Filomat 36:6 (2022), 2105–2117
https://doi.org/10.2298/FIL2206105O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Results for Robin type Problem Involving $p(x)-$Laplacian

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Abstract. This work deals with the existence and multiplicity of solutions for $p(x)$-Laplacian Robin problem without the well-known Ambrosetti-Rabinowitz type growth conditions. The uniqueness of solution is also established under some new sufficient conditions.

1. Introduction

In recent years, the study of differential equations and variational problems with variable exponent growth conditions has been an interesting topic. There are several applications concerning elastic materials, image restoration (see [33]), thermorheological and electrorheological fluids (see [29]) and also mathematical biology [16]. For the advances of the study of differential equations with variable exponents see the overview paper [18].

In this paper, we discuss the existence and multiplicity of solutions for the following Robin problem involving the $p(x)-$Laplacian

$$\begin{aligned} -\Delta_{p(x)}u = f(x,u) \quad &\text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p \in C_+(\overline{\Omega})$ where

$$C_+(\overline{\Omega}) := \Big\{p \in C(\overline{\Omega}) : \quad p^- := \inf_{x\in\overline{\Omega}} p(x) > 1\Big\},$$

$\beta \in L^\infty(\Omega)$ with $\beta^- := \inf_{x\in\Omega}\beta(x) > 0$, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplace operator and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & if \;\; p(x) < N, \\ +\infty & if \;\; p(x) \geq N. \end{cases}$$

The problem (1) is regarded in case of $f \in C(\overline{\Omega} \times \mathbb{R})$ and $1 < p^- \leq p^+ < \infty$.

Now, let us assume that f satisfy the following conditions:

2020 *Mathematics Subject Classification*. Primary 35J25, 46E35, 35D30, 35J20.
Keywords. variable exponent, generalized Sobolev spaces
Received: 10 February 2020; Accepted: 15 February 2020
Communicated by Maria Alessandra Ragusa
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(H_1) There exist $C > 0$ and $q \in C_+(\overline{\Omega})$ with $p^+ < q^- \leq q^+ < p^*(x)$ for all $x \in \overline{\Omega}$, such that f verifies

$$|f(x,s)| \leq C\Big(1 + |s|^{q(x)-1}\Big)$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$ and $f(x,t) = f(x,0) = 0$ $\forall x \in \Omega,\ t \leq 0$.

(H_2) $\lim\limits_{t\to 0} \dfrac{f(x,t)}{|t|^{p^+-1}} = l_1 < \infty$, $\lim\limits_{t\to\infty} \dfrac{f(x,t)t}{|t|^{p^+}} = \infty$.

(H_3) For a.e $x \in \Omega$, $\frac{f(x,t)}{t^{p^+-1}}$ is nondecreasing with respect to $t \geq 0$.

So, we can report our first main result:

Theorem 1.1. *1) Assume $(H_1), (H_2)$ and (H_3) hold, then problem (1) has at least a nontrivial solution.*
2)Suppose (H_1)–(H_3) are satisfied, for simplicity taking $l_1 = 0$ in condition (H_2).
Moreover, we assume that

(H_4) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.

If $q^- > p^+$, then problem (1) has a sequence of weak solutions $\{\pm u_k\}_{k=1}^{\infty}$ such that $I(\pm u_k) \to +\infty$ as $k \to +\infty$.

A lot of works have been interested in the existence of solutions for elliptic problems in this direction. For instance, we refer [1–6, 8, 10, 12, 14, 17, 20, 22–24]... and the reference therein. To be more closer to the topic, let us mention some work on the subject.

From the variational point of view, the authors in [32], have studied Robin problems involving the p-Laplacian, they proved at least four nontrivial solutions.

Papageorgiou and Radulescu studied in [25] the following problem

$$\begin{aligned} -\Delta_{p(x)}u &= \lambda f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \alpha(x)|u|^{p(x)-2}u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

so by using the truncation techniques, they proved a bifurcation-type result describing the set of positive solutions when the positive parameter λ varies.

In [8], by applying the sub-supersolution method and the variational method, the author obtained at least two positive solutions for problem (1) under $(\mathbf{H_1})$ and the following conditions:

$(\mathbf{AR})$ There exist $M > 0$ and $\theta > p^+$ such that

$$0 < \theta F(x,s) \leq f(x,s)s, \quad |s| \geq M, \quad x \in \overline{\Omega},$$

where $F(x,t) = \int_0^t f(x,s)ds$ for $x \in \Omega$ and $t \in \mathbb{R}$.

