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Some Propositonerties of a Bavrin's Family of Holomorphic Functions in $\mathbb{C}^n$

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Abstract. In the [1], [4], [3] and [2] there were examined the Bavrin's families (of holomorphic functions on bounded complete $n-$ circular domains $\mathcal{G} \subset \mathbb{C}^n$) in which the Temljakov operator $\mathcal{L}f$ was presented as a product of a holomorphic function h with a positive real part and the $(0,k)-$symmetrical part of the function f, ($k \geq 2$ is a positive integer) . In [17] there was investigated the family of the above mentioned type, where the operator $\mathcal{L}\mathcal{L}f$ was presented as a product of the same function $h \in C_{\mathcal{G}}$ and $(0,2)$-symmetrical part of the operator $\mathcal{L}f$.
These considerations can be completed by the case of the factorization $\mathcal{L}\mathcal{L}f$ by the same function h and the $(0,k)$-symmetrical part of operator $\mathcal{L}f$. In this article we will discuss the above case. In particular, we will present some estimates of a generalization of the norm of m-homogeneous polynomials $Q_{f,m}$ in the expansion of function f and we will also give a few relations between the different Bavrin's families of the above kind.

1. Introduction

Poincare [15] pointed that the Riemann mapping theorem is false in $\mathbb{C}^n, n > 1$. For this reason it is very natural to consider the holomorphicity in $\mathbb{C}^n$ on domains from a sufficiently wide class. The results in this paper concern the bounded complete circular domains, because such domains play the same role for Taylor series in $\mathbb{C}^n$ as open discs in one dimensional case.

We say that a domain $\mathcal{G} \subset \mathbb{C}^n, n \geq 2$, is complete $n-$circular if $z\lambda = (z_1\lambda_1, ..., z_n\lambda_n) \in \mathcal{G}$ for each $z = (z_1, ..., z_n) \in \mathcal{G}$ and every $\lambda = (\lambda_1, ..., \lambda_n) \in \overline{U^n}$, where U is the unit disc $\{\zeta \in \mathbb{C} \colon |\zeta| < 1\}$. From now, by $\mathcal{G}$ will be denoted a nonempty bounded complete $n-$circular domain in $\mathbb{C}^n$.

Note that the Minkowski function $\mu_{\mathcal{G}} : \mathbb{C}^n \to [0, \infty)$ of the form

$$\mu_{\mathcal{G}}(z) = inf\{t > 0 \colon \frac{1}{t} z \in \mathcal{G}\}, \quad z \in \mathbb{C}^n,$$

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gives the possibility to redefine the bounded n-circular domain $\mathcal{G}$ and its boundary $\partial\mathcal{G}$ as follows:

$$\mathcal{G} = \{z \in \mathbb{C}^n : \mu_{\mathcal{G}}(z) < 1\}, \ \ \partial\mathcal{G} = \{z \in \mathbb{C}^n : \mu_{\mathcal{G}}(z) = 1\}.$$

By $\mathcal{H}_{\mathcal{G}}(1)$ and $\mathcal{H}_{\mathcal{G}}$, let us denote the set of all holomorphic functions $f : \mathcal{G} \longrightarrow \mathbb{C}$, normalized by $f(0) = 1$ and without any normalization, respectively.

We will use the Temljakov linear operator $\mathcal{L} : \mathcal{H}_{\mathcal{G}} \longrightarrow \mathcal{H}_{\mathcal{G}}$ defined in [1] by

$$\mathcal{L}f(z) = f(z) + Df(z)(z), \ z \in \mathcal{G},$$

where $Df(z)\,(w)$ is the value of the Frechet derivative $Df(z)$ of f at the point z on a vector w ($Df(z)$ is the row vector $\left[\frac{\partial f(z)}{\partial z_1}, ..., \frac{\partial f(z)}{\partial z_n}\right]$ and w is a column vector). Of course $\mathcal{L}$ is invertible and

$$\mathcal{L}^{-1}f(z) = \int\limits_0^1 f(zt)dt, z \in \mathcal{G}.$$

Many authors (see eg. [1], [6], [7], [8], [13], [18]) considered some Bavrin's subfamilies $\mathcal{X}_{\mathcal{G}}$ of the family $\mathcal{H}_{\mathcal{G}}(1)$. In the definition of these families the main role is played by the family $C_{\mathcal{G}}$,

$$C_{\mathcal{G}} = \{f \in \mathcal{H}_{\mathcal{G}}(1) : Ref(z) > 0, \ \ z \in \mathcal{G}\}.$$

