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On Modeling Heavy Tailed Medical Care Insurance Data via a New Member of T-X Family

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Abstract. Heavy tailed distributions are worthwhile in modeling heavy tailed data. The researchers are often in search of such distributions to provide best fit to heavy tailed data. In this article, a new T-X family member called, a new exponential cosine-X family is introduced. A special sub-model of the proposed family, called, a new exponential cosine Weibull distribution is studied in detail. Some mathematical properties along with the useful series expansion of distribution and density functions of the proposed class are obtained. Two useful characterizations of this family are also provided. We consider the maximum likelihood and Bayesian estimation procedures to estimate the parameters of the proposed family. Monti Carlo simulation study is done to access the behavior of these estimators. For the illustrative purposes, a real-life application of the proposed family to a heavy tailed medical care insurance data set is provided. Finally, Bayesian analysis and performance of Gibbs sampling for the medical care insurance data are also carried out.

1. Introduction

Speaking broadly, classical distributions such as Weibull, gamma, lognormal, Beta, Pareto, Lomax, exponential and Rayleigh distributions are widely used to model data in applied fields such as engineering, sciences, actuarial, biotic studies, finance and insurance, among others. However, in a number of situations such as finance and actuarial sciences, data sets possess a behavior having extreme values yielding tails which are much heavier than those of the standard classical distributions, see for example [1] and [2].

In the recent era, there has been a great deal of study in the financial and actuarial literature on heavy tailed distributions in a number of insurance and its related frameworks. Henceforth, the introduction of the heavy tailed distributions to model real life heavy tailed insurance data is an interesting research topic. Therefore, in a series of recent papers, numerous authors have shown a deep attention toward the introduction of a number of new models possessing heavy tails. In particular, insurance data sets are frequently positively skewed, unimodal having tick tail, for detail see ([3]-[5]). The distributions having heavy tails provide adequate fits to heavy tailed insurance data sets.

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A distribution is said to have heavier tails than exponential distribution, if its survival function (sf) satisfies

$$\lim_{x\to\infty}\frac{1-F(x)}{e^{-px}}>0,$$

for all $p > 0$. For further information see [6]. Due to the prominent applications of statistical distributions in actuarial sciences, the researchers have been working to introduce new distributions for adequately modeling heavy tailed insurance data sets. For example see ([7]-[15]), among others.

The aforementioned distributions have been introduced through many different approaches such as introducing new parameters to the existing distributions. The commonly used methods are (a) transformation approach, (b) compounding of two or more distributions, (c) composition of distributions, and (d) finite mixture of models, for detail see [16].

We further carry this branch of research and propose a new approach to obtained new heavy tailed distributions to provide adequate fits to heavy tailed medical care insurance data sets. Let $p(t)$ be the density of a random variable $T \in [a_1, a_2]$ for $-\infty \le a_1 < a_2 < \infty$ and let $K[F(x;\xi)]$ be a function of cumulative distribution function (cdf) of a random variable X, satisfying

1. $K[F(x;\xi)] \in [a_1, a_2]$,
2. $K[F(x;\xi)]$ is differentiable and monotonically increasing, and
3. $K[F(x;\xi)] \to a_1$ as $x \to -\infty$ and $K[F(x;\xi)] \to a_2$ as $x \to \infty$.

Recently, [17] proposed the T-X family method by

$$G(x)=\int_{a_1}^{K[F(x;\xi)]} p(t)\ dt, \qquad x\in\mathbb{R}, \tag{1}$$

where $K[F(x;\xi)]$ fulfills the conditions stated above. The probability density function (pdf) corresponding to (1) is

$$g(x)=\left\{\frac{\partial}{\partial x}K[F(x;\xi)]\right\}p\{K[F(x;\xi)]\}, \qquad x\in\mathbb{R}.$$

Deploying the T-X approach, a number of new families of distributions have been proposed in the literature, see [18]. Recently, [19] proposed a new family called the exponential cosine-X(EC-X) family by replacing $p(t)$ with the pdf of the exponential distribution with rate parameter $\lambda = 1$ given by $p(t) = e^{-t}$ and $K[F(x;\xi)] = -\log\left[\cos\left(\frac{\pi}{2}F(x;\xi)\right)\right]$. The cdf of the EC-$X$ family is

$$G(x;\theta,\xi)=1-\left\{\cos\left(\frac{\pi}{2}F(x;\xi)\right)\right\}^{\theta}, \qquad \theta>0,\ x,\xi\in\mathbb{R}. \tag{2}$$

In the present work, we propose another member of T-X family, may be named as a new exponential cosine-X(NEC-X) family. This family is introduced by taking $p(t)$ to be the pdf of the exponential distribution exp(1) and $K[F(x;\xi)] = -\log\left[\cos\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right)\right]$ in (1). The cdf of the NEC-X family is

$$G(x;\sigma,\xi)=1-\cos\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right), \qquad \sigma>0,\ \ x,\xi\in\mathbb{R}, \tag{3}$$

where, $\bar{\sigma} = (1-\sigma)$. Clearly, for $\sigma = 1$, expression (3) is a special case of (2). The pdf and survival function (sf) corresponding to (3) are given by

$$g(x;\sigma,\xi)=\frac{\pi\sigma f(x;\xi)}{2\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^2}\sin\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right), \qquad x\in\mathbb{R}, \tag{4}$$

and

$$S(x;\sigma,\xi)=\cos\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right), \qquad x\in\mathbb{R},$$

respectively.

The main goal of this research is to model heavy tailed medical care insurance data via incorporating a single additional parameter to a family of distribution functions. The introduction of additional parameter brings more flexibility to the proposed family. Furthermore, the key motivations for using NEC-X family in the practice are

- A prominent and convenient approach of adding an extra parameter to modify the existing distributions.
- To provide best fit to heavy tailed data.
- To propose the generalized version of the distribution having closed form for the cdf.
- To provide better fits than the competing modified models having higher, lower or the same number of parameters than the proposed model.

The article is structured as follows. A special sub-model of the new family is presented in Section 2. Useful expansions for the density and cdf of the NEC-Xfamily are obtained in Section 3. Some mathematical properties are derived in Section 4. Characterizations of the NEC-X distribution are provided in Section 5. The estimation of the model parameters through maximum likelihood method and simulation study are presented in Section 6. An application to a medical care insurance data set is analyzed in Section 7. Bayesian analysis as well as the Gibbs sampling procedure for the real data set are discussed in Section 8. Finally, concluding remarks are provided in the last section.