The works [7] and [20] considered problem (1) with a particular nonlinearity such that

$$f(x,t) = \lambda V(x)|u|^{q(x)-2}u\ .$$

For example, in [7], it shows the existence of a family of eigenvalues in a neighborhood of the origin.

Tsouli et al in [30] consider the following problem

$$\begin{aligned} -\Delta_{p(x)}u &= \lambda f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \alpha(x)|u|^{p(x)-2}u &= g(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{3}$$

Under Ambrosetti– Rabinowitz type conditions on the nonlinear terms f and g, the authors obtained some existence and multiplicity results for this problem.

It is well known that (AR) condition defined by

$$(AR) \;\; p^+F(x,s) \leq f(x,s)s \;\; a.e\; x \in \Omega,$$

plays a crucial rule to guarantee that every (PS) sequence of associated functional is bounded in X. The interesting point lines in the fact that we do not need the usual Ambrosetti-Rabinowitz type condition (AR) under various assumptions on f and by a different method, as well as the assumption (H_2) is more weaken than (AR) condition. That is why, according to our knowledge, the current work is a first contribution in this direction with Robin boundary condition, then it is more interesting.

On other hand, we observe that the uniqueness for problem (1) have rarely been considered, then our next theorem gives the uniqueness of solution.

Theorem 1.2. *Suppose that the following assumption holds:*

(H_5) $f(x,u)$ is nonincreasing with respect to the second variable, for all $x \in \overline{\Omega}$, and $f(x,0) \geq 0$ with $f(x,0) \not\equiv 0$ for all $x \in \overline{\Omega}$. Then problem (1) *has a unique solution which is nontrivial.*

Finally, we deal with the nonlinearity $f(x,u) = \varepsilon h(x)g(u)$, and $\varepsilon > 0$ is small enough, we give some sufficient conditions to assure the existence of a positive solution to the problem provided h is sign-changing in Ω. So we have the third main result,

Theorem 1.3. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous with $g(0) > 0$, and*

(H_6) *there exist $\varepsilon > 0$ and $\theta > 0$ such that*

$$\lambda\Big(h^+ - (1+\varepsilon)h^-\Big) \in \Gamma_+, \;\; for\; \lambda \in (0,\theta],$$

where $\Gamma_+ = \{h \in L^\infty(\Omega) \; : \; A^{-1}h(x) > 0\}$. (see the definition of A below in the proof)

Then, problem (1) *has a positive solution.*

The remainder of the paper is organized as follows. In Section 2, we will recall the definitions and some properties of variable exponent Sobolev spaces. In Section 3, we shall establish the results of existence and uniqueness of a solution for problem (1).

2. Preliminaries

In the sequel, let us define $A = A_{p(.)} : \; W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ by

$$A(u)v = \int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla v\, dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\, d\sigma, \;\; u \;\; v \in W^{1,p(x)}(\Omega).$$

Denoting by T the inverse mapping of A, which means that $A^{-1} = T : \; (W^{1,p(x)}(\Omega))^* \to \; W^{1,p(x)}(\Omega)$ is also a strictly monotone homeomorphism, to lack of simplicity, noticing that T can be regarded as the solution operator for the following problem

$$\begin{aligned} -\Delta_{p(x)}u = b(x) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

that is $T(b(x))$ is the solution of the last problem (4), for $b \geq 0$ and $b \in L^\infty(\Omega)$.

Define $\Gamma_+ = \{b \in L^\infty(\Omega) \; : \; T(b) > 0\}$.

To discuss problem (4), we need some theory of variable exponent Lebesgue-Sobolev spaces. For convenience, we only recall some basic facts which will be used later. For details, we refer to [11, 15, 19, 21, 26–28].

For $p \in C_+(\overline{\Omega})$, we designates the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\}$$

equipped with the so called Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}dx \leq 1\Big\}.$$

Proposition 2.1. *If $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a carathéodory function and satisfies $|f(x,t)| \leq a(x) + b|t|^{\frac{p_1(x)}{p_2(x)}}$ for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$, where $p_i \in C_+(\overline{\Omega}, i = 1,2$, $a \in L^{p_2(x)}(\overline{\Omega}), a(x) \geq 0$ and $b \geq 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\overline{\Omega})$ to $L^{p_2(x)}(\overline{\Omega})$ defined by $N_f(u)(x) = f(x,u(x))$ is a continuous and bounded operator.*

As in the constant exponent case, the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

with the norm

$$\|u\|_1 = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

With such norms, $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

Let $\alpha \in L^\infty(\Omega)$, $\alpha^- := \inf_{\partial\Omega} \alpha(x) > 0$ and for any $u \in W^{1,p(x)}(\Omega)$ define

$$\|u\| = \inf\Big\{\tau > 0 \ : \ \int_\Omega \Big|\frac{\nabla u}{\tau}\Big|^{p(x)}dx + \int_{\partial\Omega} \alpha(x)\Big|\frac{u}{\tau}\Big|^{p(x)} d\sigma_x \leq 1\Big\}.$$

According to Theorem 2.1 in [8], $\|.\|$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to standard norm $\|.\|_1$.