By a Bavrin's family $\mathcal{X}_{\mathcal{G}}$ we mean a collection of functions $f \in \mathcal{H}_{\mathcal{G}}(1)$ whose the Temljakov transform $\mathcal{L}f$ has a functional factorization $\mathcal{L}f = h \cdot g$, where $h \in C_{\mathcal{G}}$ and g is from a fixed subfamily of $\mathcal{H}_{\mathcal{G}}(1)$. Below, we recall the factorizations which define a few well known Bavrin's families $\mathcal{X}_{\mathcal{G}}$, like

$$\begin{aligned} \mathcal{M}_{\mathcal{G}} : & \ \ \mathcal{L}f = h \cdot f, \ \ h \in C_{\mathcal{G}}, \\ \mathcal{N}_{\mathcal{G}} : & \ \ \mathcal{L}(\mathcal{L}f) = h \cdot \mathcal{L}f, \ \ h \in C_{\mathcal{G}}, \\ \mathcal{R}_{\mathcal{G}} : & \ \ \mathcal{L}f = h \cdot \mathcal{L}\varphi, \ \ \varphi \in \mathcal{N}_{\mathcal{G}}, \ h \in C_{\mathcal{G}}. \end{aligned}$$

Let us note that functions of these families were used to construct biholomorphic mappings in $\mathbb{C}^n$ (see eg. [9], [11], [14]). It is known that families $\mathcal{M}_{\mathcal{G}}$, $\mathcal{N}_{\mathcal{G}}$, $\mathcal{R}_{\mathcal{G}}$ correspond with the well-known classes $S^\star$, S^c, S^{cc} of univalent starlike, convex and close-to-convex normalized functions in the unit disc U. For instance: if f belongs to the class $\mathcal{M}_{\mathcal{G}}$, then the function

$$F(\zeta) = \zeta f\left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)}\right), \ \ \zeta \in U$$

belongs to the family $S^\star$ for $z \in \mathcal{G} \setminus \{0\}$.

Bavrin showed (see e.g.[1]) that $\mathcal{N}_{\mathcal{G}} \subsetneq \mathcal{M}_{\mathcal{G}}$. He proved also the following higher dimensional version of the well-known Alexander theorem: if $f \in \mathcal{N}_{\mathcal{G}}$ than $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}$ and conversely, if $f \in \mathcal{M}_{\mathcal{G}}$ then $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}$.

In [3] the authors defined the family $\mathcal{M}^2_{\mathcal{G}}$ in the following way: A function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{M}^2_{\mathcal{G}}$ if there exists a function $h \in C_{\mathcal{G}}$ such that

$$\mathcal{L}f(z) = h(z)f_{0,2}(z), \ \ z \in \mathcal{G},$$

where $f_{0,2}$ is the even part of f in the unique partition $f = f_{0,2} + f_{1,2}$ of f onto the sum of even and odd functions. In [17] the class $\mathcal{N}^2_{\mathcal{G}}$ was introduced as follows: A function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{N}^2_{\mathcal{G}}$ if there exists a function $h \in C_{\mathcal{G}}$ such that

$$\mathcal{L}\mathcal{L}f(z) = h(z)\left(\mathcal{L}f\right)_{0,2}(z),\ \ z \in \mathcal{G}.$$

In [2] the author investigated the family $\mathcal{M}^k_{\mathcal{G}},\ \ k \in \mathbb{N},\ k \geq 2$, by application a functional decomposition with respect to the group of k^{th} roots of unity.

Let $k \geq 2$ be an arbitrarily fixed integer, $\varepsilon = \varepsilon_k = \exp \frac{2\pi i}{k}$ and a set $\mathcal{D} \subset \mathbb{C}^n$ be $k-$ symmetric ($\varepsilon\mathcal{D} = \mathcal{D}$). For $j = 0, 1, ..., k-1$ we define the spaces $\mathcal{F}_{j,k} = \mathcal{F}_{j,k}(\mathcal{D})$ of functions $(j,k)-$ symmetrical, i.e., all functions $f : \mathcal{D} \to \mathbb{C}$ such that

$$f(\varepsilon z) = \varepsilon^j f(z), z \in \mathcal{D}.$$

The following result from [12] was used in this and the aforementioned article:

Theorem A For every function $f : \mathcal{D} \to \mathbb{C}$ there exists exactly one sequence of functions $f_{j,k} \in \mathcal{F}_{j,k}$, $j = 0, 1, ..., k-1$, such that

$$f = \sum_{j=0}^{k-1} f_{j,k} \tag{1}$$

Moreover,

$$f_{j,k}(z) = \frac{1}{k}\sum_{l=0}^{k-1} \varepsilon^{-jl} f\left(\varepsilon^l z\right),\ z \in \mathcal{G}. \tag{2}$$

By the uniqueness of the partition (1) the functions $f_{j,k}$ will be called $(j,k)-$ symmetrical components of the function f. Since every bounded complete n-circular domain $\mathcal{G} \subset \mathbb{C}^n$ is k-symmetric set, it is obvious that $f_{0,k} \in \mathcal{H}_{\mathcal{G}}(1)$ for $f \in \mathcal{H}_{\mathcal{G}}(1)$.

