, (F_A)_n \in \widetilde{\mathcal{A}} \Rightarrow \bigsqcup_{i=1}^{n} (F_A)_i \in \widetilde{\mathcal{A}}.$$

2. *$\widetilde{\mathcal{A}}$ is closed under countable (both finite and infinite) intersections: Let I be a nonempty countable index set and $(F_A)_j \in \widetilde{\mathcal{A}}$, $j \in I$. Then*

$$\sqcap_{j\in I}(F_A)_j \in \widetilde{\mathcal{A}}.$$

Proof.

1. follows from Definition 3.1. as

$$\bigsqcup_{i=1}^{n} (F_A)_i = (F_A)_1 \sqcup (F_A)_2 \sqcup \ldots \sqcup (F_A)_n \sqcup \Phi \sqcup \Phi \sqcup \ldots.$$

2. follows from Definition 3.1. and (1), since

$$\left(\sqcap_{j\in I}(F_A)_j\right)^c = \bigsqcup_{j\in I} (F_A)_i^c. \quad \square$$

Example 3.4. *(1) Clearly, $\{\Phi, \widetilde{X}\}$ and $\mathcal{P}(\widetilde{X})$ are soft σ-algebras on $\widetilde{X}$, where $\{\Phi, \widetilde{X}\}$ sometimes called the trivial soft σ-algebra on $\widetilde{X}$.*

(2) $\mathcal{A} = \{F_A \sqsubseteq \widetilde{X} \mid A$ countable or F_A^c countable $\}$ constitutes a soft σ-algebra on $\widetilde{X}$.

The next result will allow us to generate an abundance of soft σ-algebras:

Theorem 3.5. *[27] The soft intersection of any collection of soft σ-algebras on $\widetilde{X}$ forms again a soft σ-algebra on $\widetilde{X}$.*

Theorem 3.6. *[27] The soft difference of two soft σ-algebras on $\widetilde{X}$ is again a soft σ-algebra on $\widetilde{X}$.*

The union of (even two) soft σ-algebras is, in general, not a soft σ-algebra (see [27]). The following result provides a basic technique for the construction of σ-algebra.

Theorem 3.7. *[27] Let $\widetilde{\mathcal{G}}$ be the collection of soft subsets of $\widetilde{X}$. Then there is a smallest soft σ-algebra containing $\widetilde{\mathcal{G}}$.*

Definition 3.8. *[27] The smallest soft σ-algebra $\widetilde{\mathcal{H}}$ containing some soft collection $\widetilde{\mathcal{G}}$ of soft subsets of $\widetilde{X}$ is called the soft σ-algebra generated by $\widetilde{\mathcal{G}}$.*

Definition 3.9. *[27] Let $\widetilde{\mathcal{A}}$ be a soft σ-algebra of soft subsets over $\widetilde{X}$ and $\widetilde{\mu}$ be a soft real-valued mapping on $\widetilde{\mathcal{A}}$. Let $((F_A)_i)_{i\in\mathbb{N}}$ be a sequence of soft sets in $\widetilde{\mathcal{A}}$. The soft mapping $\widetilde{\mu}$ is called:*

- *finitely soft sub-additive, if*

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{n} (F_A)_i\right) \leqslant \sum_{i=1}^{n} \widetilde{\mu}((F_A)_i),$$

- *countably soft sub-additive, if*

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty} (F_A)_i\right) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i),$$

- *finitely soft additive, if*

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right) = \sum_{i=1}^{n}\widetilde{\mu}((F_A)_i),$$

where $(F_A)_i$*'s are pairwise soft disjoint,*

- *countably soft additive or soft* σ*-additive, if*

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) = \sum_{i=1}^{\infty}\widetilde{\mu}((F_A)_i),$$

where $(F_A)_i$*'s are pairwise soft disjoint,*

- *soft monotone, if* $F_A \sqsubseteq G_B$ *then* $\widetilde{\mu}(F_A) \leqslant \widetilde{\mu}(G_B)$*, for all* $F_A, G_B \in \widetilde{\mathcal{A}}$*.*

Definition 3.10. *[27] Let* $\widetilde{\mathcal{A}}$ *be a soft* σ*-algebra of soft subsets of a set* $\widetilde{X}$ *and* $\widetilde{\mu}$ *be an extended soft real-valued mapping on* $\widetilde{\mathcal{A}}$*. Then* $\widetilde{\mu}$ *is called a soft measure on* $\widetilde{\mathcal{A}}$*, if*

- $\widetilde{\mu}(\Phi) = 0$*,*
- $\widetilde{\mu}(F_A) \geqslant 0$ *for each* $F_A \in \widetilde{\mathcal{A}}$*,*
- $\widetilde{\mu}$ *is countably soft additive, i.e.,*

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) = \sum_{i=1}^{\infty}\widetilde{\mu}((F_A)_i),$$

$(F_A)_i$*'s being pairwise soft disjoint.*

If $\widetilde{\mu}$ *is a soft measure on a soft* σ*-algebra* $\widetilde{\mathcal{A}}$*, then the triplet* $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\mu})$ *is called a soft measure space.*

Is it always necessary to have soft σ-algebra to define a soft measure? The idea is to first define a suitable $[0, \infty]$-valued function on some set $\mathcal{E}$ that is not a soft σ-algebra and then to extend this function to a soft measure on the soft σ-algebra generated by $\mathcal{E}$. We will extend our functions from so-called soft semirings to soft σ-algebras. Useful structures arising in soft set theory are both soft rings and soft algebras.