2. Sub-Model Description

In this section, we define a special sub-case of the NEC-X family, called the new exponential cosine-Weibull (NEC-W) distribution. Let $F(x;\xi)$ be the cdf of the two parameters Weibull distribution given by $F(x;\xi) = 1 - e^{-\gamma x^{\alpha}},\ x \geq 0,\ \alpha, \gamma > 0$, where $\xi = (\alpha, \gamma)$. Then, the distribution function of the NEC-W distribution is

$$G(x;\sigma,\xi) = 1 - \cos\left(\frac{\pi}{2}\left\{\frac{\left(1 - e^{-\gamma x^{\alpha}}\right)}{1 - \bar{\sigma}e^{-\gamma x^{\alpha}}}\right\}\right), \quad x \geq 0, \sigma > 0, \xi \in \mathbb{R}, \tag{5}$$

and the pdf corresponding to (5) is given by

$$g(x;\sigma,\xi) = \frac{\pi\sigma\alpha\gamma x^{\alpha-1}e^{-\gamma x^{\alpha}}}{2\left(1 - \bar{\sigma}e^{-\gamma x^{\alpha}}\right)^2}\sin\left(\frac{\pi}{2}\left\{\frac{\left(1 - e^{-\gamma x^{\alpha}}\right)}{1 - \bar{\sigma}e^{-\gamma x^{\alpha}}}\right\}\right), \qquad x > 0. \tag{6}$$

For selected values of the model parameters, different plots of the pdf of the NEC-W distribution are sketched in Figure 1.

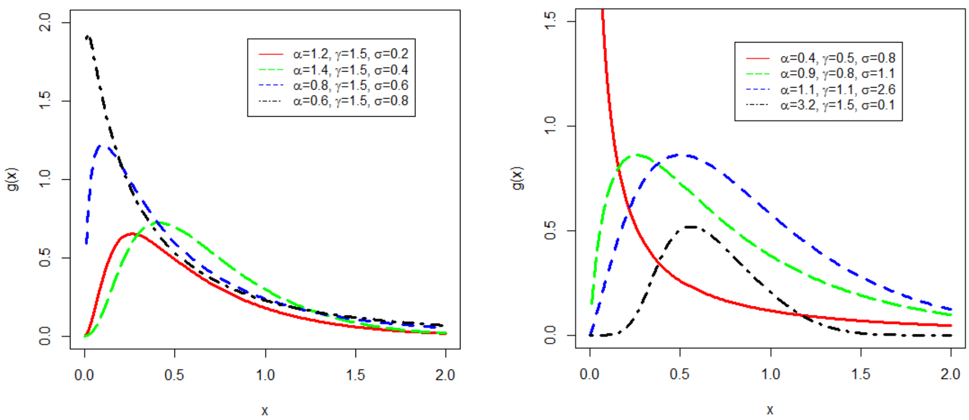

Figure 1: Some possible shapes for the pdf of the NEC-W distribution.

3. Mixture Representation

The following section offers a mixture representation of the cdf (3) and pdf (4).

3.1. Mixture representation of the density function

The Maclaurin series expansion for $\sin(x)$ is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \tag{7}$$

Let $x = \frac{\pi}{2}\left(\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right)$, from (5), we have

$$\sin\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\left(\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right)^{2n+1}. \tag{8}$$

Using (8) in (4), we obtain

$$g(x;\sigma,\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n \sigma f(x;\xi)}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+2} \frac{F(x;\xi)^{2n+1}}{\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^{2n+3}}. \tag{9}$$

Also, we know that

$$\frac{1}{(1-x)^{v}} = \sum_{m=0}^{\infty}\binom{m+v-1}{v-1} x^m. \tag{10}$$

Letting $v = 2n+1$ and $x = \bar{\sigma}\bar{F}(x;\xi)$, from (9), we have

$$\frac{1}{\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^{2n+3}} = \sum_{m=0}^{\infty}\binom{m+2n+2}{2n+2}\bar{\sigma}^m \left(\bar{F}(x;\xi)\right)^m. \tag{11}$$

Using (11) in (9), we arrive at

$$g(x;\sigma,\xi) = \sum_{n,m=0}^{\infty} \frac{(-1)^n \sigma\bar{\sigma}^m}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+2}\binom{m+2n+2}{2n+2} f(x;\xi)\, F(x;\xi)^{2n+1}\left(\bar{F}(x;\xi)\right)^m,$$

and finally, we have the following representation form

$$g(x;\sigma,\xi) = \sum_{n,m,k=0}^{\infty} \eta_{n,m,k} f(x;\xi) F(x;\xi)^{2n+k+1}, \tag{12}$$

where $\eta_{n,m,k} = \frac{(-1)^{n+k}\sigma\bar{\sigma}^m}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+2}\left(\begin{array}{c} m \\ k \end{array}\right)\left(\begin{array}{c} m+2n+2 \\ 2n+2 \end{array}\right)$.

3.2. Series representation of the distribution function

The Maclaurin series expansion for $\cos(x)$ is

$$\cos(x) = \frac{d}{dx}\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n}. \tag{13}$$

Letting $x = \frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}$, from (13), we have

$$\cos\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!}\left(\frac{\pi}{2}\right)^{2n} \frac{F(x;\xi)^{2n}}{\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^{2n}} \ . \tag{14}$$

Using (11) in (14), we have

$$\begin{aligned}\cos\left(\tfrac{\pi}{2}\left\{\tfrac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right) = &\sum_{n,m=0}^{\infty} \tfrac{(-1)^n(\bar{\sigma})^m}{2n!}\left(\tfrac{\pi}{2}\right)^{2n}\left(\begin{array}{c} m+2n-1 \\ 2n-1 \end{array}\right) F(x;\xi)^{2n} \\ &\times \left(\bar{F}(x;\xi)\right)^{2n}.\end{aligned} \tag{15}$$

Finally, we get

$$\cos\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right) = \sum_{n,m,k=0}^{\infty} \eta^{/}_{n,m,k} F(x;\xi)^{2n+k},$$

where $\eta^{/}_{n,m,k} = \frac{(-1)^{n+k}(\bar{\sigma})^m}{2n!}\left(\frac{\pi}{2}\right)^{2n}\left(\begin{array}{c} 2n \\ k \end{array}\right)\left(\begin{array}{c} m+2n-1 \\ 2n-1 \end{array}\right)$. Hence, the final form of the series representation of the cdf (3) is given by

$$G(x;\sigma,\xi) = 1 - \sum_{n,m,k=0}^{\infty} \eta^{/}_{n,m,k} F(x;\xi)^{2n+k}.$$

4. Mathematical Properties

In this section, we discuss some mathematical properties of the NEC-X family.

