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# Mean Exact Finite-Approximate Contollability of Linear Stochastic Equations in Hilbert Spaces

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**Abstract.** We introduce the concepts of mean exact/mean exact finite dimensional/mean square approximate controllability in the framework of linear fractional stochastic evolution Ito equations. We show that mean square approximate controllability is equivalent to simultaneous mean exact finite dimensional and mean square approximate controllability.

## 1. Introduction

The concept of controllability is an important feature of a control system that plays an important role in many control problems. For this reason, control problems have been explored in recent years for linear stochastic dynamic systems in various publications (see [3]-[9]). For linear fractional stochastic evolution systems it should be noted that the controllability theory is still in its initial phase. There are a small number of studies on approximate controllability problems for different types of fractional stochastic evolution systems, see [4]-[12] .

The purpose of this paper is to introduce the mean exact/mean exact finite dimensional/mean square approximate controllability concepts and study relationship between them in the framework of linear fractional stochastic evolution Ito equations.

We are given a probability space $(\Omega, \mathfrak{F}, P)$ together with a normal filtration $(\mathfrak{F}_t)_{t\geq 0}$. We consider three real separable spaces $K, H$ and $U$, and $Q$-Wiener process on $(\Omega, \mathfrak{F}, P)$ with covariance operator $Q \in L(K)$ such that tr$Q < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $K$, a bounded sequence of nonnegative real numbers $\lambda_k$ such that $Qe_k = \lambda_k e_k,\ \ k = 1, 2, ...,$ and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \langle e_k, e\rangle\, \beta_k(t),\ e \in K,\ t \in [0, T],$$

and $\mathfrak{F}_t = \mathfrak{F}_t^w$, where $\mathfrak{F}_t^w$ is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$, that is $\mathfrak{F}_t^w = \sigma\{w(s) : 0 \leq s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of $P$-null sets of $\mathfrak{F}$.

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- $L_2^0 = L_2\left(Q^{1/2}K;H\right)$ is the space of all Hilbert-Schmidt operators from $Q^{1/2}K$ to $H$ with the inner product $\left\langle \psi,\phi\right\rangle_{L_2^0} = \mathrm{tr}\left[\psi Q\phi\right]$.

- $L^p(\mathfrak{F}_T, H)$ is the Banach space of all $\mathfrak{F}_T$-measurable $p$th power integrable variables with values in $H$.

- $L^p_{\mathfrak{F}}(0,T;H), p \geq 2$ is the Banach space of all $p$th power integrable and $\mathfrak{F}_t$-adapted processes with values in $H$.

- $C(0,T;L^p(\mathfrak{F}_T,H))$ is the Banach space of continuous maps from $[0,T]$ into $L^p(\mathfrak{F}_T,H)$ satisfying the condition $\sup\left\{\mathbb{E}\left\|\varphi(t)\right\|_H^p : t\in[0,T]\right\} < \infty$.

- $C_{\mathfrak{F}}(0,T;L^p(\mathfrak{F},H))$ is the closed subspace of $C(0,T;L^p(\mathfrak{F}_T,H))$ consisting of measurable and $\mathfrak{F}_t$-adapted $H$-valued processes $\varphi \in C(0,T;L^p(\mathfrak{F},H))$ endowed with the norm $\left\|\varphi\right\|_{C_{\mathfrak{F}}} = \left(\sup_{0\leq t\leq T}\mathbb{E}\left\|\varphi(t)\right\|_H^p\right)^{\frac{1}{p}}$.

- $A : D(A) \subset H \to H$ is the infinitesimal generator of a $C_0$-semigroup $S : H \to H$ and $B \in L(U,H)$.

We consider the fractional linear stochastic Ito equation of the form

$$\begin{aligned} {}^C D_t^\alpha y(t) &= \left[Ay(t) + Bu(t)\right]dt + \sigma(t)\,dw(t), \\ y(0) &= y_0 \in H \end{aligned} \tag{1}$$

in a separable Hilbert space $H$. Here ${}^C D_t^\alpha$ is the Caputo derivative, $\frac{1}{2} < \alpha \leq 1$, $y : [0,T]\times\Omega \to H$ is the state function, $u : [0,T]\times\Omega \to U$ is the control function, $U$ is a separable Hilbert space, $A$ is an infinitesimal generator of strongly continuous compact semigroup, $B : U \to H$ is a linear bounded operator, $\sigma : [0,T]\times\Omega \to L_2^0$ is a stochastic process.