We say (see [2]) that a function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{M}^k_{\mathcal{G}},\ \ k \in \mathbb{N},\ k \geq 2$, if there exists a function $h \in C_{\mathcal{G}}$ such that

$$\mathcal{L}f(z) = h(z)f_{0,k}(z),\ \ z \in \mathcal{G}. \tag{3}$$

The family $\mathcal{M}^k_{\mathcal{G}}$ corresponds to the well-known class S^{*k} [16] of normalized univalent functions, starlike with respect to k -symmetric points.

These considerations can be completed by the case of the factorization $\mathcal{L}\mathcal{L}f$ by the same function h and the $(0,k)$-symmetrical part of operator $\mathcal{L}f$.

It is known ([2]) that for every function $f \in \mathcal{H}_{\mathcal{G}}(1)$

$$(\mathcal{L}f)_{0,k} = \mathcal{L}(f_{0,k}). \tag{4}$$

Let us define the class $\mathcal{N}^k_{\mathcal{G}}$ $k \in \mathbb{N},\ k \geq 2$ as a family of functions $f \in \mathcal{H}_{\mathcal{G}}(1)$ for which there exists a function

$h \in C_{\mathcal{G}}$ such that

$$\mathcal{L}\mathcal{L}f(z) = h(z)\mathcal{L}f_{0,k}(z),\ z \in \mathcal{G}. \tag{5}$$

2. Main results

Between functions from the class $\mathcal{M}^k_{\mathcal{G}}$, $\mathcal{N}^k_{\mathcal{G}}$ there holds the following generalization of the Alexander's relation:

Theorem 2.1. *Let $k \in \mathbb{N},\ k \geq 2$. If $f \in \mathcal{N}_{\mathcal{G}}^{k}$, then $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}^{k}$ and conversely, if $f \in \mathcal{M}_{\mathcal{G}}^{k}$, then $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}^{k}$.*

Proof. If $f \in \mathcal{N}_{\mathcal{G}}^{k}$ $k \in \mathbb{N},\ k \geq 2$, then $f \in \mathcal{H}_{\mathcal{G}}(1),\ \ \mathcal{L}f \in \mathcal{H}_{\mathcal{G}}(1)$ and there exists $h \in C_{\mathcal{G}}$ such that the condition (5) holds, where $f_{0,k}$ is $(0,k)$-symmetrical part of f. In view of (4) the condition (5) can be written in the form

$$\mathcal{L}\left(\mathcal{L}f(z)\right) = h(z)\left(\mathcal{L}f\right)_{0,k}(z), \quad z \in \mathcal{G},$$

therefore, by (3) $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}^{k}$.
Now, let f belongs to the class $\mathcal{M}_{\mathcal{G}}^{k}$, i.e. $f \in H_{\mathcal{G}}(1)$ and there exists $h \in C_{\mathcal{G}}$ such that the condition (3) holds. Hence,

$$\mathcal{L}\left(\mathcal{L}\mathcal{L}^{-1}f(z)\right) = h(z)\mathcal{L}\mathcal{L}^{-1}f_{0,k}(z), \quad z \in \mathcal{G},$$

$$\mathcal{L}\mathcal{L}\left(\mathcal{L}^{-1}f(z)\right) = h(z)\mathcal{L}\left((\mathcal{L}^{-1}f)_{0,k}(z)\right), \quad z \in \mathcal{G}.$$

Therefore, $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}^{k}$. □

Let us observe that $\mathcal{N}_{\mathcal{G}}^{k}$ $k \in \mathbb{N},\ k \geq 2$ are nonempty classes. Indeed, the function $f = 1$ belongs to $\mathcal{N}_{\mathcal{G}}^{k}$, because it satisfies the factorization (5) with $h = 1 \in C_{\mathcal{G}}$.

It is known that the constructs of functions of several complex variables are very difficult. We will give an example of non-trivial hypergeometric function belonging to the class $\mathcal{N}_{\mathcal{G}}^{k}$.