Proposition 2.2. *Let $\rho(u) = \int_\Omega |\nabla u|^{p(x)}dx + \int_{\partial\Omega} \alpha(x)|u|^{p(x)} d\sigma_x$. For $u, u_n \in W^{1,p(x)}(\Omega), n = 1,2,...$, we have*

1. $\rho\Big(u\big/|u|_{p(x)}\Big) = 1.$
2. $\|u\| < 1(= 1, > 1) \iff \rho(u) < 1(= 1 > 1).$
3. $\|u\| < 1 \implies \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}.$
4. $\|u\| > 1 \implies \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}.$
5. *Then the following statements are equivalent each other:*
   (a) $\lim_{n\to\infty} \|u_n - u\| = 0.$
   (b) $\lim_{n\to\infty} \rho(u_n - u) = 0.$
   (c) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = I(u).$

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf_{x\in A} p(x), p^+(A) = \sup_{x\in A} p(x)$.

Recall the following embedding theorem.

Theorem 2.3. *If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*(x)$ (resp. $q(x) < p^*(x)$) for $x \in \overline{\Omega}$, then there is a continuous (resp.compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Let us recall the following interesting result:

Proposition 2.4. *Let X a Banach space. If $J \in C^1(X,\mathbb{R})$ is bounded from below and satisfies (PS) condition, then $c = \inf_X J$ is a critical value of J.*

Related to problem (1), the associated functional $I: X \to \mathbb{R}$ is given by

$$I(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x) \frac{1}{p(x)} |u_n|^{p(x)}\, d\sigma - \int_\Omega F(x,u)\, dx.$$

From the continuous embedding

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega), \ \ \forall s(x) \in [1, p^*(x)],$$

which implies that $I \in C^1(W^{k,p(x)}(\Omega), \mathbb{R})$.

Proposition 2.5. *Putting*

$$\phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u)|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x) \frac{1}{p(x)} |u|^{p(x)}\, d\sigma,$$

then $\phi \in C^1(X, \mathbb{R})$ derivative operator ϕ' of ϕ is

$$\phi'(u).v = \int_\Omega |\nabla u|^{p(x)-2} \nabla u . \nabla v\, dx + \int_{\partial\Omega} \beta(x) \frac{1}{p(x)} |u|^{p(x)-2} uv\, d\sigma.$$

(i) The functional ϕ' is of (S_+) type, where ϕ' is the Gâteaux derivative of the functional ϕ.
(ii) $\phi' : X \to X^*$ is a bounded homeomorphism and strictly monotone operator.

The proof is similar to that in [14] with slight modification.

3. proofs

Lemma 3.1. *Assume that $(H_1), (H_2)$ and (H_3) hold, then*
i) There exists $v \in X$ with $v > 0$ such that $I(tv) \to -\infty$ as $t \to \infty$.
ii) There exist $\alpha, \delta > 0$ such that $I(u) \geq \delta$ for all $u \in X$ with $\|u\| = \alpha$.

Proof. i) In view of the condition (H_2), we may choose a constant $K > 0$ such that

$$F(x,s) > K|s|^{p^+} \qquad \text{uniformly in} \quad x \in \Omega, \ \ |s| > C_K. \tag{5}$$

Let $t > 1$ large enough and $v \in X$ with $v > 0$, from (5) we get

$$\begin{aligned} I(tv) &\leq \int_\Omega \frac{1}{p(x)} |\nabla tv|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x) \frac{1}{p(x)} |tv|^{p(x)}\, d\sigma - \int_{|tv|>C_K} F(x,tv) dx - \int_{|tv|\leq C_K} F(x,tv) dx \\ &\leq t^{p^+} \frac{1}{p^-} \Big(\int_\Omega |\nabla v|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x) |v|^{p(x)}\, d\sigma \Big) - K t^{p^+} \int_\Omega |v|^{p^+} dx - \int_{|tv|\leq C_K} F(x,tv) dx \\ &\leq t^{p^+} \frac{1}{p^-} \Big(\int_\Omega |\nabla v|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x) |v|^{p(x)}\, d\sigma \Big) - K t^{p^+} \int_\Omega |v|^{p^+} dx + C_1, \end{aligned}$$

where $C_1 > 0$ is a constant, taking K sufficiently large to ensure that

$$\frac{1}{p^-} \Big(\int_\Omega |\nabla v|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x) |v|^{p(x)}\, d\sigma \Big) - K \int_\Omega |v|^{p^+} dx < 0$$

which implies that

$$I(tv) \to -\infty \quad \text{as } t \to +\infty.$$

ii) for $\| u \|< 1$ we have

$$I(u) \geq \frac{1}{p^+}\Big(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}\,d\sigma\Big) - \int_\Omega F(x,u)\,dx$$
$$\geq \frac{1}{p^+}\|u\|^{p^+} - \int_\Omega F(x,u)\,dx.$$