**Definition 1.1.** *[13] The fractional integral of order $\alpha$ with the lower limit $a \in R$ for a function $f : [a,\infty) \to R$ is*

$$I_{a,t}^\alpha = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}\,ds}, \quad t > a, \quad \alpha > 0,$$

*provided that the right-hand side is point-wise defined on $[a,\infty)$, where $\Gamma$ is the gamma function. The Riemann-Liouville derivative of order $\alpha$ with the lower limit zero for a function $f : [0,\infty) \to R$ is*

$$\begin{aligned} {}^L D_{0,t}^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}\,ds}, \\ &\quad t > 0, \quad n-1 < \alpha < n. \end{aligned}$$

*The Caputo derivative of order $\alpha$ for a function $f : [0,\infty) \to R$ is*

$$\begin{aligned} {}^C D_{0,t}^\alpha f(t) &= {}^L D_{0,t}^\alpha\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \\ &\quad t > 0, \quad n-1 < \alpha < n. \end{aligned}$$

If $f$ is an abstract function with values in $X$, then the integrals in the definition are understood in the Bochner sense.

Suppose $M_S = \sup\{\|S(t)\| : t \geq 0\}$ and define

$$\mathfrak{T}(t) = \int_0^\infty \xi_\alpha(\theta) S(t^\alpha \theta)\, d\theta,$$

$$\mathfrak{S}(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) S(t^\alpha \theta)\, d\theta, \quad t \geq 0,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\varpi_\alpha\left(\theta^{-1/\alpha}\right) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}$$

$$\times\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta > 0,$$

where $\xi_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty \xi_\alpha(\theta)\, d\theta = 1.$$

It is known that [14]

(i) For any fixed $t \geq 0$ the operators $\mathfrak{T}(t)$ and $\mathfrak{S}(t)$ are linear bounded.

(ii) $\{\mathfrak{T}(t) : t \geq 0\}$ and $\{\mathfrak{S}(t) : t \geq 0\}$ are strongly continuous.

(iii) $\{\mathfrak{T}(t) : t > 0\}$ and $\{\mathfrak{S}(t) : t > 0\}$ are compact operators if $S(t)$, $t > 0$, is compact.

The paper is organized as follows. In Section 2 we introduce mean exact/mean exact finite dimensional/mean square approximate controllability/mean exact finite-approximate controllability concepts and prove that the mean square approximate controllability on $[0,T]$ is equivalent to the mean exact finite-approximate controllability on $[0,T]$. Section 3 is devoted to mean exact finite-approximate controllability of the linear stochastic heat equation.

## 2. Mean exact finite-approximate controllability

**Definition 2.1.** *A stochastic process $y \in C_{\mathfrak{F}}\left(0,T;L^2(\mathfrak{F},H)\right)$ is said to be a mild solution of the system (1) if for any $u \in L^p_{\mathfrak{F}}(0,T;U)$ the following stochastic integral equation is satisfied:*

$$\begin{aligned} y(t) &= \mathfrak{T}(t) y_0 \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathfrak{S}(t-s) Bu(s)\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathfrak{S}(t-s)\sigma(s)\, dw(s). \end{aligned}$$

We introduce the following concepts of controllability for the system (1):

**Definition 2.2.** *System (1) is said to fulfill the property of mean exact controllability or to be exactly controllable in mean if* $\operatorname{Im}\mathbf{E}L_0^T = H$*,where*

$$L_0^T u = \int_0^T (T-s)^{\alpha-1}\mathfrak{S}(T-s) Bu(s)\, ds.$$

It is clear that $L_0^T : L^2_{\mathfrak{F}}(0,T;U) \to L^2(\mathfrak{F}_T, H)$ is a linear bounded operator for $\frac{1}{2} < \alpha \leq 1$. In what follows we assume that $\frac{1}{2} < \alpha \leq 1$.

**Definition 2.3.** *Let $M$ be a finite dimensional subspace of $H$ and denote by $\pi_M$ the orthogonal projection from $H$ onto $M$. System (1) is said to fulfill the property of mean exact finite dimensional controllability or to be exactly controllable in mean to finite dimensional subspace if* $\operatorname{Im}\pi_M \mathbf{E} L_0^T = M$.