Definition 3.11. *A collection* $\widetilde{\mathcal{S}}$ *of soft subsets of* $\widetilde{X}$ *is called a soft semiring on* $\widetilde{X}$ *if, and only if, it satisfies the following conditions:*

- $\Phi \in \widetilde{\mathcal{S}}$*,*
- *if* $F_A, G_B \in \widetilde{\mathcal{S}}$*, then* $F_A \sqcap G_B \in \widetilde{\mathcal{S}}$*,*
- *if* $F_A, G_B \in \widetilde{\mathcal{S}}$*, then there exist soft disjoint* $(C_H)_1, \ldots, (C_H)_n \in \widetilde{\mathcal{S}}$*,* $n \in \mathbb{N}$*, such that*

$$F_A \backslash G_B = \bigsqcup_{i=1}^{n}(C_H)_i.$$

Theorem 3.12. *Let* $\widetilde{\mathcal{S}}$ *be a soft semiring on* $\widetilde{X}$ *and* $F_A, (G_B)_1, \ldots, (G_B)_n \in \widetilde{\mathcal{S}}$*,* $n \in \mathbb{N}$*. Then there exists* $m \in \mathbb{N}$ *and soft disjoint* $(C_H)_1, \ldots, (C_H)_m \in \widetilde{\mathcal{S}}$ *such that*

$$F_A \backslash \left(\bigsqcup_{i=1}^{n}(G_B)_i\right) = \bigsqcup_{i=1}^{m}(C_H)_i.$$

Proof. Proof by induction on n. The base case holds, since $\widetilde{\mathcal{S}}$ is a soft semiring. For the induction step, let $F_A, (G_B)_1, \ldots, (G_B)_n, (G_B)_{n+1} \in \widetilde{\mathcal{S}}$, $n \in \mathbb{N}$. By the induction hypothesis, there exist soft disjoint $(C_H)_1, \ldots, (C_H)_m \in \widetilde{\mathcal{S}}$ such that $F_A \setminus (\bigsqcup_{i=1}^{n} (G_B)_i) = \bigsqcup_{i=1}^{m} (C_H)_i$.

Then

$$\begin{aligned} F_A \setminus \left(\bigsqcup_{i=1}^{n+1} (G_B)_i \right) &= F_A \sqcap \left(\bigsqcup_{i=1}^{n+1} (G_B)_i \right)^c = F_A \sqcap \left((G_B)_1^c \sqcap (G_B)_2^c \sqcap \cdots \sqcap (G_B)_{n+1}^c \right) \\ &= (F_A \sqcap (G_B)_1^c \sqcap \cdots (G_B)_n^c) \sqcap (G_B)_{n+1}^c = \left(F_A \sqcap \left(\bigsqcup_{i=1}^{n} (G_B)_i \right)^c \right) \sqcap (G_B)_{n+1}^c \\ &= \left(F_A \setminus \left(\bigsqcup_{i=1}^{n} (G_B)_i \right) \right) \sqcap (G_B)_{n+1}^c = \left(\bigsqcup_{i=1}^{m} (C_H)_i \right) \sqcap (G_B)_{n+1}^c \\ &= \bigsqcup_{i=1}^{m} ((C_H)_i \sqcap (G_B)_{n+1}^c) = \bigsqcup_{i=1}^{m} ((C_H)_i \setminus (G_B)_{n+1}). \end{aligned}$$

Since each of the $(C_H)_i \setminus (G_B)_{n+1}$ is a soft disjoint finite union of soft sets from $\widetilde{\mathcal{S}}$, the induction is complete. □

Definition 3.13. *A collection $\widetilde{\mathcal{R}}$ of soft subsets of $\widetilde{X}$ is called a soft ring on $\widetilde{X}$ if, and only if, it satisfies the following conditions:*

- $\Phi \in \widetilde{\mathcal{R}}$,
- *if $F_A, G_B \in \widetilde{\mathcal{R}}$, then $F_A \sqcup G_B \in \widetilde{\mathcal{R}}$,*
- *if $F_A, G_B \in \widetilde{\mathcal{R}}$, then $F_A \setminus G_B \in \widetilde{\mathcal{R}}$.*

Theorem 3.14. *The soft intersection of any collection of soft rings on $\widetilde{X}$ forms again a soft ring on $\widetilde{X}$.*

Proof. Let $\{\widetilde{\mathcal{R}_i}\}_{i \in I}$ be a collection of soft rings on $\widetilde{X}$. Let $\widetilde{\mathcal{R}} = \sqcap_{i \in I} \widetilde{\mathcal{R}_i}$.

Since $\Phi \in \widetilde{\mathcal{R}_i}$ for each $i \in I$, then $\Phi \in \widetilde{\mathcal{R}}$.