4.1. Quantile function

Suppose X follows the NEC-X distribution, then the quantile function of X can be obtained via inverting $G\left(x_q;\sigma,\xi\right) = q$ in (3). We obtain

$$x_q = F^{-1}\left(\frac{(\pi)\,arc\cos(1-q)}{2-(\bar{\sigma}\pi)\,arc\cos(1-q)}\right). \tag{16}$$

The expression (16) has a closed form solution in x_q, which makes it easier to generate random numbers.

4.2. Moments

Let X has pdf (4), then the r^{th} moment of X denoted by $\mu_r^{/}$ is

$$\mu_r^{/} = \int_{-\infty}^{\infty} x^r g\left(x; \sigma, \xi\right) dx, \tag{17}$$

and using (3) in (16), we have

$$\mu_r^{/} = \sum_{n,m,k=0}^{\infty} \eta_{n,m,k} \int_{-\infty}^{\infty} x^r f\left(x; \xi\right) F\left(x; \xi\right)^{2n+k+1} dx. \tag{18}$$

Using (18), we can easily derive the moments for any sub-model of the NEC-X family. Furthermore, with (18), we can derive the moment generating function of X, $M_x\left(t\right)$, is given by

$$M_x\left(t\right) = \sum_{n,m,k=0}^{\infty} \frac{\eta_{n,m,k}}{r!} t^r \int_{-\infty}^{\infty} x^r f\left(x; \xi\right) F\left(x; \xi\right)^{2n+k+1} dx. \tag{19}$$

4.3. Residual life

Let X have pdf (4), then the residual life of X is

$$\varepsilon\left(x\right) = \frac{S\left(x+t\right)}{S\left(x\right)},$$

$$\varepsilon\left(x\right) = \frac{\cos\left(\frac{\pi}{2}\left\{\frac{F(x+t;\xi)}{1-\bar{\sigma}\bar{F}(x+t;\xi)}\right\}\right)}{\cos\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right)}.$$

4.4. Reverse residual life

The reverse residual lifetime of X represented by $\bar{\varepsilon}\left(x\right)$ is

$$\bar{\varepsilon}\left(x\right) = \frac{S\left(x-t\right)}{S\left(x\right)},$$

$$\bar{\varepsilon}\left(x\right) = \frac{\cos\left(\frac{\pi}{2}\left\{\frac{F(x-t;\xi)}{1-\bar{\sigma}\bar{F}(x-t;\xi)}\right\}\right)}{\cos\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right)}.$$

5. Characterizations

This section deals with certain characterizations of the NEC-X distribution in two directions: (i) based on a simple relationship between two truncated moments; (ii) in terms of the hazard function. We present our characterizations (i) and (ii) in two subsections.

5.1. Characterizations based on a simple relationship between two truncated moments

In this subsection we present characterizations of the NEC-X distribution in terms of a simple relationship between two truncated moments. The first characterization result employs Theorem 1 of [20]. As shown in [21], this characterization is stable in the sense of weak convergence.

Theorem 5.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d; e]$ be an interval for some $d<e$ ($d = -\infty$; $e = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function G and let q_1 and q_2 be two real functions defined on H such that

$$E\left(q_2(X)\,|X \geq x\right) = E\left(q_1(X)\,|X \geq x\right)\eta(x), \qquad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H), \eta \in C^2(H)$ and G is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then G is uniquely determined by the functions q_1, q_2 and η, particularly

$$G(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,q_1(u) - q_2(u)}\right| \exp(-s(u))\,du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Remark 5.1. The goal in Theorem 5.1, is to have $\eta(x)$ as simple as possible.

Proposition 5.1. Suppose X is a continuous random variable. Let $q_1(x) = \left[\sin\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right)\right]^{-1}$ and $q_2(x) = q_1(x)\left[1 - \bar{\sigma}\bar{F}(x;\xi)\right]^{-1}$ for $x \in \mathbb{R}$. Then X has density function (4) if and only if the function η defined in Theorem 5.1 is given by

$$\eta(x) = \frac{1}{2}\left[1 + \left(1 - \bar{\sigma}\bar{F}(x;\xi)\right)^{-1}\right], \qquad x \in \mathbb{R}.$$

Proof. Suppose X is a random variable with density function (4), then we have

$$(1 - G(x))\,E\left(q_1(X)\,|X \geq x\right) = \frac{\pi\sigma}{2\bar{\sigma}}\left[\left(1 - \bar{\sigma}\bar{F}(x;\xi)\right)^{-1} - 1\right], \qquad x \in \mathbb{R},$$

and

$$(1 - G(x))\,E\left(q_2(X)\,|X \geq x\right) = \frac{\pi\sigma}{4\bar{\sigma}}\left[\left(1 - \bar{\sigma}\bar{F}(x;\xi)\right)^{-2} - 1\right], \qquad x \in \mathbb{R},$$

and finally

$$\eta(x)\,q_1(x) - q_2(x) = \frac{1}{2}q_1(x)\left[1 - \left(1 - \bar{\sigma}\bar{F}(x;\xi)\right)^{-1}\right] > 0, \qquad \text{for } x \in \mathbb{R}.$$

Conversely, if η is of the above form, then

$$s'(x) = \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \frac{\bar{\sigma}f(x;\xi)\left(1 - \bar{\sigma}\bar{F}(x;\xi)\right)^{-2}}{1 - \left(1 - \bar{\sigma}\bar{F}(x;\xi)\right)^{-1}}, \qquad x \in \mathbb{R},$$

and consequently

$$s(x) = -\log\left[1 - \left(1 - \bar{\sigma}\bar{F}(x;\xi)\right)^{-1}\right], \qquad x \in \mathbb{R}.$$

Now, according to Theorem 5.1, X has pdf (4).

Corollary 5.1. Let X: $\Omega \to \mathbb{R}$ be a continuous random variable and $q_1(x)$ be as in Proposition 5.1. Then X has density function (4) if and only if there exist functions q_2 and η defined in Theorem 5.1 satisfying the following first order differential equation

$$\frac{\eta^{/}(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\bar{\sigma} f(x;\xi)\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^{-2}}{1-\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^{-1}}, \quad x \in \mathbb{R}.$$

Corollary 5.2. The differential equation in Corollary 5.1 has the following general solution

$$\eta(x) = \left[1-\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^{-1}\right]^{-1}\left[\int \bar{\sigma} f(x;\xi)\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^{-2}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

in which D is a constant. We like to mention that a set of functions satisfying the above first order differential equation is given in Proposition 5.1 with D =1/2. Clearly, there are other triplets satisfying the conditions of Theorem 5.1.