**Definition 2.4.** *System (1) fulfills the property of mean square approximate controllability or is approximately controllable in mean square if given any $y_0 \in H$, $y_T \in L^2(\mathfrak{F}_T, H)$ and $\varepsilon > 0$, there exists a control $u_\varepsilon \in L^2_{\mathfrak{F}}(0, T; U)$ such that the solutions to (1) satisfy*

$$\mathbf{E}\left\|y(T; u_\varepsilon) - y_T\right\|^2 < \varepsilon^2.$$

**Definition 2.5.** *Let $M$ be a finite dimensional subspace of $H$ and denote by $\pi_M$ the orthogonal projection from $H$ onto $M$. System (1) is said to be mean exact finite dimensional and mean square approximately controllable if for given any $y_0 \in H$, $y_T \in L^2(\mathfrak{F}_T, H)$ and $\varepsilon > 0$, there exists a control $u_\varepsilon \in L^2_{\mathfrak{F}}(0, T; U)$ such that the solutions to (1) satisfy*

$$\mathbf{E}\left\|y(T; u_\varepsilon) - y_T\right\|^2 < \varepsilon^2, \tag{2}$$

$$\pi_M \mathbf{E} y(T; u_\varepsilon) = \pi_M \mathbf{E} y_T. \tag{3}$$

Simultaneous mean exact finite dimensional and mean square approximate controllability is referred to as mean exact finite-approximate controllability. This means that the control $u$ can be chosen such that $y(T; u_\varepsilon)$ satisfies (2) and simultaneously a finite number of constrains (3).

**Theorem 2.6.** *[10] The control system (1) is mean square approximately controllable on $[0, T]$ if and only if any one of the following conditions holds.*

**(a)** $\overline{\operatorname{Im} L_0^T} = L^2(\mathfrak{F}_T, H)$;

**(b)** $B^* \mathfrak{S}^*(T - s)\mathbf{E}\{\varphi \mid \mathfrak{F}_s\} = 0$, $0 < s < T$ *implies that* $\varphi = 0$;

**(c)** $\Pi_0^T = L_0^T \left(L_0^T\right)^* : L^2(\mathfrak{F}_T, H) \to L^2(\mathfrak{F}_T, H)$ *is positive, that is,* $\mathbf{E}\left\langle \Pi_0^T \varphi, \varphi\right\rangle > 0$ *for all nonzero* $\varphi \in L^2(\mathfrak{F}_T, H)$.

Given the finite dimensional subspace $M \subset H$, the target $y_T \in L^2(\mathfrak{F}_T, H)$. For any $\varepsilon > 0$, we introduce the following functional

$$\begin{aligned} J_\varepsilon(\varphi) = \frac{1}{2}\int_0^T &(T - s)^{\alpha-1} \\ &\times \mathbf{E}\left\|B^* \mathfrak{S}^*(T - s)\mathbf{E}\{\varphi \mid \mathfrak{F}_s\}\right\|^2 ds \\ &+ \varepsilon\sqrt{\mathbf{E}\left\|\varphi - \pi_M \mathbf{E}\varphi\right\|^2} - \mathbf{E}\langle \varphi, h\rangle, \end{aligned} \tag{4}$$

where $h = y_T - \mathfrak{T}(T) y_0 - \int_0^T (T - s)^{\alpha-1} \mathfrak{S}(T - s)\sigma(s) dw(s)$. From the definition of $J_\varepsilon$, we see immediately that it is continuous and strictly convex. Next lemma shows that $J_\varepsilon(\varphi)$ is coercive.

**Lemma 2.7.** *Assume that the system (1) is mean square approximately controllable on $[0, T]$. For any sequence $\left\{\varphi_n \in L^2(\mathfrak{F}_T, H) : n \in \mathbb{N}\right\}$ converging to $\infty$ in $L^2(\mathfrak{F}_T, H)$ we have*

$$\lim_{n\to\infty} \frac{J_\varepsilon(\varphi_n)}{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}} \geq \varepsilon. \tag{5}$$

*Proof.* Suppose for the sake of contradiction that it is not true. Then there exists sequences $\{\varphi_n\} \subset L^2(\mathfrak{F}_T, H)$, with $\mathbf{E}\left\|\varphi_n\right\|^2 \to \infty$, such that

$$\lim_{n\to\infty} \frac{J_\varepsilon(\varphi_n)}{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}} < \varepsilon. \tag{6}$$