Example It is known that the function $H(a,b,c,\zeta)$ of the form

$$H(a,b,c,\zeta) = \sum_{\nu=0}^{\infty} \frac{(a)_\nu\,(b)_\nu}{(c)_\nu}\frac{\zeta^\nu}{\nu!},\ a,b,c,\in \mathbb{C},\ \zeta \in U, \tag{6}$$

where $(a)_\nu = a(a+1)...(a+\nu-1)$, $\nu = 1,2,\ldots$ and $(a)_0 = 1$ is called the hypergeometrical function.

Let $I : \mathbb{C}^n \to \mathbb{C}$ be a linear operator of the form

$$I\left(z\right) = \frac{1}{\mu_{\mathcal{G}}(\widehat{I})}\widehat{I}\left(z\right),$$

where

$$\widehat{I}(z) = \sum_{j=1}^{n} z_j,\ \ z = (z_1,...,z_n) \in \mathbb{C}^n. \tag{7}$$

and

$$\mu_{\mathcal{G}}(\widehat{I}) = \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{\left|\widehat{I}\left(w\right)\right|}{\mu_G(w)} = \sup_{v\in\partial\mathcal{G}}\left|\widehat{I}(v)\right|. \tag{8}$$

The quantity $\mu_{\mathcal{G}}(\widehat{I})$ is called a $\mathcal{G}$-balance of the linear functional $\widehat{I}$.

The $\mathcal{G}$-balance of the form (8) coincides with the $\Delta = \Delta(\mathcal{G})$ - characteristic of the domain $\mathcal{G}$, which was introduced by Bavrin (see [1]) by the formula $\Delta = \sup_{z=(z_1,z_2,...,z_n)\in\mathcal{G}}\left|\sum_{j=1}^{n} z_j\right|$. If $\mathcal{G}$ is a convex bounded complete n-circular domain, then $\mu_{\mathcal{G}}(\widehat{I}) = \|\,\widehat{I}\,\|$.

To show that function

$$f\left(z\right) = H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},I^k\left(z\right)\right),\ z \in \mathcal{G},\ k \in \mathbb{N}, k \geq 2 \tag{9}$$

belongs to the class $\mathcal{N}_{\mathcal{G}}^{k}$, we will use the following known propositonerties of the hypergeometrical function:

Remark 2.2. *If $H(a,b,c,\zeta)$, $a,b,c,\in\mathbb{C}$, $\zeta\in U$, is the hypergeometrical function of the form (6), then*

(i) $\zeta\frac{d}{d\zeta}H(a,b,c,\zeta)=aH(a+1,b,c,\zeta)-aH(a,b,c,\zeta)$;

(ii) $H(a,b,c,\zeta)=H(b,a,c,\zeta)$;

(iii) $H(l,b,b,\zeta)=(1-\zeta)^{-l}$.

To determine the transform $\mathcal{L}f$ of function f, let us write

$$f(z)=T(I(z)),\quad z\in\mathcal{G},$$

where $T:\mathcal{U}\to\mathbb{C}$ is defined in the following way:

$$T(\zeta)=H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},\zeta^k\right),\ \zeta\in U,\ k\in\mathbb{N},k\geq 2. \tag{10}$$

Now, we will find explicit form of $\mathcal{L}f$. We start with the following equality

$$\mathcal{L}f(z)=\frac{d}{d\zeta}(\zeta T(\zeta))|_{\zeta=I(z)}.$$

Next, in view of the form (10) of T, we have at $\zeta\in U$

$$\begin{aligned}\frac{d}{d\zeta}(\zeta T(\zeta)) &= H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},\zeta^k\right)+\zeta\frac{d}{d\zeta}\left(H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},\zeta^k\right)\right)=\\ &= H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},\zeta^k\right)+\zeta\frac{d}{d\zeta^k}\frac{d\zeta^k}{d\zeta}\left(H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},\zeta^k\right)\right)=\\ &= H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},\zeta^k\right)+k\zeta^k\frac{d}{d\zeta^k}\left(H\left(\frac{1}{k},\frac{2}{k},1+\frac{1}{k},\zeta^k\right)\right).\end{aligned}$$

Thus, by propositonerty 2.2 (i), we have for $\zeta\in U$

$$\frac{d}{d\zeta}(\zeta T(\zeta))=\left(H\left(\frac{1}{k}+1,\frac{2}{k},1+\frac{1}{k},\zeta^k\right)\right).$$

Hence, using propositonerty 2.2 (ii) and (iii) of H, we conclude that

$$\frac{d}{d\zeta}(\zeta T(\zeta))=\frac{1}{(1-\zeta^k)^{\frac{2}{k}}},\quad \zeta\in U.$$

Finally,

$$\mathcal{L}f(z)=\frac{1}{(1-I^k(z))^{\frac{2}{k}}},\quad z\in\mathcal{G}.$$

In the paper [4] the authors showed, that function

$$g(z)=\frac{1}{(1-I^k(z))^{\frac{2}{k}}},\quad z\in\mathcal{G},$$

so according to the Theorem 2.1, the function of the form (9) belongs to the $\mathcal{N}_{\mathcal{G}}^k$.