5.2. *Characterization based on hazard function*

Clearly, a twice differentiable distribution function, G, satisfies the following differential equation

$$\frac{g^{/}(x)}{g(x)} = \frac{h_G^{/}(x)}{h_G(x)} - h_G(x).$$

The following proposition provides a non-trivial characterization of NEC-X distribution.

Proposition 5.2. Suppose X is a continuous random variable. Then X has density function (4) if and only if its hazard function $h_G(x)$ satisfies the following first order differential equation

$$h_G^{/}(x) = \frac{f^{/}(x;\xi)}{f(x;\xi)} h_G(x) = \frac{\pi\sigma}{2} g(x;\xi)\, \frac{d}{dx}\left\{\frac{\tan\left(\frac{\pi}{2}\left\{\frac{F(x;\xi)}{1-\bar{\sigma}\bar{F}(x;\xi)}\right\}\right)}{\left(1-\bar{\sigma}\bar{F}(x;\xi)\right)^2}\right\}, \quad x \in \mathbb{R}.$$

Proof. Is straightforward and hence omitted.

6. Maximum Likelihood Estimation and Simulation

In this section, we derive the maximum likelihood estimators of the unknown parameters of the NEC-X family. Furthermore, we also provide a simulation study to assess the performance of these estimators.

6.1. *Maximum likelihood estimation*

Let $x_1, x_1, ..., x_n$ be the observed values of a random sample from pdf (4) with parameters σ and ξ. The log-likelihood function is

$$\begin{aligned}\log L(x;\sigma,\xi) = n\log\left(\tfrac{\pi}{2}\right) + n\log(\sigma) + \sum_{i=1}^{n}\log\left[f(x_i;\xi)\right] - 2\sum_{i=1}^{n}\log(A_i) \\ + \sum_{i=1}^{n}\log\left(\sin\left[\tfrac{\pi}{2}\left\{\tfrac{F(x_i;\xi)}{A_i}\right\}\right]\right).\end{aligned} \tag{20}$$

The partial derivatives of $\log L(x;\alpha,\xi)$ are

$$\begin{aligned}\frac{\partial \log L(x;\sigma,\xi)}{\partial\sigma} &= \frac{n}{\sigma} - \frac{\pi}{2}\sum_{i=1}^{n}\left(\cot\left[\frac{\pi}{2}\left\{\frac{F(x_i;\xi)}{A_i}\right\}\right]\right)\left\{\frac{F(x_i;\xi)\bar{F}(x_i;\xi)}{[A_i]^2}\right\} \\ &- 2\sum_{i=1}^{n}\log(A_i),\end{aligned} \tag{21}$$

and

$$\begin{aligned}\frac{\partial \log L(x;\sigma,\xi)}{\xi} &= \sum_{i=1}^{n}\frac{\partial f(x_i;\xi)/\partial\xi}{f(x_i;\xi)} + \frac{\pi\sigma}{2}\sum_{i=1}^{n}\cot\left[\frac{\pi}{2}\left\{\frac{F(x_i;\xi)}{A_i}\right\}\right]\partial F(x_i;\xi)/\partial\xi \\ &- 2\bar{\sigma}\sum_{i=1}^{n}\frac{\partial F(x_i;\xi)/\partial\xi}{A_i},\end{aligned} \tag{22}$$

where, $A_i = 1 - \bar{\sigma}\bar{F}(x_i;\xi)$. The maximum likelihood estimates of σ and ξ are numerical solutions of (21) and (22) simultaneously.

6.2. Simulation study

In this sub-section, the maximum likelihood estimates are evaluated through Monte Carlo simulation. The simulation is done using R software and is based on the following steps:

- We generate N =500 random samples of sizes $n = 25, 50, ..., 500$ from the NEC-W model.
- Compute the maximum likelihood estimates for the parameters of the NEC-W distribution.
- Compute the mean square error (MSE) and biases given by $MSE(n) = \frac{1}{500}\sum_{i=1}^{500}(\hat{w}_i - w)^2$ and $Bias(n) = \frac{1}{500}\sum_{i=1}^{500}(\hat{w}_i - w)$ for $w = (\alpha, \sigma, \gamma)$, respectively.

The simulation results of the NEC-W are provided in Tables 1 and 2. Corresponding to each Table, the graphical representation of the simulation results is also provided.

Table 1: Simulation results for NEC-W distribution.

Set 1: $\alpha = 1.4, \sigma = 0.7, \gamma = 1$				
n	parameters	MLE	MSE	Bias
25	$\hat{\alpha}$	1.650759	0.3737015	0.2507585
	$\hat{\sigma}$	1.780067	5.2498591	1.1800679
	$\hat{\gamma}$	1.999261	4.7896870	0.9992605
100	$\hat{\alpha}$	1.527348	0.1529964	0.1273478
	$\hat{\sigma}$	1.245407	2.5371046	0.6454071
	$\hat{\gamma}$	1.354976	1.5080290	0.3549759
200	$\hat{\alpha}$	1.460271	0.0725317	0.0602710
	$\hat{\sigma}$	0.872451	0.7602116	0.2724514
	$\hat{\gamma}$	1.152294	0.5415998	0.1522937
300	$\hat{\alpha}$	1.438300	0.0440006	0.0383002
	$\hat{\sigma}$	0.764876	0.3897828	0.1648760
	$\hat{\gamma}$	1.127544	0.3343627	0.1275436
400	$\hat{\alpha}$	1.448867	0.0375601	0.0488669
	$\hat{\sigma}$	0.779101	0.3351656	0.1791017
	$\hat{\gamma}$	1.033442	0.1592310	0.0334415
500	$\hat{\alpha}$	1.429324	0.0276946	0.0293243
	$\hat{\sigma}$	0.716434	0.2182770	0.1164344
	$\hat{\gamma}$	1.050925	0.1437044	0.0509253

In support of Table 1, the simulation results are displayed graphically in Figures 2 and 3.