Furthermore, in view of (H_1) and (H_2),

$$|f(x,u)| \leq \varepsilon|u|^{p^+-1} + C(\varepsilon)|u|^{q(x)-1}, \ \ \forall (x,u) \in \Omega \times \mathbb{R}.$$

By the continuous embedding from X into $L^{q(x)}(\Omega)$ and $L^{p^+}(\Omega)$ there exist $c_1, c_2 > 0$ such that

$$|u|_{L^{p^+}(\Omega)} \leq c_1\|u\|, \quad |u|_{L^{q^+}(\Omega)}, |u|_{L^{q^-}(\Omega)} \leq c_2\|u\| \tag{6}$$

for all $u \in X$. Hence

$$\begin{aligned}\int_\Omega F(x,u)\,dx &\leq \int_\Omega \frac{\varepsilon}{p^+}|u|^{p^+}\,dx + \int_\Omega \frac{C(\varepsilon)}{q(x)}|u|^{q(x)}\,dx \\ &\leq \varepsilon c_1^{p^+}\|u\|^{p^+} + c_2^{q^-}\frac{C(\varepsilon)}{q^-}\|u\|^{q^-}\end{aligned} \tag{7}$$

for all $x \in \Omega$ and all $u \in \mathbb{R}$.

Therefore,

$$I(u) \geq \left(\frac{1}{p^+} - C(\varepsilon)c_2^{q^-}\|u\|^{q^- - p^+} - \varepsilon c_1^{p^+}\right)\|u\|^{p^+},$$

since $1 < p^+ < q^-$, then for r sufficiently small we take $\sigma > 0$ such that

$$I(u) \geq \sigma, \ \ \forall u \in X \quad \text{with} \quad \|u\| = r.$$

□

Lemma 3.2. *For the functional I and for any $(u_n)_n \in X$ and $t \in]0,1[$, then we have*

$$I(tu_n) \leq \frac{t^{p^-}}{p^-}\Big[\frac{1}{n} + \int_\Omega \frac{1}{p^-} f(x,u_n)u_n\,dx\Big] - \int_\Omega F(x,u_n)\,dx.$$

Proof. Consider a function ψ such that

$$\psi(t) = \frac{1}{p^-}t^{p^-} f(x,u_n)u_n - F(x,tu_n),$$

then

$$\begin{aligned}\psi'(t) &= t^{p^- -1} f(x,u_n)u_n - f(x,tu_n)u_n \\ &= t^{p^- -1}u_n\Big(f(x,u_n) - \frac{f(x,tu_n)}{t^{p^- -1}}\Big),\end{aligned}$$

which means that $\psi'(t) \geq 0$ for $t \in]0,1]$ and $\psi'(t) \leq 0$ when $t \geq 1$, it follows that

$$\psi(t) \leq \psi(1), \ \ \forall t > 0. \tag{8}$$

Since $I'(u_n).u_n \to 0$ we can see, for any $n > 1$, that

$$-\frac{1}{n} < I'(u_n).u_n = \int_\Omega |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)}\,d\sigma - \int_\Omega f(x,u_n)u_n\,dx < \frac{1}{n}. \tag{9}$$

Using the formulas (8) and (9) we obtain

$$\begin{aligned} I(tu_n) &= \int_\Omega \frac{1}{p(x)}|\nabla tu_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|tu_n|^{p(x)}\,d\sigma - \int_\Omega F(x,tu_n)\,dx \\ &< \frac{t^{p^-}}{p^-}\left[\frac{1}{n} + \int_\Omega f(x,u_n)u_n\,dx\right] - \int_\Omega F(x,tu_n)\,dx. \end{aligned} \tag{10}$$

□

Proof. **[Proof of Theorem 1.1:]** 1) To this end, let $(u_n)_n \subset X$ satisfying the proposition 2.4, then

$$I(u_n) = \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma - \int_\Omega F(x,u_n)\,dx = c + o(1)$$

and

$$\Big(1 + \|u_n\|\Big)\|\phi'(u_n)\| \to 0$$

then

$$\|u_n\| - \int_\Omega f(x,u_n)u_n\,dx = o(1)$$

and also

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n.\nabla\varphi + \int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u\varphi\,d\sigma - \int_\Omega f(x,u_n)\varphi = o(1)$$

$\forall \varphi \in X$.