If $F_A, G_B \in \widetilde{\mathcal{R}} = \sqcap_{i \in I} \widetilde{\mathcal{R}_i}$, then $F_A, G_B \in \widetilde{\mathcal{R}_i}$ for every $i \in I$. Since each $\widetilde{\mathcal{R}_i}$ is a ring on $\widetilde{X}$, then $F_A \sqcup G_B \in \widetilde{\mathcal{R}_i}$ and $F_A \setminus G_B \in \widetilde{\mathcal{R}_i}$ for every $i \in I$. This implies that $F_A \sqcup G_B \in \widetilde{\mathcal{R}}$ and $F_A \setminus G_B \in \widetilde{\mathcal{R}}$. □

Theorem 3.15. *Let $\widetilde{\mathcal{E}}$ be the collection of soft subsets of $\widetilde{X}$. Then there is a smallest soft ring containing $\widetilde{\mathcal{E}}$.*

Proof. Let $\widetilde{\mathcal{F}} = \{\widetilde{\mathcal{R}} \mid \widetilde{\mathcal{R}}$ is a soft ring and $\widetilde{\mathcal{E}} \sqsubseteq \widetilde{\mathcal{R}}\}$. Then $\widetilde{\mathcal{F}} \neq \Phi$, because $\mathcal{P}(\widetilde{X})$ is a soft ring such that $\widetilde{\mathcal{E}} \sqsubseteq \mathcal{P}(\widetilde{X})$.

Let's write $\rho(\widetilde{\mathcal{R}}) = \sqcap_{\widetilde{\mathcal{R}} \in \widetilde{\mathcal{F}}} \widetilde{\mathcal{R}}$. Since arbitrary soft intersection of soft rings is also a soft rings, $\rho(\widetilde{\mathcal{R}})$ is a soft ring. $\rho(\widetilde{\mathcal{R}})$ is a soft ring containing $\widetilde{\mathcal{E}}$, because $\widetilde{\mathcal{E}} \sqsubseteq \widetilde{\mathcal{R}}$ for all $\widetilde{\mathcal{R}} \in \widetilde{\mathcal{F}}$. Suppose that $\rho_0(\widetilde{\mathcal{R}})$ is a soft ring containing $\widetilde{\mathcal{E}}$. It follows by definition of $\widetilde{\mathcal{F}}$ that $\rho_0(\widetilde{\mathcal{R}}) \in \widetilde{\mathcal{F}}$. Thus, $\sqcap_{\widetilde{\mathcal{R}} \in \widetilde{\mathcal{F}}} \widetilde{\mathcal{R}} \sqsubseteq \rho_0(\widetilde{\mathcal{R}})$ gives that $\rho(\widetilde{\mathcal{R}}) \sqsubseteq \rho_0(\widetilde{\mathcal{R}})$. So $\rho(\widetilde{\mathcal{R}})$ is the smallest soft ring containing $\widetilde{\mathcal{E}}$. □

Definition 3.16. *The smallest soft ring $\rho(\widetilde{\mathcal{R}})$ containing some soft collection $\widetilde{\mathcal{E}}$ of soft subsets of $\widetilde{X}$ (whose existence is guaranteed in Theorem 3.15.) is called the soft ring generated by $\widetilde{\mathcal{E}}$.*

Theorem 3.17. *If $\widetilde{\mathcal{S}}$ is a soft semiring on $\widetilde{X}$ and $\rho(\widetilde{\mathcal{S}}) = \{\bigsqcup_{i=1}^{n} (F_A)_i \mid n \in \mathbb{N}, (F_A)_1, \ldots, (F_A)_n \in \widetilde{\mathcal{S}}$ soft disjoint$\}$, then $\rho(\widetilde{\mathcal{S}})$ is a soft ring.*

Proof. Since $\Phi \in \widetilde{\mathcal{S}}$, then $\Phi \in \rho(\widetilde{\mathcal{S}})$. Let $m, n \in \mathbb{N}$, $(F_A)_1, \ldots, (F_A)_m \in \widetilde{\mathcal{S}}$ soft disjoint, $(G_B)_1, \ldots, (G_B)_n \in \widetilde{\mathcal{S}}$ soft disjoint, $F_A = \bigsqcup_{i=1}^{m}(F_A)_i$ and $G_B = \bigsqcup_{i=1}^{n}(G_B)_i$. We have to show $F_A \sqcup G_B \in \rho(\widetilde{\mathcal{S}})$ and $F_A \backslash G_B \in \rho(\widetilde{\mathcal{S}})$. It is

$$F_A \backslash G_B = \bigsqcup_{i=1}^{m}\left((F_A)_i \backslash \left(\bigsqcup_{j=1}^{n}(G_B)_j\right)\right).$$

Since, by Theorem 3.12., each of the $(F_A)_i \backslash \left(\bigsqcup_{j=1}^{n}(G_B)_j\right)$ is a soft disjoint finite soft union of soft sets from $\widetilde{\mathcal{S}}$, we have $F_A \backslash G_B \in \rho(\widetilde{\mathcal{S}})$. We also have

$$F_A \sqcap G_B = \bigsqcup_{i=1}^{m}\bigsqcup_{j=1}^{n}((F_A)_i \sqcap (G_B)_j) \in \rho(\widetilde{\mathcal{S}}),$$

since each $(F_A)_i \sqcap (G_B)_j \in \widetilde{\mathcal{S}}$ by Definition 3.11.