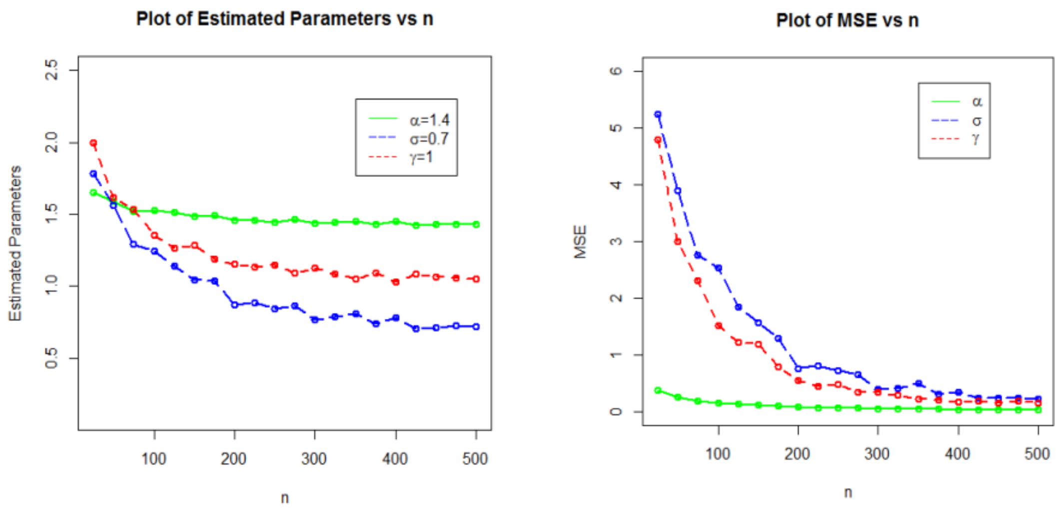

Figure 2: Corresponding to Table 1, plots of the estimated parameters and MSEs of the NEC-Weibull distribution.

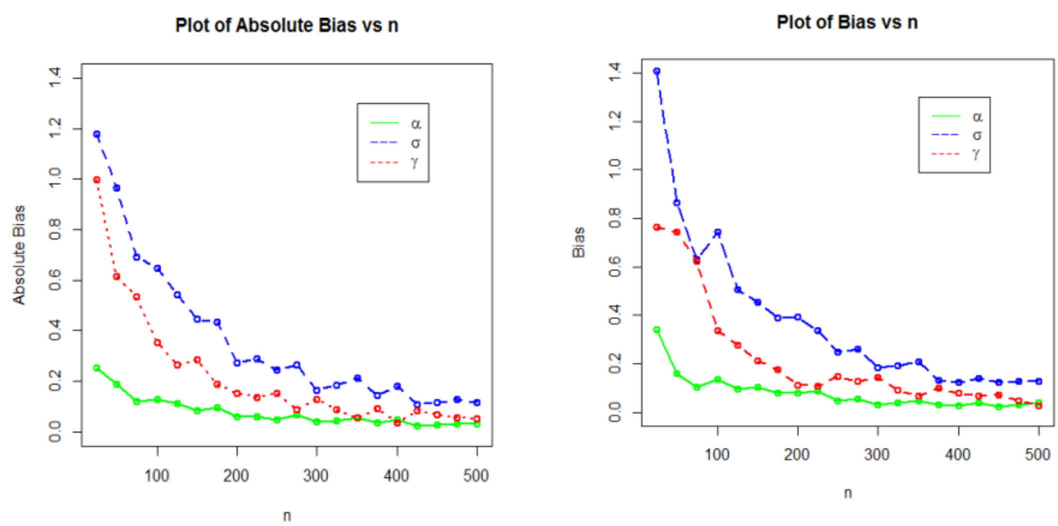

Figure 3: Corresponding to Table 1, plots of Absolute Biases and Biases for NEC-Weibull distribution.

Table 2: Simulation results for NEC-W distribution.

Set 2: $\alpha = 0.4, \sigma = 1.5, \gamma = 1$				
n	parameters	MLE	MSE	Bias
25	$\hat{\alpha}$	0.462856	1.267×10^{-02}	0.06285640
	$\hat{\sigma}$	2.791048	5.16634516	1.29104787
	$\hat{\gamma}$	1.205851	1.42384517	0.20585158
100	$\hat{\alpha}$	0.418350	2.349×10^{-03}	0.01835031
	$\hat{\sigma}$	1.889642	1.20315322	0.38964235
	$\hat{\gamma}$	0.961707	0.06228977	-0.0382926
200	$\hat{\alpha}$	0.404230	4.737×10^{-04}	0.00423072
	$\hat{\sigma}$	1.592749	0.26005562	0.09274858
	$\hat{\gamma}$	0.976693	0.00468954	-0.0133068
300	$\hat{\alpha}$	0.403043	1.413×10^{-04}	0.00104325
	$\hat{\sigma}$	1.546722	0.08987322	0.02672245
	$\hat{\gamma}$	0.986608	0.00144524	-0.0033914
400	$\hat{\alpha}$	0.400375	3.861×10^{-05}	0.00037590
	$\hat{\sigma}$	1.507690	0.01660442	0.00768966
	$\hat{\gamma}$	0.998740	0.00041469	-0.0012591
500	$\hat{\alpha}$	0.400108	0.00041469	0.00013442
	$\hat{\sigma}$	1.503864	2.8618e-07	0.00012239
	$\hat{\gamma}$	0.999086	0.01063522	0.00000001

In support of Table 2, the simulation results of the NEC-W distribution are displayed graphically in Figures 4 and 5.

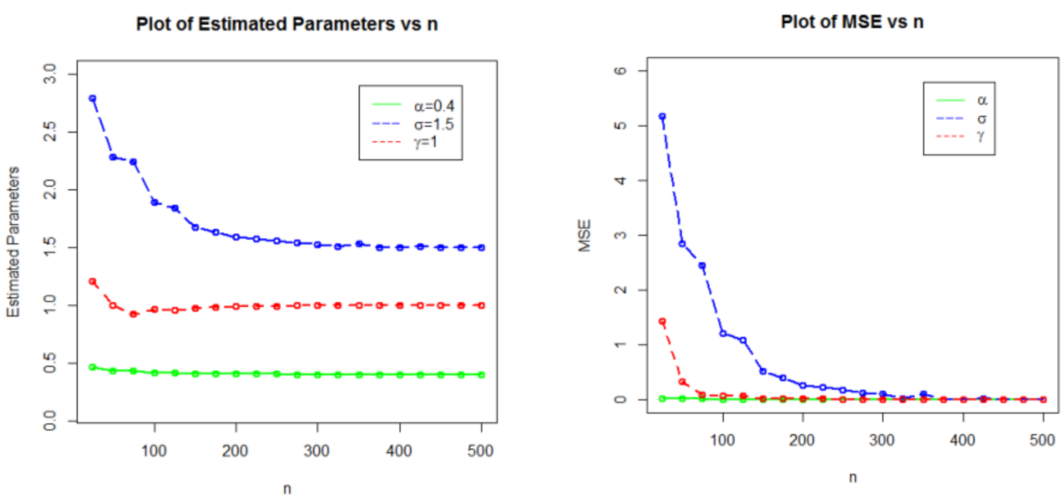

Figure 4: Corresponding to Table 2, plots of the estimated parameters and MSEs of the NEC-Weibull distribution.