We prove that such $(u_n)_n$ is a bounded sequence in X.

Define

$$t_n = \frac{(2p^+c)^{1/p^+}}{\|u_n\|} \in]0,1[$$

and

$$\omega_n = t_n u_n.$$

Because $\|\omega_n\| = (2p^+c)^{1/p^+}$ so ω_n is bounded in X, therefore, up to a subsequence still denoted by ω_n we have

$$\omega_n \rightharpoonup \omega \ \ in \ \ X$$

$$\omega_n \to \omega \ \ in \ \ L^{q(x)}(\Omega), \ \ q(x) \in [p^-, p^*(x))$$

and

$$\omega_n \to \omega \ \ a.e \ \ in \ \ \Omega.$$

Suppose that $\|u_n\| \to \infty$ and then we confirm that $\omega \equiv 0$. Indeed, putting

$$\Omega_1 = \{x \in \Omega \ : \ \ \omega(x) = 0\}$$

and

$$\Omega_2 = \{x \in \Omega : \ \ \omega(x) \neq 0\}.$$

Easily we can see that $|u_n(x)| \to \infty \ \ a.e \ \ in \ \ \Omega_2$.

From the assumption (H_2) and for n large enough, we have that

$$\frac{f(x,u_n)u_n}{|u_n|^{p^+}} > k \ \ uniformely \ \ x \in \Omega_2$$

for k large enough. Thus,

$$\begin{aligned} 2p^+c &= \lim_{n\to\infty} \|\omega_n\|^{p^+} \\ &= \lim_{n\to\infty} \int_\Omega \frac{|f(x,u_n)|}{|u_n|^{p^+-1}} |\omega_n|^{p^+} \, dx \\ &> k \lim_{n\to\infty} \int_{\Omega_2} |\omega_n|^{p^+} \, dx \\ &= k \int_{\Omega_2} |\omega|^{p^+} \, dx. \end{aligned} \tag{11}$$

The fact that $2p^+c$ is constant and k is sufficiently large so we infer that $|\Omega_2| = 0$ and then $\omega \equiv 0$ in Ω.

Furthermore, since $\omega = 0$ and in view of the continuity of the Nemitskii operator we get

$$F(.,\omega_n) \to 0 \ \ in \ \ L^1(\Omega)$$

what implies that

$$\lim_n F(x,\omega_n)\, dx = 0,$$

then,

$$\begin{aligned} I(\omega_n) &\geq \frac{1}{p^+} t_n^{p^+} \Big[\int_\Omega |\nabla u_n|^{p(x)} \, dx + \int_{\partial\Omega} |u_n|^{p(x)} \, d\sigma \Big] - o(1) \\ &\geq \frac{1}{p^+} \, 2p^+c - o(1) = 2c - o(1) \\ &> c. \end{aligned} \tag{12}$$

On the other hand, for certain $n > 1$ we have

$$\frac{-1}{n} < \frac{p^-}{p^+} \langle I'(u_n), u_n \rangle < \frac{1}{n}.$$

Hence,

$$\begin{aligned} I(u_n) &= \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx \int_{\partial\Omega} |u_n|^{p(x)} \, d\sigma - \int_\Omega F(x,u_n)\, dx \\ &\geq \frac{1}{p^+} \frac{p^+}{p^-} \left(\frac{-1}{n} + \int_\Omega f(x,u_n)u_n \, dx \right) - \int_\Omega F(x,u_n)\, dx \end{aligned} \tag{13}$$

that is,

$$I(u_n) + \frac{1}{np^-} \geq \int_\Omega \Big(\frac{1}{p^-} f(x,u_n)u_n - F(x,u_n) \Big) \, dx. \tag{14}$$

Meanwhile, from Lemma 3.2,

$$I(tu_n) \leq \frac{t^{p^-}}{np^-} + \int_\Omega \Big(\frac{1}{p^-} f(x,u_n)u_n - F(x,u_n) \Big) \, dx. \tag{15}$$

By virtue of (14) and (15), we have

$$I(\omega_n) \leq \frac{t^{p^-} + 1}{np^-} + I(u_n) \to c,$$

which is contradictory with (12). Therefore, $(u_n)_n$ is bounded in X.