Thus,

$$F_A \sqcup G_B = (F_A \backslash G_B) \sqcup (G_B \backslash F_A) \sqcup (F_A \sqcap G_B) \in \rho(\widetilde{\mathcal{S}})$$

completing the proof that $\rho(\widetilde{\mathcal{S}})$ is a soft ring. □

Example 3.18. *Let $X = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2\}$. Assume that*
$(F_A)_1 = \Phi$,
$(F_A)_2 = \{(e_1, \{h_1\}), (e_2, \emptyset)\}$,
$(F_A)_3 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_2\})\}$,
Then $\widetilde{\mathcal{S}} = \{(F_A)_i \mid i = 1, 2, 3\}$ is a soft semiring over X.
Since,

$$(F_A)_2 \sqcup (F_A)_3 = \{(e_1, \{h_1, h_2, h_3\}), (e_2, \{h_1, h_2\})\} \notin \widetilde{\mathcal{S}},$$

a collection $\widetilde{\mathcal{S}}$ is not a soft ring over X.

Now, if we use the Theorem 3.17. we have that it is a collection

$$\rho(\widetilde{\mathcal{S}}) = \{(F_A)_1, (F_A)_2, (F_A)_3, (F_A)_4\}$$

is a soft ring over X, where is $(F_A)_4 = \{(e_1, \{h_1, h_2, h_3\}), (e_2, \{h_1, h_2\})\}$.

Definition 3.19. *A collection $\widetilde{\mathcal{A}}$ of soft subsets of $\widetilde{X}$ is called a soft algebra on $\widetilde{X}$ if, and only if, $\widetilde{\mathcal{A}}$ is a soft ring and $\widetilde{X} \in \widetilde{\mathcal{A}}$.*

Example 3.20. *It is obvious that the collection $\rho(\widetilde{\mathcal{S}})$ from the example above is not a soft algebra over X. Assume that*
$(F_A)_1 = \Phi$,
$(F_A)_2 = \{(e_1, \{h_1\}), (e_2, \emptyset)\}$,
$(F_A)_3 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_2\})\}$,
$(F_A)_4 = \{(e_1, \{h_1, h_2, h_3\}), (e_2, \{h_1, h_2\})\}$,
$(F_A)_5 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_2, h_3\})\}$,
$(F_A)_6 = \{(e_1, \{h_1\}), (e_2, \{h_3\})\}$,
$(F_A)_7 = \{(e_1, \emptyset), (e_2, \{h_3\})\}$,
$(F_A)_8 = \widetilde{X}$.
Then $\widetilde{\mathcal{A}} = \{(F_A)_i \mid i = 1, 2, 3, 4, 5, 6, 7, 8\}$ is a soft algebra over X.

Definition 3.21. *Let $\widetilde{\mathcal{S}}$ be a soft semiring of soft subsets of a set $\widetilde{X}$ and $\widetilde{\nu}$ be an extended soft real-valued mapping on $\widetilde{\mathcal{S}}$. Then $\widetilde{\nu}$ is called a soft content on $\widetilde{\mathcal{S}}$, if*

- $\widetilde{\nu}(\Phi) = 0$,
- *$\widetilde{\nu}$ is finitely soft additive, i.e., if $n \in \mathbb{N}$ and $(F_A)_1, (F_A)_2, \ldots, (F_A)_n \in \widetilde{\mathcal{S}}$ are soft disjoint soft sets such that $\bigsqcup_{i=1}^{n}(F_A)_i \in \widetilde{\mathcal{S}}$, then*
$$\widetilde{\nu}\left(\bigsqcup_{i=1}^{n}(F_A)_i\right) = \sum_{i=1}^{n}\widetilde{\nu}((F_A)_i).$$

The soft content $\widetilde{\nu}$ is called soft finite or soft bounded if, and only if, $\widetilde{\nu}(F_A) < \infty$ for each $F_A \in \widetilde{\mathcal{S}}$; it is called soft σ-finite if, and only if, there exists a sequence $((F_A)_i)_{i\in\mathbb{N}}$ in $\widetilde{\mathcal{S}}$ such that $\widetilde{X} = \bigsqcup_{i=1}^{\infty}(F_A)_i$ and $\widetilde{\nu}((F_A)_i) < \infty$ for each $i \in \mathbb{N}$.

Definition 3.22. *A soft content $\widetilde{\nu}$ is called a soft premeasure if, and only if, it is countably soft additive (also called soft σ-additive), i.e., if $((F_A)_i)_{i\in\mathbb{N}}$ is a sequence in $\widetilde{\mathcal{S}}$ consisting of soft disjoint soft sets such that $\bigsqcup_{i=1}^{n}(F_A)_i \in \widetilde{\mathcal{S}}$, then*

$$\widetilde{\nu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) = \sum_{i=1}^{\infty}\widetilde{\nu}((F_A)_i).$$

Example 3.23. *A soft measure is a soft premeasure that is defined on a soft σ-algebra.*

As an important result of this paper, we present a theorem showing that any soft content over a soft semiring can be uniquely extended to a soft content on the generated soft ring.