From (4) it follows that

$$\begin{aligned} \frac{J_\varepsilon(\varphi_n)}{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}} &= \frac{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}}{2}\int_0^T (T-s)^{\alpha-1} \\ &\quad\times \mathbf{E}\left\|B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widetilde{\varphi}_n \mid \mathfrak{F}_s\}\right\|^2 ds \\ &\quad+\varepsilon\sqrt{\mathbf{E}\left\|\widetilde{\varphi}_n-\pi_M\mathbf{E}\widetilde{\varphi}_n\right\|^2}-\mathbf{E}\langle\widetilde{\varphi}_n,h\rangle \\ &\geq \varepsilon\sqrt{\mathbf{E}\left\|\widetilde{\varphi}_n-\pi_M\mathbf{E}\widetilde{\varphi}_n\right\|^2}-\mathbf{E}\langle\widetilde{\varphi}_n,h\rangle, \end{aligned} \tag{7}$$

where $\widetilde{\varphi}_n = \dfrac{\varphi_n}{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}}$. Since $\sqrt{\mathbf{E}\left\|\widetilde{\varphi}_n\right\|^2} = 1$, we can extract a subsequence (still denoted by $\widetilde{\varphi}_n$), which weakly converges in $L^2(\mathfrak{F}_T, H)$ to an element $\widetilde{\varphi}$ in $L^2(\mathfrak{F}_T, H)$. Compactness of $S(t)$, $t > 0$, implies that

$$B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widetilde{\varphi}_n \mid \mathfrak{F}_s\} \to B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widetilde{\varphi} \mid \mathfrak{F}_s\} \quad \text{strongly} \tag{8}$$

in $C\left(0,T;L^2(\mathfrak{F},H)\right)$. From (4), it follows that

$$\begin{aligned} \frac{J_\varepsilon(\varphi_n)}{\mathbf{E}\left\|\varphi_n\right\|^2} &= \frac{1}{2}\int_0^T (T-s)^{\alpha-1} \\ &\quad\times \mathbf{E}\left\|B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widetilde{\varphi}_n \mid \mathfrak{F}_s\}\right\|^2 ds \\ &\quad+\frac{\varepsilon}{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}}\sqrt{\mathbf{E}\left\|\varphi_n-\pi_M\mathbf{E}\varphi_n\right\|^2} \\ &\quad-\frac{1}{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}}\mathbf{E}\langle\widetilde{\varphi}_n,h\rangle. \end{aligned}$$

Thus, noting that $\mathbf{E}\left\|\varphi_n\right\|^2 \to \infty$, by (6)-(8) and the Fatou lemma

$$\begin{aligned} &\int_0^T (T-s)^{\alpha-1}\,\mathbf{E}\left\|B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widetilde{\varphi} \mid \mathfrak{F}_s\}\right\|^2 ds \\ &\leq \lim_{n\to\infty}\int_0^T (T-s)^{\alpha-1} \\ &\times\mathbf{E}\left\|B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widetilde{\varphi}_n \mid \mathfrak{F}_s\}\right\|^2 ds = 0. \end{aligned}$$

Mean square approximate controllability implies that $\widetilde{\varphi} = 0$, and we deduce that

$$\widetilde{\varphi}_n \rightharpoonup 0 \;\text{ weakly in } L^2(\mathfrak{F}_T, H) \quad\text{and}\quad \mathbf{E}\widetilde{\varphi}_n \rightharpoonup 0 \;\text{ weakly in } H.$$

Since $M$ is finite-dimensional and the orthogonal projection $\pi_M : H \to M$ is compact, $\pi_M\mathbf{E}\widetilde{\varphi}_n \longrightarrow 0$ strongly in $H$. Hence, from (7) it follows that

$$\begin{aligned} &\lim_{n\to\infty}\frac{J_\varepsilon(\varphi_n)}{\sqrt{\mathbf{E}\left\|\varphi_n\right\|^2}} \\ &\geq \lim_{n\to\infty}\left(\varepsilon\sqrt{\mathbf{E}\left\|\widetilde{\varphi}_n-\pi_M\mathbf{E}\widetilde{\varphi}_n\right\|^2}-\mathbf{E}\langle\widetilde{\varphi}_n,h\rangle\right) = \varepsilon, \end{aligned}$$

which contradicts with (6) and proves the claim (5). □

**Theorem 2.8.** *Let* $\frac{1}{2} < \alpha \leq 1$. *If the system (1) is mean square approximately controllable on* $[0, T]$, *then it is mean exact finite-approximately controllable on* $[0, T]$ *with the control* $u_\varepsilon(\cdot) = B^*\mathfrak{S}^*(T - \cdot)\,\mathbf{E}\{\widehat{\varphi}_\varepsilon \mid \mathfrak{F}_s\}$, *where* $\widehat{\varphi}_\varepsilon$ *is the minimizer of the functional* $J_\varepsilon(\varphi)$.