Since $(u_n)_n$ is bounded and $f(x,t)$ verifies the sub-critical growth condition, then by using the compactness of Sobolev embedding and Proposition 2.5, it follows that there exists a subsequence of $(u_n)_n$ which converges strongly to a nontrivial critical point of I and the proof of the first assertion is completed. □

2) Since X is a reflexive and separable Banach space, it is worth to recall that there exist $e_j \in X$ and $e_j^* \in X^*$ (j=1,2...) such that

$$e_j^*(e_i) = \begin{cases} 1 \ if \ i = j, \\ 0 \ if \ i \neq j. \end{cases}$$

$X = \overline{span\{e_1, e_2, ...\}}, \quad X^* = \overline{span\{e_1^*, e_2^*, ...\}},$

$$X_i = span\{e_i\}, \quad Y_k = \bigoplus_{i=1}^{k} X_i, \quad Z_k = \overline{\bigoplus_{i=1}^{\infty} X_i}. \tag{16}$$

Theorem 3.3 (Fountain Theorem, [31]). *X is a Banach space, $I \in C^1(X, \mathbb{R})$ is an even functional an satisfies the (P.S) condition, the subspaces Y_k and Z_k are defined in (16).*

If for each k=1,2..., there exists $\rho_k > d_k > 0$ such that

(a) $\max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0.$

(b) $\inf_{u \in Z_k, \|u\| = d_k} I(u) \to \infty$ *as* $k \to \infty$.

Then I has an unbounded sequence of critical values.

Now, via Fountain Theorem, we are to prove that problem (1) has infinitely many solutions.

Lemma 3.4. *Si $q(x) \in C_+(\Omega)$, $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k = \sup\{|u|_{q(x)}; \ \|u\| = 1, \ u \in Z_k\},$$

then $\lim_{k \to \infty} \beta_k = 0$.

(a) For $u \in Z_k$ such that $\|u\| = r_k > 1$ (r_k will be specified below), by condition (H_1), we have

$$\begin{aligned}
I(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} |u_n|^{p(x)}\, d\sigma - \int_\Omega F(x,u)\, dx \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - \int_\Omega C(|u| + |u|^{q(x)}) dx \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - C|u|_{q(x)}^{q(\xi)} - C\|u\|, \quad \text{where } \ \xi \in \Omega, \\
&\geq \begin{cases} \frac{1}{p^+} \|u\|^{p^-} - C - C\|u\| & \text{if } |u|_{q(x)} \leq 1 \\ \frac{1}{p^+} \|u\|^{p^-} - C(\beta_k \|u\|)^{q^+} - C\|u\| & \text{if } |u|_{q(x)} > 1 \end{cases} \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - C(\beta_k \|u\|)^{q^+} - C\|u\| - C \\
&= r_k^{p^-} \left(\frac{1}{p^+} - C\beta_k^{q^+} r_k^{q^+ - p^-} \right) - Cr_k - C.
\end{aligned}$$

We fix r_k as follows

$$r_k = \left(\frac{Cq^+ \beta_k^{q^+}}{1} \right)^{\frac{1}{p^- - q^+}},$$

then

$$I(u) \geq r_k^{p^-} \left(\frac{1}{p^+} - \frac{1}{q+} \right) - Cr_k - C.$$

Using Lemma 3.4 and the fact $p^+ < q^+$, it follows $r_k \to +\infty$, as $k \to +\infty$. Consequently, $I(u) \to +\infty$ as $\|u\| \to +\infty$ with $u \in Z_k$.

(b)

Since $dim Y_k < \infty$ and all norms are equivalent in the finite-dimensional space, there exists $d_k > 0$, for all $u \in Y_k$ with $\|u\| \geq 1$, we have

$$\begin{aligned} I(u) &\leq d_k |u|_{p^+}^{p^+} - 2d_k |u|_{p^+}^{p^+} + \varepsilon |u|^{p^+} \\ &\leq -C_2 \|u\|^{p^+} + \varepsilon C_3 \|u\|^{p^+}, \end{aligned}$$

Therefore, for $\varepsilon > 0$ small enough and ρ_k large enough ($\rho_k > r_k$), we get from the above that

$$a_k := \max\{I(u) : \ u \in Y_k, \|u\| = \rho_k\} \leq 0. \qquad \square$$

Proof. [**Proof of Theorem 1.2**]:
a) **Existence:** From the fact that $f \in C(\Omega, \mathbb{R})$ and by (H_5), there exists a constant m such that

$$f(x, 0) \leq m, \ \forall x \in \partial\Omega.$$

Then

$$\begin{aligned} &\Delta_{p(x)} u = m \quad \text{in } \Omega, \\ &|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2} u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{17}$$

has unique L^∞ solution u_1 which is nonegative (see [8].