Theorem 3.24. *Let $\widetilde{\mathcal{S}}$ be a soft semiring of soft subsets of a set $\widetilde{X}$ and let $\widetilde{\mu} : \widetilde{\mathcal{S}} \to [0,\infty]$ be a soft content. Furthermore, let $\widetilde{\mathcal{R}} = \rho(\widetilde{\mathcal{S}})$. Then there is a unique extension $\widetilde{\nu} : \widetilde{\mathcal{R}} \to [0,\infty]$ of $\widetilde{\mu}$ such that $\widetilde{\nu}$ is a soft content. Moreover $\widetilde{\nu}$ is a soft premeasure if, and only if, $\widetilde{\mu}$ is a soft premeasure.*

Proof. Uniqueness: Assume $\widetilde{\nu}$ is a soft content on $\widetilde{\mathcal{R}}$ such that $\widetilde{\nu} \restriction_{\widetilde{\mathcal{S}}} = \widetilde{\mu}$. If $F_A \in \widetilde{\mathcal{R}}$, then, by Theorem 3.17. there exist soft disjoint $(F_A)_1, (F_A)_2, \ldots, (F_A)_n \in \widetilde{\mathcal{S}}$, $n \in \mathbb{N}$, such that $F_A = \bigsqcup_{i=1}^{n}(F_A)_i$. Thus, as $\widetilde{\nu}$ is finitely soft additive, $\widetilde{\nu}(F_A) = \sum_{i=1}^{n}\widetilde{\mu}((F_A)_i)$ showing that $\widetilde{\nu}$ is uniquely determined by $\widetilde{\mu}$.

Existence: As we have seen that $\widetilde{\nu}(F_A) = \sum_{i=1}^{n}\widetilde{\mu}((F_A)_i)$ must hold if F_A and $(F_A)_1, (F_A)_2, \ldots, (F_A)_n$ are as above, we use this formula to define the function $\widetilde{\nu}$ on $\widetilde{\mathcal{R}}$. We now need to verify that $\widetilde{\nu}$ is well-defined, i.e. that, if we pick, possibly different, soft disjoint $(G_B)_1, (G_B)_2, \ldots, (G_B)_m \in \widetilde{\mathcal{S}}$ with $F_A = \bigsqcup_{j=1}^{m}(G_B)_j$, then $\sum_{i=1}^{n}\widetilde{\mu}((F_A)_i) = \sum_{j=1}^{m}\widetilde{\mu}((G_B)_j)$. Indeed, we have, since $\widetilde{\mu}$ is a soft content on $\widetilde{\mathcal{S}}$,

$$\begin{aligned}
\sum_{i=1}^{n}\widetilde{\mu}((F_A)_i) &= \sum_{i=1}^{n}\widetilde{\mu}\left((F_A)_i \sqcap \left(\bigsqcup_{j=1}^{m}(G_B)_j\right)\right) = \sum_{i=1}^{n}\widetilde{\mu}\left(\bigsqcup_{j=1}^{m}((F_A)_i \sqcap (G_B)_j)\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\widetilde{\mu}\big((F_A)_i \sqcap (G_B)_j\big) = \sum_{j=1}^{m}\widetilde{\mu}\left(\bigsqcup_{i=1}^{n}((F_A)_i \sqcap (G_B)_j)\right)\\
&= \sum_{j=1}^{m}\widetilde{\mu}\left((G_B)_j \sqcap \left(\bigsqcup_{i=1}^{n}(F_A)_i\right)\right)\\
&= \sum_{j=1}^{m}\widetilde{\mu}((G_B)_j)
\end{aligned}$$

proving that $\widetilde{\nu}$ is well-defined. Then $\widetilde{\nu} \restriction_{\widetilde{\mathcal{S}}} = \widetilde{\mu}$ is also clear. To prove that $\widetilde{\nu}$ is a soft content, we still need to check that it is finitely soft additive. To this end, let $F_A, G_B \in \widetilde{\mathcal{R}}$ be soft disjoint. Then there are soft disjoint

$(F_A)_1, (F_A)_2, \ldots, (F_A)_n, (G_B)_1, (G_B)_2, \ldots, (G_B)_m \in \widetilde{\mathcal{S}}$, $n, m \in \mathbb{N}$, such that $F_A = \bigsqcup_{i=1}^{n} (F_A)_i$ and $G_B = \bigsqcup_{j=1}^{m} (G_B)_j$. In consequence,

$$\widetilde{\nu}(F_A \sqcup G_B) = \sum_{i=1}^{n} \widetilde{\mu}((F_A)_i) + \sum_{j=1}^{m} \widetilde{\mu}((G_B)_j) = \widetilde{\nu}(F_A) + \widetilde{\nu}(G_B),$$

as needed.