Now, we consider an extremal problemma for the family $\mathcal{N}_{\mathcal{G}}^k$.

It is known that each function $f \in \mathcal{H}_{\mathcal{G}}(1)$ can be developed into series of m-homogeneous polynomials $Q_{f,m}$, $m \in \mathbb{N}$ of the form

$$f(z) = 1 + \sum_{m=1}^{\infty} Q_{f,m}(z), \quad z \in \mathcal{G} \tag{11}$$

where

$$Q_{f,m}(z) = \sum_{\alpha_1+\ldots+\alpha_n=m} c_{\alpha_1\ldots\alpha_n} z_1^{\alpha_1}\ldots z_n^{\alpha_n}, \qquad z = (z_1,\ldots,z_n) \in \mathbb{C}^n,$$

and the coefficients $c_{\alpha_1\ldots\alpha_n}$, $\alpha_l \in \mathbb{N}\cup\{0\}$, $l = 1,\ldots,n$ are defined by the partial derivatives as follows:

$$c_{\alpha_1\ldots\alpha_n} = \frac{1}{\alpha_1!\cdot\ldots\cdot\alpha_n!}\frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial z_1^{\alpha_1}\ldots\partial z_n^{\alpha_n}}(0)$$

Bearing in mind that for the considered domains $\mathcal{G}$, $\mu_{\mathcal{G}}$ is a seminorm in $\mathbb{C}^n$ and it is a norm in $\mathbb{C}^n$ in the case if $\mathcal{G}$ is also convex, we will use a generalization $\mu_{\mathcal{G}}(Q_{f,m})$ of the norm of m-homogeneous polynomials $Q_{f,m}$. Putting for $m \in \mathbb{N}$

$$\mu_{\mathcal{G}}(Q_{f,m}) = \sup_{w\in\mathbb{C}^n\setminus\{0\}} \frac{\left|Q_{f,m}(w)\right|}{(\mu_{\mathcal{G}}(w))^m}$$

and using the m-homogeneity of $Q_{f,m}$ and the maximum principle for modulus of holomorphic functions of several variables we have

$$\mu_{\mathcal{G}}(Q_{f,m}) = \sup_{v\in\partial\mathcal{G}}\left|Q_{f,m}(v)\right| = \sup_{u\in\mathcal{G}}\left|Q_{f,m}(u)\right|$$

It is easy to see that

$$\left|Q_{f,m}(w)\right| \leq \mu_{\mathcal{G}}(Q_{f,m})(\mu_{\mathcal{G}}(w))^m, \quad w \in \mathbb{C}^n,\ m \in \mathbb{N},$$

and the above estimate generalizes the well-known inequality

$$\left|Q_{f,m}(w)\right| \leq \| Q_{f,m} \| \cdot \| w \|^m, \quad w \in \mathbb{C}^n,\ m \in \mathbb{N}.$$

By the above considerations and in view of the fact that every complete n-circular domain is balanced, the quantities $\mu_{\mathcal{G}}(z)$ and $\mu_{\mathcal{G}}(Q_{f,m})$ are called $\mathcal{G}$-balance of the point z and $\mathcal{G}$-balances of m-homogeneous polynomials $Q_{f,m}$, respectively.

In the next theorem we give the sharp estimates of $\mathcal{G}$-balances of m-homogeneous polynomials which appear in the Taylor series development of the form (11).

Theorem 2.3. *If the expansion of the function $f \in \mathcal{N}_{\mathcal{G}}^k$ into a series of m−homogeneous polynomials $Q_{f,m}$ has the form (11) then for the $\mathcal{G}$−balances $\mu_G(Q_{f,m})$ of polynomials $Q_{f,m}$ there hold the following sharp estimates:*

$$\mu_{\mathcal{G}}(Q_{f,\,m}) \leq \begin{cases} \frac{2}{m(m+1)} \prod\limits_{p=1}^{\frac{m}{k}-1}\left(1+\frac{2}{pk}\right) \text{ for } m = k, 2k, 3k, ... \\ \frac{2}{(m+1)^2} \prod\limits_{p=1}^{\lfloor \frac{m}{k} \rfloor}\left(1+\frac{2}{pk}\right) \text{ for remaining } m \in \mathbb{N} \end{cases}, \tag{12}$$

where $\lfloor q \rfloor$ – means the integral part of the number q.