Denote

$$\widetilde{f}(x, u) = \begin{cases} f(x, 0) \ \ if \ \ u < 0, \\ f(x, u) \ \ if \ \ 0 \leq u \leq u_1, \\ f(x, u_1) \ \ if \ \ u > u_1 \end{cases}$$

Hence, $-\infty < \widetilde{f}(x, u) \leq m, \ \forall x \in \Omega$ and $u \in \mathbb{R}$. Thus,

$$|\widetilde{F}(x, u)| \leq K|u|, \ \ for \ \ x \in \Omega,$$

where $\widetilde{F}(x, u) = \int_0^u \widetilde{f}(x, s)\, ds$.

Let us consider

$$\psi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)}\, d\sigma - \int_\Omega \widetilde{F}(x, u)\, dx,$$

for $u \in W^{1,p(x)}(\Omega)$.

A standard argument shows that $\psi \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$, since $p^- > 1$ and $\widetilde{f}$ is bounded.

When $\|u\| > 1$ we have

$$\psi(u) \geq \frac{1}{p^+} \|u\|^{p^-} - K_1 \|u\|,$$

with K_1 is a positive constant. Then ψ is coercive, and since it is sequentially weakly lower continuous, we conclude that ψ has a global minimizer $\widetilde{u} \in W^{1,p(x)}(\Omega)$ s.t $\psi'(\widetilde{u}) = 0$. Thereby, $\widetilde{u}$ verifies

$$\int_{\Omega} |\nabla \widetilde{u}|^{p(x)-2} \nabla \widetilde{u}.\nabla v \, dx + \int_{\partial\Omega} \beta(x) |\widetilde{u}|^{p(x)-2} \widetilde{u} v \, d\sigma - \int_{\Omega} \widetilde{f}(x, \widetilde{u}) v \, dx,$$

for all $v \in W^{1,p(x)}(\Omega)$.
Taking $\widetilde{u}^-$ as test function and keeping in mind that $\widetilde{f}(x,u) = f(x,0)$, for $u < 0$, then we have

$$\int_{\Omega} |\nabla \widetilde{u}^-|^{p(x)} \, dx + \int_{\partial\Omega} \beta(x) |\widetilde{u}^-|^{p(x)} \, d\sigma - \int_{\Omega} \widetilde{f}(x, \widetilde{u}^-) \widetilde{u}^- \, dx = 0$$

As we have $\widetilde{f}(x, \widetilde{u}^-)\widetilde{u}^- \leq 0$, so we get

$$\int_{\Omega} |\nabla \widetilde{u}^-|^{p(x)} \, dx + \int_{\partial\Omega} \beta(x) |\widetilde{u}^-|^{p(x)} \, d\sigma \leq 0,$$

which implies that $\widetilde{u}^- = 0$ and then $\widetilde{u} \geq 0$.

Meanwhile, $\widetilde{f}(x, \widetilde{u}) \leq m$, according to comparison principle [9], we have $\widetilde{u} \leq u_1$. Hence,

$$\widetilde{f}(x, \widetilde{u}) = f(x, \widetilde{u})$$

and then $\widetilde{u}$ is a solution of (1.1), which is nontrivial because $f(x,0) \not\equiv 0$.

b) **Uniqueness:**

Let recall the following formulas:

$\forall x, y \in \mathbb{R}^N$

$$| x - y |^{\gamma} \leq 2^{\gamma} (| x |^{\gamma-2} x - | y |^{\gamma-2} y).(x - y) \ \ if \ \ \gamma \geq 2,$$

$$| x - y |^2 \leq \frac{1}{\gamma - 1} (| x | + | y |)^{2-\gamma} (| x |^{\gamma-2} x - | y |^{\gamma-2} y).(x - y) \ \ if \ \ 1 < \gamma < 2,$$

where $x.y$ is the inner product in $\mathbb{R}^N$.

Let u and v two solutions of (1.1), viewing the last inequalities, we have

$$\begin{aligned}
0 &\leq \int_{[u>v]} \Big| \nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v \Big) \Big(\nabla u - \nabla v \Big) \, dx + \\
&\quad \int_{\partial\Omega} \beta(x) \Big(|u|^{p(x)-2} u - |v|^{p(x)-2} v \Big) \Big(u - v \Big) \, d\sigma \\
&\leq \int_{\Omega} \Big| \nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v \Big) \nabla (u - v)^+ \, dx + \\
&\quad \int_{\partial\Omega} \beta(x) \Big(|u|^{p(x)-2} u - |v|^{p(x)-2} v \Big) \Big(u - v \Big)^+ \, d\sigma \\
&= \int_{\Omega} \Big(f(x,u) - f(x,v) \Big) (u - v)^+ \, dx \leq 0.
\end{aligned} \tag{18}$$

Thus, $\nabla u(x) = \nabla v(x)$ for a.e $[u > v] = \Omega_1$.