Since $\widetilde{\nu} \upharpoonright_{\widetilde{\mathcal{S}}} = \widetilde{\mu}$, if $\widetilde{\nu}$ is a soft premeasure, so is $\widetilde{\mu}$. For the converse, assume $\widetilde{\mu}$ to be a soft premeasure and let $(F_A)_1, (F_A)_2, \ldots$ be a sequence of soft disjoint sets in $\widetilde{\mathcal{R}}$ such that $F_A = \bigsqcup_{i=1}^{\infty} (F_A)_i \in \widetilde{\mathcal{R}}$. Then there are soft disjoint sets $(G_B)_1, (G_B)_2, \ldots, (G_B)_n \in \widetilde{\mathcal{S}}$, $n \in \mathbb{N}$, with $F_A = \bigsqcup_{i=1}^{n} (G_B)_i$, and, for each $i \in \mathbb{N}$, there are soft disjoint sets $(H_C)_{i1}, (H_C)_{i2}, \ldots, (H_C)_{in_i} \in \widetilde{\mathcal{S}}$, $n_i \in \mathbb{N}$ with $(F_A)_i = \bigsqcup_{j=1}^{n_i} (H_C)_{ij}$. Then, for all $i \in \{1, 2, \ldots, n\}$

$$(G_B)_i = (G_B)_i \sqcap F_A = \bigsqcup_{j=1}^{\infty} ((G_B)_i \sqcap (F_A)_j) = \bigsqcup_{j=1}^{\infty} \bigsqcup_{k=1}^{n_j} \underbrace{((G_B)_i \sqcap (H_C)_{jk})}_{\in \widetilde{\mathcal{S}}}.$$

Thus, as $\widetilde{\mu}$ is a soft premeasure, for all $i \in \{1, 2, \ldots, n\}$

$$\widetilde{\mu}((G_B)_i) = \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \widetilde{\mu}((G_B)_i \sqcap (H_C)_{jk}) = \sum_{j=1}^{\infty} \widetilde{\nu}((G_B)_i \sqcap (F_A)_j),$$

implying

$$\begin{aligned} \widetilde{\nu}(F_A) &= \sum_{i=1}^{n} \widetilde{\mu}((G_B)_i) = \sum_{i=1}^{n} \sum_{j=1}^{\infty} \widetilde{\nu}((G_B)_i \sqcap (F_A)_j) \\ &= \sum_{j=1}^{\infty} \widetilde{\nu}(F_A \sqcap (F_A)_j) = \sum_{j=1}^{\infty} \widetilde{\nu}((F_A)_j), \end{aligned}$$

proving $\widetilde{\nu}$ to be a soft premeasure. □

As in the classical measure theory, with soft measures we want to extend soft premeasures to mappings that will be soft measures. An essential tool for achieving this goal is the special mappings introduced in article [27], called soft outer measures.

Definition 3.25. *[27] A non-negative soft extended real-valued set function $\widetilde{\mu}^*$ defined on $\mathcal{P}(\widetilde{X})$ is called a soft outer measure, if $\widetilde{\mu}^*$ satisfies the following conditions (1) - (3):*

1. $\widetilde{\mu}^*(\Phi) = 0$.
2. *$\widetilde{\mu}^*$ is soft monotone, i.e., $F_A \sqsubseteq G_B \ \Rightarrow\ \widetilde{\mu}^*(F_A) \leqslant \widetilde{\mu}^*(G_B)$.*
3. *$\widetilde{\mu}^*$ is countably soft sub-additive (also called σ-subadditive), i.e.,*

$$\widetilde{\mu}^*\left(\bigsqcup_{i=1}^{\infty} (F_A)_i\right) \leqslant \sum_{i=1}^{\infty} \widetilde{\mu}^*((F_A)_i).$$

Remark 3.26. *Since we know soft measures to be soft monotone and soft σ-subadditive, every soft measure on $\mathcal{P}(\widetilde{X})$ is a soft outer measure. However, the converse is not true in general: For example*

$$\widetilde{\mu}^* : \mathcal{P}(\widetilde{X}) \to [0, \infty], \quad \widetilde{\mu}^*(F_A) = \begin{cases} 0, & F_A = \Phi, \\ 1, & F_A \neq \Phi, \end{cases}$$

defines a soft outer measure that is not a soft measure if X has more than one element.

Definition 3.27. *We call $F_A \in \mathcal{P}(\widetilde{X})$ soft $\widetilde{\mu}^*$-measurable if, and only if,*

$$(\forall Q_P \in \mathcal{P}(\widetilde{X}))\widetilde{\mu}^*(Q_P) \geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c).$$

Lemma 3.28. *Let $\widetilde{\mu}^*$ be a soft outer measure and $F_A \in \mathcal{P}(\widetilde{X})$.*

1. *If $\widetilde{\mu}^*(F_A) = 0$ or $\widetilde{\mu}^*(F_A^c) = 0$, then F_A is soft $\widetilde{\mu}^*$-measurable.*
2. *F_A is soft $\widetilde{\mu}^*$-measurable if, and only if,*

$$(\forall Q_P \in \mathcal{P}(\widetilde{X}))\widetilde{\mu}^*(Q_P) = \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c).$$

Proof. (1) If $\widetilde{\mu}^*(F_A) = 0$, then the soft monotonicity of $\widetilde{\mu}^*$ implies, for each $Q_P \in \mathcal{P}(\widetilde{X})$, $\widetilde{\mu}^*(Q_P \sqcap F_A) = 0$ and, thus $\widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c) = \widetilde{\mu}^*(Q_P \sqcap F_A^c) \leqslant \widetilde{\mu}^*(Q_P)$, showing F_A to be soft $\widetilde{\mu}^*$-measurable.