We agree, as usual, that the product $\prod\limits_{l=l_1}^{l_2} a_l$ is equal to 1 for $l_2 < l_1$.

Proof. Let $f \in \mathcal{N}_{\mathcal{G}}^k$ be arbitrarily fixed. Then from the generalization of Alexander's theorem (see Theorem 2.1) $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}^k$. If f has the form (11), then

$$\mathcal{L}f(z) = 1 + \sum_{m=1}^{\infty} (m+1)\, Q_{f,m}(z), \quad z \in \mathcal{G}.$$

Hence, we have

$$\mu_{\mathcal{G}}(Q_{\mathcal{L}f,\,m}) = \sup_{z\in\mathcal{G}} \left|Q_{\mathcal{L}f,\,m}(z)\right| = \sup_{z\in\mathcal{G}} \left|(m+1)\, Q_{f,\,m}(z)\right| = (m+1)\,\mu_{\mathcal{G}}(Q_{f,\,m})\,. \tag{13}$$

It is known ([4]) that for $g \in \mathcal{M}_{\mathcal{G}}^k$, there hold the following sharp estimates of the $\mathcal{G}$−balances $\mu_G(Q_{g,m})$ of polynomials $Q_{g,m}$:

$$\mu_{\mathcal{G}}(Q_{g,\,m}) \le \begin{cases} \frac{2}{m} \prod\limits_{p=1}^{\frac{m}{k}-1} \left(1 + \frac{2}{pk}\right) \text{ for } m = k, 2k, 3k, ... \\ \frac{2}{m+1} \prod\limits_{p=1}^{\left\lfloor \frac{m}{k} \right\rfloor} \left(1 + \frac{2}{pk}\right) \text{ for remaining } m \in \mathbb{N} \end{cases}. \tag{14}$$

In view of (13) and (14) the proof is complete. □

The next lemmama will show the connection between the families $\mathcal{N}_{\mathcal{G}}^k$ and $\mathcal{M}_{\mathcal{G}}$.

Lemma 2.4. *Let $k \in \mathbb{N},\ k \ge 2$. For every function $f \in \mathcal{N}_{\mathcal{G}}^k$ its $(0,k)$-symmetrical part $f_{0,k}$ belongs to $\mathcal{N}_{\mathcal{G}}$. Moreover, $f_{0,k} \in \mathcal{N}_{\mathcal{G}}^k$.*

Proof. Let $f \in \mathcal{N}_{\mathcal{G}}^k$ and let z be arbitrarily fixed. There exists the function $h \in \mathcal{C}_{\mathcal{G}}$ such that (5) holds. In view of the propositonerties of $\mathcal{G}$ we have $f_{0,k} \in \mathcal{H}_{\mathcal{G}}(1)$ and

$$f_{0,k}\left(\varepsilon^l z\right) = f_{0,k}\,(z)\,.$$

Hence, we obtain the system of equations of the form

$$\mathcal{L}\mathcal{L}f(\varepsilon^l z) = h(\varepsilon^l z)\mathcal{L}f_{0,k}(z), \quad z \in \mathcal{G}, \quad l = 0, 1, ..., k-1.$$

Summing up the above equalities, we have

$$\frac{1}{k}\sum_{l=0}^{k-1} \mathcal{L}\mathcal{L}f(\varepsilon^l z) = \mathcal{L}f_{0,k}(z)\frac{1}{k}\sum_{l=0}^{k-1} h(\varepsilon^l z)$$

and according to theorem A

$$(\mathcal{L}\mathcal{L}f)_{0,k}\,(z) = h_{0,k}\,(z)\,\mathcal{L}f_{0,k}(z).$$

From (4) we obtain

$$(\mathcal{L}\mathcal{L}f)_{0,k}\,(z) = \mathcal{L}\mathcal{L}(f_{0,k}(z))$$

and

$$\mathcal{L}\mathcal{L}(f_{0,k}\,(z)) = h_{0,k}\,(z)\,\mathcal{L}f_{0,k}(z).$$

Note that if $h \in \mathcal{C}_{\mathcal{G}}$, then $h_{0,k} \in \mathcal{C}_{\mathcal{G}}$, so $f_{0,k}$ fulfils the condition defining the family $\mathcal{N}_{\mathcal{G}}$.