*Proof.* We know that $J_\varepsilon$ is strictly convex. Then $J_\varepsilon(\varphi)$ has a unique critical point which is a unique minimizer:

$$\widehat{\varphi}_\varepsilon \in X : J_\varepsilon(\widehat{\varphi}_\varepsilon) = \min_{\varphi\in L^2(\mathfrak{F}_T, H)} J_\varepsilon(\varphi).$$

Given any $\psi \in L^2(\mathfrak{F}_T, H)$ and $\lambda \in R$ we have

$$J_\varepsilon(\widehat{\varphi}_\varepsilon) \leq J_\varepsilon(\widehat{\varphi}_\varepsilon + \lambda\psi)$$

or, in other words,

$$\begin{aligned}
&\varepsilon\sqrt{\mathbf{E}\left\|\widehat{\varphi}_\varepsilon - \pi_M\mathbf{E}\widehat{\varphi}_\varepsilon\right\|^2} \\
&\leq \frac{\lambda^2}{2}\int_0^T (T-s)^{\alpha-1} \\
&\times \mathbf{E}\left\|B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\psi \mid \mathfrak{F}_s\}\right\|^2 ds \\
&+ \lambda\int_0^T (T-s)^{\alpha-1}\,\mathbf{E}\left\langle B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widehat{\varphi}_\varepsilon \mid \mathfrak{F}_s\},\right. \\
&\left. B^*\mathfrak{S}^*(T-s)\,\psi\right\rangle ds \\
&+ \varepsilon\sqrt{\mathbf{E}\left\|\widehat{\varphi}_\varepsilon + \lambda\psi - \pi_M\mathbf{E}(\widehat{\varphi}_\varepsilon + \lambda\psi)\right\|^2} \\
&- \lambda\mathbf{E}\langle\psi, h\rangle.
\end{aligned}$$

Dividing this inequality by $\lambda > 0$ and letting $\lambda \longrightarrow 0^+$ we obtain that

$$\begin{aligned}
&\mathbf{E}\langle\psi, h\rangle \\
&\leq \int_0^T (T-s)^{\alpha-1}\,\mathbf{E}\left\langle B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widehat{\varphi}_\varepsilon \mid \mathfrak{F}_s\},\right. \\
&\left. B^*\mathfrak{S}^*(T-s)\,\psi\right\rangle ds \\
&+ \varepsilon\underline{\lim}_{\lambda\longrightarrow 0^+}\frac{1}{\lambda}\left(\sqrt{\mathbf{E}\left\|\widehat{\varphi}_\varepsilon + \lambda\psi - \pi_M\mathbf{E}(\widehat{\varphi}_\varepsilon + \lambda\psi)\right\|^2}\right. \\
&\left. - \sqrt{\mathbf{E}\left\|\widehat{\varphi}_\varepsilon - \pi_M\mathbf{E}\widehat{\varphi}_\varepsilon\right\|^2}\right) ds \\
&\leq \int_0^T (T-s)^{\alpha-1}\,\mathbf{E}\left\langle B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widehat{\varphi}_\varepsilon \mid \mathfrak{F}_s\},\right. \\
&\left. B^*\mathfrak{S}^*(T-s)\,\psi\right\rangle ds \\
&+ \varepsilon\sqrt{\mathbf{E}\left\|\psi - \pi_M\mathbf{E}\psi\right\|^2}.
\end{aligned}$$

Repeating this argument with $\lambda < 0$ we obtain finally that

$$\begin{aligned}
&\left|\int_0^T (T-s)^{\alpha-1}\,\mathbf{E}\left\langle B^*\mathfrak{S}^*(T-s)\,\mathbf{E}\{\widehat{\varphi}_\varepsilon \mid \mathfrak{F}_s\},\right.\right. \\
&\left.\left. B^*\mathfrak{S}^*(T-s)\,\psi\right\rangle ds - \mathbf{E}\langle\psi, h\rangle\right| \\
&\leq \varepsilon\sqrt{\mathbf{E}\left\|\psi - \pi_M\mathbf{E}\psi\right\|^2}. \qquad (9)
\end{aligned}$$