Let $x \in \Omega \setminus \Omega_1$, then $(u - v)^+(x) = 0$ and $\nabla(u - v)^+(x) = 0$ for a.e $\Omega \setminus \Omega_1$ thereby, $(u - v)^+(x) = 0$ and $\nabla(u - v)^+(x) = 0$ for a.e Ω, so $(u - v)^+ = 0$ for a.e $x \in \Omega$, that means $u \leq v \ \ for \ \ a.e \ \ x \in \Omega$.

Similarly, we prove $v \leq u \ \ a.e \ \ x \in \Omega$, hence, $u = v$. □

With similar arguments as those used in [13], we can obtain the following result,

Proposition 3.5. *1) For every $b \in L^{\infty}(\Omega)$, problem (4) has a unique solution $T(b)$ and $T(b) \in L^{\infty}$.*

2) The mapping $T : L^{\infty}(\Omega) \to L^{\infty}(\Omega)$, is increasing, that is, when $b(x) \leq d(x)$ we have $T(b(x)) \leq T(c(x))$ in Ω.

Proof. [**Proof of Theorem 1.3**]:

Suppose that $f(x,u) = \varepsilon h(x)g(u)$.

Let us consider the function

$$\widetilde{g}(u) = \begin{cases} g(1) \;\; for \;\; u > 1, \\ g(-1) \;\; for \;\; u < -1, \\ g(u) \;\; for \;\; -1 \leq u \leq 1, \end{cases}$$

and define $\widetilde{G}(u) = \int_0^u g(t)\,dt$ for $u \in \mathbb{R}$.

Let put

$$\psi_\varepsilon(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)\frac{1}{p(x)}|u|^{p(x)}\,d\sigma - \varepsilon\int_\Omega h(x)g(u),\; u \in W^{1,p(x)}(\Omega).$$

From the definition of $\widetilde{g}$, there exists a positive constant C such that

$$\widetilde{g}(u) < C,\; u \in \mathbb{R},$$

then

$$|\widetilde{G}(u)| \leq C|u|.$$

The fact that $p^- > 1$ and $h \in L^\infty(\Omega)$, we can see that ψ_ε is coercive and sequentially weakly lower semi-continuous, since $W^{1,p(x)}(\Omega)$ is reflexive, so ψ_ε possess a global minimizer u_ε which is a weak solution of the following problem

$$\begin{aligned} -\Delta_{p(x)}u = \varepsilon h(x)g(u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{19}$$

We point out that when $\varepsilon \to 0$, we have $|\varepsilon h(x)g(u)| \to 0$ and $|u_\varepsilon|_{L^\infty}(\Omega) \to 0$. Taking $\varepsilon > 0$ sufficiently small in order to have $|u_\varepsilon|_{L^\infty}(\Omega) \leq r < 1$.

From the definition of $\widetilde{g}$ we can see that $\widetilde{g}(u_\varepsilon) = g(u_\varepsilon)$ and accordingly u_ε is a solution of problem (1).

On the other side, from the continuity of g, particulary in 0, there exists $r > 0$ such that

$$|g(s) - g(0)| < g(0)\epsilon,\; for \;\; |s| < r$$

with $\epsilon_k = \frac{\varepsilon}{k+\varepsilon},\;\; for\; k \geq 2$

For $\varepsilon > 0$ is small enough and the last inequality, we have

$$\begin{aligned} \varepsilon h(x)g(u_\varepsilon) &= \varepsilon h^+(x)g(u_\varepsilon) - \varepsilon h^-(x)g(u_\varepsilon) \\ &\geq \varepsilon(1-\epsilon_k)g(0)\Big(h^+(x) - \frac{1+\epsilon_k}{1-\epsilon_k}h^-(x)\Big) \\ &\geq \varepsilon(1-\epsilon_k)g(0)(h^+(x) - (1+\varepsilon)h^-(x)). \end{aligned} \tag{20}$$

Hence, for $\varepsilon \in (0, \frac{\theta}{(1-\epsilon_k)f(0)})$ we have that $\lambda = \varepsilon(1-\epsilon)g(0) \leq \theta$. In virtue of the assumption (H_6),

$$\varepsilon(1-\epsilon_k)g(0)(h^+(x) - (1+\varepsilon)h^-(x)) > 0.$$

In view of the comparison principle, u_ε is a positive solution of problem (1). □

Acknowledgements: The author would like to thank the anonymous referee for the suggestions and helpful comments.

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