Since $f_{0,k}$ is $(0,k)$-symmetrical part of f and $\mathcal{L}f_{0,k}$ is a $(0,k)$-symmetrical part of $\mathcal{L}f$ then the condition $(\mathcal{L}f_{0,k}(z))_{0,k} = \mathcal{L}f_{0,k}(z)$ holds and by (4) there exists the function $h \in \mathcal{C}_{\mathcal{G}}$ such that

$$\mathcal{L}\mathcal{L}f_{0,k}(z) = h(z)\,\mathcal{L}\left(f_{0,k}(z)\right)_{0,k}.$$

Hence, $f_{0,k} \in \mathcal{N}^k_{\mathcal{G}}$. □

In the proof of next theorem, we will use the following result form the paper [10]:

Theorem B *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n-circular domain. Let us assume that $F \in H_{\mathcal{G}}(1)$, $H \in \mathcal{M}_{\mathcal{G}}$ and ρ is a relation defined as follows*

$$F\rho H \Longleftrightarrow \mathrm{Re}\frac{F(z)}{H(z)} > 0, \quad z \in \mathcal{G}. \tag{15}$$

If $(\mathcal{L}F)\rho(\mathcal{L}H)$, then $F\rho H$.

Theorem 2.5. *For every $k \in \mathbb{N}$, $k \geq 2$ there holds the inclusion*

$$N^k_{\mathcal{G}} \subsetneq \mathcal{M}^k_{\mathcal{G}}.$$

Proof. Since, the relation $N^2_{\mathcal{G}} \subsetneq \mathcal{M}^2_{\mathcal{G}}$ is true (see [17]), we can assume that $k > 2$.
First, we will prove that $N^k_{\mathcal{G}} \subset \mathcal{M}^k_{\mathcal{G}}$. Let f belongs to $N^k_{\mathcal{G}}$. It means that $f \in \mathcal{H}_{\mathcal{G}}(1)$ and $\mathcal{L}f \in \mathcal{H}_{\mathcal{G}}(1)$ and (see the lemmama 2.4) $f_{0,k} \in N_{\mathcal{G}}$, so $f_{0,k} \in M_{\mathcal{G}}$. Let as put $F = \mathcal{L}f$ and $H = f_{0,k}$. The condition (5) is equivalent to inequality $\mathrm{Re}\frac{\mathcal{L}\mathcal{L}f(z)}{\mathcal{L}f_{0,k}(z)} > 0$, so in the terminology of Theorem B, $\mathcal{L}F\rho\mathcal{L}H$. Hence, we have that $F\rho H$, which is equivalent to the condition $\mathrm{Re}\frac{\mathcal{L}f(z)}{f_{0,k}(z)} > 0$, $z \in \mathcal{G}$. It gives that $f \in M^k_{\mathcal{G}}$.
Now, we will show that $N^k_{\mathcal{G}} \neq \mathcal{M}^k_{\mathcal{G}}$. For this purpose, let us remind that for the function $f \in N^k_{\mathcal{G}}$ there hold the sharp estimates $\mu_{\mathcal{G}}(Q_{f,m})$ given by (12), while for function $f \in M^k_{\mathcal{G}}$ these estimates are presented in the formula (14). Therefore, the function $f \in M^k_{\mathcal{G}}$ that meets equality (14) does not belong to $N^k_{\mathcal{G}}$. □

Next results concern the topological propositonerties of the family $\mathcal{N}^k_{\mathcal{G}}$.

Theorem 2.6. *The family $\mathcal{N}^k_{\mathcal{G}}$ is not convex for any $k \in \mathbb{N}$, $k \geq 2$.*

Proof. Let us consider the mapping $f = \frac{1}{2}(f_1 + f_2)$, where

$$f_1(z) = H\left(\frac{1}{k}, \frac{2}{k}, 1 + \frac{1}{k}, I^k(z)\right), \quad z \in \mathcal{G},$$
$$f_2(z) = f_1(\sqrt[k]{-1}z), \quad z \in \mathcal{G}.$$