Analogously, we see F_A to be soft $\widetilde{\mu}^*$-measurable for $\widetilde{\mu}^*(F_A^c) = 0$.

(2) Is clear, since $\leqslant$ always holds in Definition 3.27. by the soft subadditivity of $\widetilde{\mu}^*$. □

Theorem 3.29. *Let $\widetilde{\mu}^*$ be a soft outer measure. Define*

$$\mathcal{A}_{\widetilde{\mu}^*} = \{F_A \sqsubseteq \widetilde{X} \mid F_A \text{ is soft } \widetilde{\mu}^*\text{-measurable}\}.$$

Then $\mathcal{A}_{\widetilde{\mu}^*}$ is a soft σ-algebra and $\widetilde{\mu}^* \restriction_{\mathcal{A}_{\widetilde{\mu}^*}}$ is a soft measure.

Proof. We first show $\mathcal{A}_{\widetilde{\mu}^*}$ to be a soft algebra: Let $Q_P \sqsubseteq \widetilde{X}$. Since

$$\widetilde{\mu}^*(Q_P) = \widetilde{\mu}^*(Q_P \sqcap \widetilde{X}) + \widetilde{\mu}^*(Q_P \sqcap \Phi),$$

$\Phi \in \mathcal{A}_{\widetilde{\mu}^*}$ and $\widetilde{X} \in \mathcal{A}_{\widetilde{\mu}^*}$.

It is immediate from Definition 3.27. that $F_A \in \mathcal{A}_{\widetilde{\mu}^*}$ implies $F_A^c \in \mathcal{A}_{\widetilde{\mu}^*}$.

Now let $F_A, G_B \in \mathcal{A}_{\widetilde{\mu}^*}$. Then

$$\begin{aligned}
\widetilde{\mu}^*(Q_P) &\geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c)\\
&\geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c \sqcap G_B) + \widetilde{\mu}^*(Q_P \sqcap F_A^c \sqcap G_B^c)\\
&\geqslant \widetilde{\mu}^*((Q_P \sqcap F_A) \sqcup (Q_P \sqcap F_A^c \sqcap G_B)) + \widetilde{\mu}^*(Q_P \sqcap (F_A \sqcup G_B)^c)\\
&= \widetilde{\mu}^*(Q_P \sqcap (F_A \sqcup G_B)) + \widetilde{\mu}^*(Q_P \sqcap (F_A \sqcup G_B)^c)
\end{aligned}$$

proving $F_A \sqcup G_B \in \mathcal{A}_{\widetilde{\mu}^*}$. Now

$$F_A \backslash G_B = F_A \sqcap G_B^c = (F_A^c \sqcap G_B)^c \in \mathcal{A}_{\widetilde{\mu}^*},$$

establishing $\mathcal{A}_{\widetilde{\mu}^*}$ to be a soft algebra.

Now let $(F_A)_1, (F_A)_2, \ldots, (F_A)_n, \ldots$ be a sequence of soft disjoint sets in $\mathcal{A}_{\widetilde{\mu}^*}$ and let $F_A = \bigsqcup_{i=1}^{\infty}(F_A)_i$. We claim that $F_A \in \mathcal{A}_{\widetilde{\mu}^*}$ and

$$\widetilde{\mu}^*(F_A) = \sum_{i=1}^{\infty} \widetilde{\mu}^*((F_A)_i).$$

To prove the claim, fix $Q_P \sqsubseteq \widetilde{X}$. We first show for all $n \in \mathbb{N}$

$$\widetilde{\mu}^*\left(Q_P \sqcap \left(\bigsqcup_{i=1}^{n}(F_A)_i\right)\right) = \sum_{i=1}^{n} \widetilde{\mu}^*(Q_P \sqcap (F_A)_i),$$

via induction on n.

For $n = 1$, there is nothing to prove.

For $n > 1$, we apply $\widetilde{\mu}^*(Q_P) = \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c)$ with $Q_P \sqcap (\bigsqcup_{i=1}^n (F_A)_i)$ instead of Q_P and $\bigsqcup_{i=1}^{n-1}(F_A)_i$ instead of F_A (we know $\bigsqcup_{i=1}^{n-1}(F_A)_i \in \mathcal{A}_{\widetilde{\mu}^*}$, as $\mathcal{A}_{\widetilde{\mu}^*}$ is a soft algebra) to obtain

$$\begin{aligned}\widetilde{\mu}^*\left(Q_P \sqcap \left(\bigsqcup_{i=1}^n (F_A)_i\right)\right) &= \widetilde{\mu}^*\left(Q_P \sqcap \left(\bigsqcup_{i=1}^{n-1} (F_A)_i\right)\right) + \widetilde{\mu}^*(Q_P \sqcap (F_A)_n)\\ &= \sum_{i=1}^n \widetilde{\mu}^*(Q_P \sqcap (F_A)_i),\end{aligned}$$