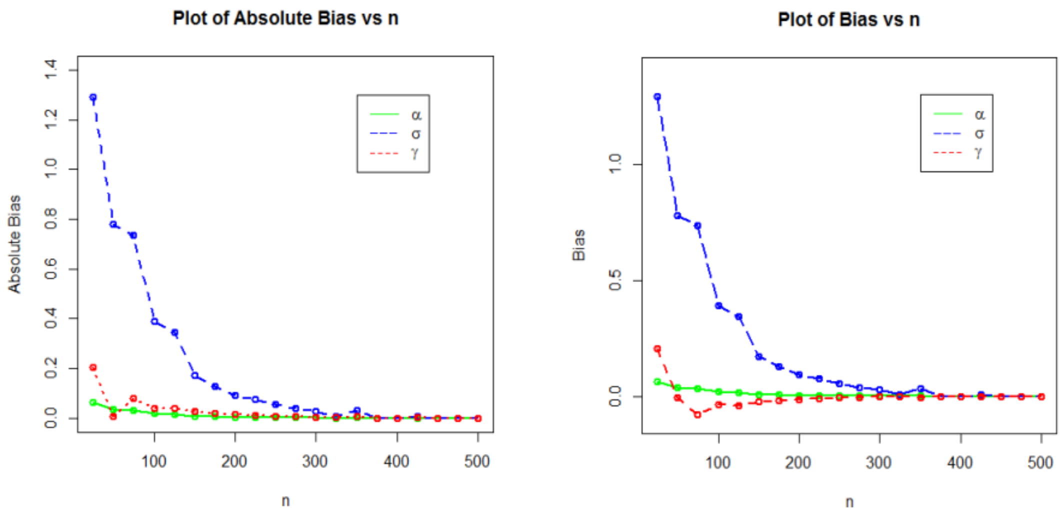

Figure 5: Corresponding to Table 2, plots of Absolute Biases and Biases for NEC-Weibull distribution.

7. An Application to Medical Care Insurance Data

The main applications of the heavy tail models are the so-called extreme value theory or insurance loss phenomena. in this section, we illustrate the potentiality of the proposed model via a real life application taken from actuarial sciences. The data set is available at: https: instruction.bus.wisc.edujfreesjfreesbooksRegression. The comparison of the NEC-W distribution is made with two parameters, three parameters and four parameters models. The cdf's of the competitive distributions are:

- The Weibull

$$G(x;\alpha,\gamma) = \left(1 - e^{-\gamma x^{\alpha}}\right), \qquad x > 0, \ \alpha,\gamma > 0.$$

- Marshall-Olkin Weibull (MOW) distribution

$$G(x;\alpha,\gamma,\sigma) = \frac{\left(1 - e^{-\gamma x^{\alpha}}\right)}{\sigma + (1-\sigma)\left(1 - e^{-\gamma x^{\alpha}}\right)}, \qquad x > 0, \ \alpha,\gamma,\sigma > 0.$$

- Kumaraswamy Weibull (Ku-W) distribution

$$G(x;\alpha,\gamma,a,b) = 1 - \left\{1 - \left(1 - e^{-\gamma x^{\alpha}}\right)^{a}\right\}^{b}, \qquad x > 0, \ \alpha,\gamma,a,b > 0.$$

To decide about the goodness of fit between the NEC-W and other applied distributions, we consider certain analytical measures including the Bayesian information criterion (BIC), Akaike information criterion (AIC), minus two times maximized log-likelihood under the model $-2\hat{\ell}$, Anderson-Darling (AD) test statistic and Kolmogorov-Smirnov (KS) with the corresponding p-value. A distribution with lower values of these analytical measures is considered to be a good candidate model for the underlying data set. The analytical measures are given by

- The AIC is given by

$$AIC = 2k - 2\ell.$$

- The BIC is given by

$BIC = k\log(n) - 2\ell.$

where ℓ denotes the log-likelihood function evaluated at the MLEs, k is the number of model parameters and n is the sample size.

- The AD test statistic

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log G(x_i) + \log\{1 - G(x_{n-i+1})\}\right],$$

where:
n = the sample size,
x_i = the i^{th} sample, calculated when the data is sorted in ascending order.

- The KS test statistic is given by

$$KS = sup_x\left[G_n(x) - G(x)\right],$$

where $G_n(x)$is the empirical cdf and $\sup_x$ is the supremum of the set of distances.

The maximum likelihood estimates with standard error (in parenthesis) of the models for the analyzed data are presented in Table 3. Whereas, the analytical measures of the NEC-W and other considered models are provided in Table 4. Form Table 4, we can see the NEC-W model has lower values than the other distributions applied in comparison. The estimated densities of the fitted distributions are plotted in Figure 6. Whereas, the estimated distribution functions are sketched in Figure 7.

Table 3: The estimated values of the parameters with standard errors (in parenthesis) of the fitted distributions.

Dist.	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\sigma}$	$\hat{a}$	$\hat{b}$
NEC-W	1.436	0.094	0.277		
	(0.3110)	(0.1109)	(0.3080)		
Weibull	1.236	0.132			
	(0.0458)	(0.0119)			
MOW	2.176	0.008	0.092		
	(0.0954)	(0.0029)	(0.0259)		
Ku-W	0.615	0.986		5.922	1.414
	(0.2401)	(0.2795)		(3.3065)	(1.2110)

Table 4: Analytical measures of the fitted models.