On the other hand, with $u_\varepsilon = B^*\Im^*(T-s)\,\mathbf{E}\{\widehat{\varphi}_\varepsilon \mid \mathfrak{F}_s\}$ we have

$$\begin{aligned}&\int_0^T (T-s)^{\alpha-1}\,\mathbf{E}\left\langle B^*\Im^*(T-s)\,\mathbf{E}\{\widehat{\varphi}_\varepsilon \mid \mathfrak{F}_s\},\right.\\ &\left.B^*\Im^*(T-s)\,\psi\right\rangle ds - \mathbf{E}\langle\psi,h\rangle\\ &= \mathbf{E}\langle y(T;u_\varepsilon)-y_T,\psi\rangle.\end{aligned} \tag{10}$$

Then, combining (9) and (10) we obtain that

$$\left|\mathbf{E}\langle y(T;u_\varepsilon)-y_T,\psi\rangle\right| \le \varepsilon\sqrt{\mathbf{E}\left\|\psi-\pi_M\mathbf{E}\psi\right\|^2}$$

holds for any $\psi\in L^2(\mathfrak{F}_T,H)$. Thus

$$\begin{aligned}&\left|\mathbf{E}\langle y(T;u_\varepsilon)-y_T,\psi\rangle\right| \le \varepsilon\sqrt{\mathbf{E}\left\|\psi\right\|^2}\\ &\qquad\Longrightarrow \sqrt{\mathbf{E}\left\|y(T;u_\varepsilon)-y_T\right\|^2}\le\varepsilon.\end{aligned}$$

Moreover, for any $\psi\in H$

$$\begin{aligned}&\left|\mathbf{E}\langle y(T;u_\varepsilon)-y_T,\pi_M\psi\rangle\right|\\ &= \left|\langle\pi_M\mathbf{E}(y(T;u_\varepsilon)-y_T),\psi\rangle\right| = 0\\ &\Longrightarrow \pi_M\mathbf{E}(y(T;u_\varepsilon)-y_T)=0.\end{aligned}$$

□

## 3. Example

Consider the stochastic control system governed by the fractional stochastic linear heat equation

$$\begin{aligned}&{}^CD_t^\alpha y(t,\theta) = \frac{\partial^2}{\partial\theta^2}y(t,\theta)+Bu(t,\theta)+\sigma\frac{dw(t)}{dt},\\ &\qquad y(t,0)=y(t,\pi)=0,\ 0\le t\le T,\\ &\qquad y(0,\theta)=f(\theta)\in L^2[0,\pi],\ \ 0\le\theta\le\pi.\end{aligned} \tag{11}$$

Let $H=L^2[0,\pi]$. Operator $A:D(A)\subset H\to H$ is defined by $Az=z''$ with the domain

$$\begin{aligned}&D(A)=\{z\in H: z,z' \text{ are absolutely continuous},\\ &\qquad z''\in H, z(0)=z(\pi)=0\}.\end{aligned}$$

Then

$$Az=\sum_{n=1}^{\infty}\left(-n^2\right)\langle z,z_n\rangle z_n,$$

where $z,z_n\in H$, $z_n(\theta)=\sqrt{2/\pi}\sin(n\theta)$, $n=1,2,\ldots$, is the orthogonal set of eigenvectors of $A$. It is well known that $A$ is the infinitesimal generator of a compact analytic semigroup $S(t)$, $t>0$. Define an infinite dimentional control space $U$ by

$$U=\left\{u=\sum_{n=2}^{\infty}u_nz_n(\theta):\|u\|^2=\sum_{n=2}^{\infty}u_n^2<\infty\right\}.$$

Let $B : U \to H$ be the control operator defined by

$$Bu(\theta) = 2u_2 z_1(\theta) + \sum_{n=2}^{\infty} u_n z_n(\theta), \quad u = \sum_{n=2}^{\infty} u_n z_n(\theta).$$

It is known that the fractional stochastic linear system of (11) is mean square approximately controllable on $[0, T]$, see [11]. Thus by Theorem 2.8 the system (11) is mean exact finite-approximately controllable on $[0, T]$.

## 4. Conclusion

In this paper, we have focused on concepts of approximate controllability for linear fractional stochastic systems in Hilbert spaces. We introduced the concepts of mean exact/mean exact finite dimensional/mean square approximate controllability in the framework of linear fractional stochastic evolution equations. We have proved that mean square approximate controllability is equivalent to simultaneous mean exact finite dimensional and mean square approximate controllability.

One possible direction in which to generalise the results of this paper is by looking at approximate controllability concepts for nonlinear fractional stochastic evolution equations. Another direction in which we would like to extend is to consider approximate controllability problems for nonlinear stochastic systems driven by Rosenblatt process, nonlinear stochastic delay systems with delayed perturbation of matrices.

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