as needed. Now we prove $F_A \in \mathcal{A}_{\widetilde{\mu}^*}$ and

$$\widetilde{\mu}^*(F_A) = \sum_{i=1}^\infty \widetilde{\mu}^*((F_A)_i).$$

For each $n \in \mathbb{N}$, since

$$\bigsqcup_{i=1}^n (F_A)_i \in \mathcal{A}_{\widetilde{\mu}^*}$$

we can apply $\widetilde{\mu}^*(Q_P \sqcap (\bigsqcup_{i=1}^n (F_A)_i)) = \sum_{i=1}^n \widetilde{\mu}^*(Q_P \sqcap (F_A)_i)$, to estimate

$$\begin{aligned}\widetilde{\mu}^*(Q_P) &\geqslant \widetilde{\mu}^*\left(Q_P \sqcap \left(\bigsqcup_{i=1}^n (F_A)_i\right)\right) + \widetilde{\mu}^*\left(Q_P \sqcap \left(\bigsqcup_{i=1}^n (F_A)_i\right)^c\right)\\ &\geqslant \sum_{i=1}^n \widetilde{\mu}^*(Q_P \sqcap (F_A)_i) + \widetilde{\mu}^*(Q_P \sqcap F_A^c),\end{aligned}$$

implying

$$\begin{aligned}\widetilde{\mu}^*(Q_P) &\geqslant \sum_{i=1}^n \widetilde{\mu}^*(Q_P \sqcap (F_A)_i) + \widetilde{\mu}^*(Q_P \sqcap F_A^c)\\ &\geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c)\\ &\geqslant \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c)\\ &\geqslant \widetilde{\mu}^*(Q_P).\end{aligned}$$

Thus, all terms must be equal, proving $F_A \in \mathcal{A}_{\widetilde{\mu}^*}$ and, for $Q_P = F_A$, $\widetilde{\mu}^*(F_A) = \sum_{i=1}^\infty \widetilde{\mu}^*((F_A)_i)$.

Due to $\widetilde{\mu}^*(F_A) = \sum_{i=1}^\infty \widetilde{\mu}^*((F_A)_i)$, $\widetilde{\mu}^* \upharpoonright_{\mathcal{A}_{\widetilde{\mu}^*}}$ is a soft measure, provided $\mathcal{A}_{\widetilde{\mu}^*}$ is a soft σ-algebra. So, finally, let be $(F_A)_1, (F_A)_2, \ldots, (F_A)_n, \ldots$ a sequence in $\mathcal{A}_{\widetilde{\mu}^*}$ (now the $(F_A)_i$ do not have to be soft disjoint) and let $F_A = \bigsqcup_{i=1}^\infty (F_A)_i$. Then

$$F_A = \bigsqcup_{i=1}^\infty \underbrace{\left((F_A)_i \setminus \left(\bigsqcup_{k=1}^{i-1} (F_A)_k\right)\right)}_{\in \mathcal{A}_{\widetilde{\mu}^*}} \in \mathcal{A}_{\widetilde{\mu}^*}$$

showing $\mathcal{A}_{\widetilde{\mu}^*}$ to be a soft σ-algebra, completing the proof of the proposition. □

Example 3.30. *Let $\widetilde{\mu}^* : \mathcal{P}(\widetilde{X}) \to [0, \infty]$ be defined as*

$$\widetilde{\mu}^*(F_A) = \begin{cases} 0, & F_A = \Phi, \\ 1, & F_A \neq \Phi. \end{cases}$$

Then clearly, $\widetilde{\mu}^*$ is a soft outer measure (see [27]). If $(F_A)_i$ consists of more than one soft point, then $\widetilde{\mu}^*$ is not countably soft additive. For an illustration, let $X = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2\}$. Suppose that

$(F_A)_1 = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$,

$(F_A)_2 = \{(e_1, \{h_2\}), (e_2, \{h_1\})\}$ *and*

$(F_A)_3 = \{(e_1, \{h_3\}), (e_2, \{h_3\})\}$. *Clearly,* $\widetilde{\mathcal{A}} = \{(F_A)_1, (F_A)_2, (F_A)_3\}$ *is not a soft σ-algebra and*

$$\widetilde{\mu}^*\left(\bigsqcup_{i=1}^{3}(F_A)_i\right) \neq \sum_{i=1}^{3}\widetilde{\mu}^*((F_A)_i).$$

By the Theorem 3.29. we have

$$\mathcal{A}_{\widetilde{\mu}^*} = \{\Phi, \widetilde{X}\}.$$

Then $\mathcal{A}_{\widetilde{\mu}^*}$ is a soft σ-algebra and $\widetilde{\mu}^* \upharpoonright_{\mathcal{A}_{\widetilde{\mu}^*}}$ is a soft measure.

4. Conclusion

Molodtsov presented several possible applications of soft set theory [20], and many applied soft sets in different areas. This paper continues research in the field of soft measure theory and defines specific concepts such as soft content and soft premeasure. Hopefully, properties given in this paper will help many researchers improve and promote the soft set theory, especially soft measure and soft outer measure. This paper creates an important framework for an even better application of soft set theory.

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