Dist.	AIC	BIC	$-2\hat{\ell}$	AD	KS	p-value
NEC-W	2281.75	2294.54	1137.876	0.423	0.058	0.557
Weibull	2321.13	2329.66	1158.564	0.943	0.098	0.249
MOW	2285.13	2299.93	1147.876	0.485	0.083	0.308
Ku-W	2283.11	2297.17	1141.876	0.452	0.062	0.436

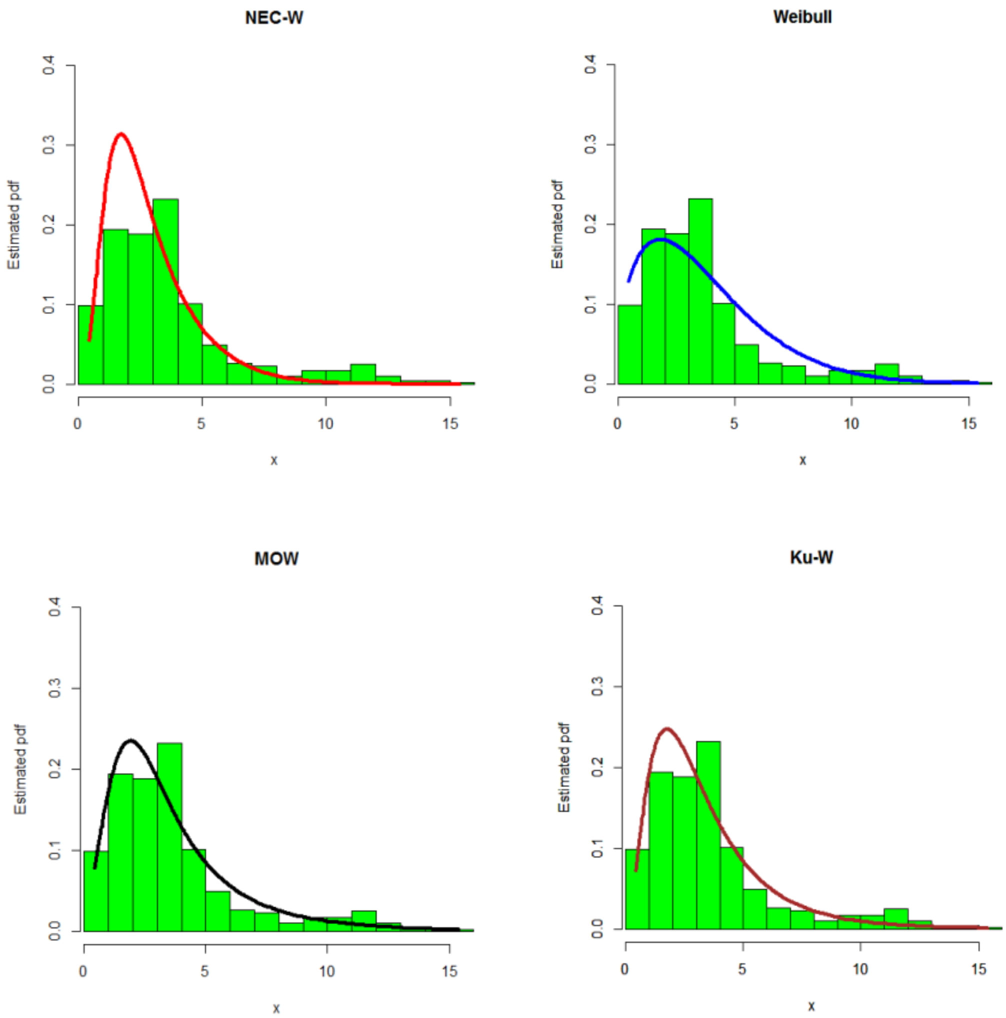

Figure 6: Estimated densities of the fitted models corresponding to medical care insurance data.

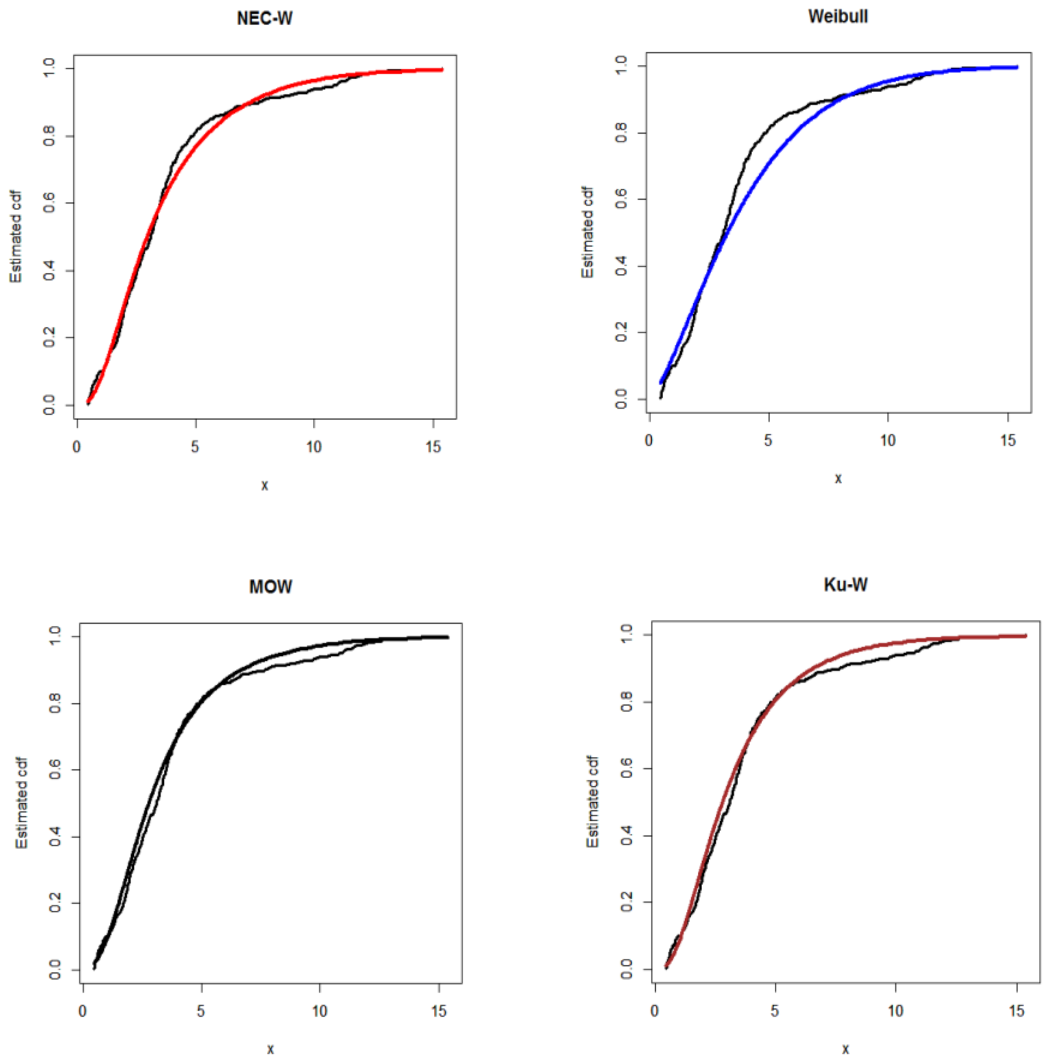

Figure 7: Estimated distribution functions of the fitted models for the medical care insurance data set.