(in both above formulas the branches of the power function $x^{\frac{1}{k}}$ are such that $1^{\frac{1}{k}} = 1$ and the root is arbitrarily fixed) In view of the earlier consideration, the functions $f_1, f_2 \in \mathcal{N}^k_{\mathcal{G}}$.
Now, we will show that f does not belong to the family $\mathcal{N}^k_{\mathcal{G}}$. To this aim, we will show that $\mathcal{L}f$ does not belong to the family $\mathcal{M}^k_{\mathcal{G}}$.
We have

$$f_1(z) = \mathcal{L}^{-1}\left[\frac{1}{(1 - I^k(z))^{\frac{2}{k}}}\right], \quad z \in \mathcal{G},$$
$$f_2(z) = \mathcal{L}^{-1}\left[\frac{1}{(1 + I^k(z))^{\frac{2}{k}}}\right], \quad z \in \mathcal{G},$$

hence

$$\mathcal{L}f(z) = \frac{1}{2}\left[\left(1-z^k\right)^{-\frac{2}{k}} + \left(1+z^k\right)^{-\frac{2}{k}}\right],\ z \in \mathcal{G}$$

and

$$\frac{\mathcal{L}f(z)}{f_{0,k}(z)} = \frac{1}{1-z^{2k}} \frac{\left(1+z^k\right)^{\frac{2}{k}+2} + \left(1-z^k\right)^{\frac{2}{k}+2}}{(1+z^k)^{\frac{2}{k}} + (1-z^k)^{\frac{2}{k}}},\ z \in \mathcal{G}.$$

Denoting by a an elemmaent of $\sqrt[k]{i}$, with $Arga\in\left(0,\frac{\pi}{2}\right)$, we get the continuity $\frac{\mathcal{L}f}{f_{0,k}}$ at a point $\widehat{z}=(a,0,...,0) \in \partial\mathcal{G}$. We will also show that

$$\mathrm{Re}\frac{\mathcal{L}f(\widehat{z})}{f_{0,k}(\widehat{z})} < 0.$$

Since

$$\mathrm{Re}\frac{\mathcal{L}f(\widehat{z})}{f_{0,k}(\widehat{z})} = -\mathrm{Im}\frac{i^{\frac{2}{k}}-1}{i^{\frac{2}{k}}+1},$$

we use the fact that the homography $\frac{\zeta-1}{\zeta+1}$ transforms the unit circle onto the imaginary axis, wherein the upper semicircle onto the upper semiaxis. The above propositonerty gives the equality

$$\frac{i^{\frac{2}{k}}-1}{i^{\frac{2}{k}}+1} = bi, b > 0,$$

because the point $i^{\frac{2}{k}} = a^2$ belongs to the upper unit semicircle.
Therefore, Re $\frac{\mathcal{L}f}{f_{0,k}}$ is negative at the point $\widehat{z} \in \partial\mathcal{G}$. Hence, from continuity of the functions Re $\frac{\mathcal{L}f}{f_{0,k}}$ at the point $\widehat{z}$, we obtain that Re$\frac{\mathcal{L}f}{f_{0,k}}$ is negative also in some points $z \in \mathcal{G}$ close to the above point $\widehat{z} \in \partial\mathcal{G}$. Summing up, we proved that $\mathcal{L}f \notin \mathcal{M}^k_{\mathcal{G}}$. Hence, f does not belong to the family $\mathcal{N}^k_{\mathcal{G}}$, $k \geq 2$. Therefore, $\mathcal{N}^k_{\mathcal{G}}$ is not convex. □

Now, we give a topological propositonerty of the family $\mathcal{N}^k_{\mathcal{G}}, k \geq 2$ in the space $\mathcal{H}_{\mathcal{G}}$ of holomorphic mappings $f : \mathcal{G} \longrightarrow \mathbb{C}^n$ with a topology introduced by the closure operation.

We say, as usual, that a mapping f belongs to the closure $\overline{Y}$ of a set $Y \subset \mathcal{H}_{\mathcal{G}}$ if there exists a sequence of mappings $f_\nu \in Y$ convergent to f almost uniformly in $\mathcal{G}$, i.e., convergent uniformly on every compact subset of $\mathcal{G}$. Of course, it sufficies to guarantee the uniform convergence on every domain $\overline{r\mathcal{G}} \subset \mathcal{G}, r \in (0,1)$.

The announced topological propositonerty of the family $\mathcal{N}^k_{\mathcal{G}}, k \geq 2$, we present in the following:

Theorem 2.7. *The family* $N^k_{\mathcal{G}}, k \geq 2$, *is a path connected set in* $H_{\mathcal{G}}$ *with a topology given by the closure operation. Hence,* $N^k_{\mathcal{G}}$, $k \geq 2$, *is also connected.*

Proof. There is enough to use the fact that $\mathcal{N}^k_{\mathcal{G}}$ is a subset of $\mathcal{M}^k_{\mathcal{G}}$ and estimates $\mu_{\mathcal{G}}(Q_{f,m}) \leq 1, \ \ m \in \mathbb{N}$ in the family $\mathcal{M}^k_{\mathcal{G}}$ and the proof of the path - connectness is similar to the investigations in [4]. □

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