From Figures 6 and 7, we one can easily detect that proposed model fit the estimated pdf and cdf very closely indicating best fitting of the model.

8. Bayesian Estimation

Bayesian inference procedure has been taken into consideration by many statistical researchers, especially those in the field of survival analysis and reliability engineering. In this section, a complete sample data is analyzed through Bayesian point of view. We assume that the parameters α, γ and σ of NEC-Weibull distribution have independent prior distributions as

$$\alpha \sim \text{Gamma}\,(a, b)\,,\ \gamma \sim \text{Gamma}\,(c, d)\ \text{and}\ \sigma \sim \text{Gamma}\,(e, f)\,,$$

where a, b, c, d, e and f are positive. Hence, the joint prior density function is formulated as follows:

$$\pi(\alpha,\gamma,\sigma)=\frac{b^{\alpha}d^{c}f^{e}}{(\Gamma a)(\Gamma c)(\Gamma e)}\alpha^{a-1}\gamma^{c-1}\sigma^{e-1}\exp\{-(b\alpha+d\gamma+f\sigma)\}. \tag{23}$$

In the Bayesian estimation, we do not know the actual value of the parameter. Some well-known loss functions along with Bayesian estimators and the corresponding posterior risk are provided in Table 5.

Table 5: Bayes estimator and posterior risk under different loss functions.

loss function	Bayes estimator	Posterior risk
$L_1 = SELF = (\theta-d)^2$	$E(\theta\|x)$	$Var(\theta\|x)$
$L_2 = WSELF = \frac{(\theta-d)^2}{\theta}$	$(E(\theta^{-1}\|x))^{-1}$	$E(\theta\|x)-(E(\theta^{-1}\|x))^{-1}$
$L_3 = MSELF = \left(1-\frac{d}{\theta}\right)^2$	$\frac{E(\theta^{-1}\|x)}{E(\theta^{-2}\|x)}$	$1-\frac{E(\theta^{-1}\|x)^2}{E(\theta^{-2}\|x)}$
$L_4 = PLF = \frac{(\theta-d)^2}{d}$	$\sqrt{E(\theta^2\|x)}$	$2\left(\sqrt{E(\theta^2\|x)}-E(\theta\|x)\right)$
$L_5 = KLF = \left(\sqrt{\frac{d}{\theta}}-\sqrt{\frac{\theta}{d}}\right)$	$\sqrt{\frac{E(\theta\|x)}{E(\theta^{-1}\|x)}}$	$2\left(\sqrt{E(\theta\|x)E(\theta^{-1}\|x)}-1\right)$

For more details see [2]. Next, we provide the posterior probability distributions for a complete data set. Let us define the function φ as

$$\varphi(\alpha,\gamma,\sigma)=\alpha^{a-1}\gamma^{c-1}\sigma^{e-1}\exp\{-(b\alpha+d\gamma+f\sigma)\},\quad \alpha>0,\ \gamma>0,\ \sigma>0.$$

The joint posterior distribution in terms of a given likelihood function L (data) and joint prior distribution $\pi(\alpha,\gamma,\sigma)$ is defined as

$$\pi^{*}(\alpha,\gamma,\sigma|\text{data})\,\alpha\ \pi(\alpha,\gamma,\sigma)\,L(\text{data}) \tag{24}$$

Hence, the joint posterior density of parameters α, γ and σ for complete sample data is obtained by combining the likelihood function and joint prior density (23). Therefore, the joint posterior density function is given by

$$\pi^{*}(\alpha,\gamma,\sigma|x)=K\varphi(\alpha,\gamma,\sigma|x)\prod_{i=1}^{n}\frac{\alpha\gamma\sigma\pi}{2}\frac{x_i^{\alpha-1}\exp\left(-\gamma x_i^{\alpha}\right)}{\left[1-\bar{\sigma}\exp\left(-\gamma x_i^{\alpha}\right)\right]}\sin\left(\frac{\pi}{2}\frac{\left\{1-\exp\left(-\gamma x_i^{\alpha}\right)\right\}}{\left[1-\bar{\sigma}\exp\left(-\gamma x_i^{\alpha}\right)\right]}\right), \tag{25}$$

where

$$K^{-1}=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\phi(\alpha,\gamma,\sigma)\prod_{i=1}^{n}\frac{\alpha\gamma\sigma\pi}{2}\frac{x_i^{\alpha-1}\exp\left(-\gamma x_i^{\alpha}\right)}{\left[1-\bar{\sigma}\exp\left(-\gamma x_i^{\alpha}\right)\right]}$$
$$\times\sin\left(\frac{\pi}{2}\frac{\left\{1-\exp\left(-\gamma x_i^{\alpha}\right)\right\}}{\left[1-\bar{\sigma}\exp\left(-\gamma x_i^{\alpha}\right)\right]}\right)d\alpha\,d\gamma\,d\sigma.$$

It is clear from the equation (25) that there is no closed form for the Bayesian estimators under the five loss functions described in Table 5, so we suggest using an MCMC procedure based on 10000 replicates to compute Bayesian estimators. The corresponding Bayesian point and interval estimation and posterior risk are provided in Tables 6 and 7. Table 7 provides 95% credible and HPD intervals for each parameter of the NEC-W distribution. The posterior samples extracted by using Gibbs sampling technique. Moreover, we provide the posterior summary plots in Figures 8 and 9. These plots confirm that the sampling process is of the prime quality and convergence is occurred

Table 6: Bayesian estimates along with posterior risks of the parameters using different loss functions based on Medical Care Insurance data.

Data	Medical Care Insurance data					
Bayes	$\widehat{\alpha}$		$\widehat{\gamma}$		$\widehat{\sigma}$	
Loss functions	Estimate	Risk	Estimate	Risk	Estimate	Risk
SELF	1.1869	0.0004	4.6431e-05	9.2079e-11	0.4224	0.00009
WSELF	1.1865	0.0004	4.5107e-05	1.3234e-06	0.4221	0.00022
MSELF	1.1861	0.0003	4.4236e-05	0.01932340	0.4219	0.00054
PLF	1.1870	0.0004	4.7412e-05	1.9621e-06	0.4225	0.00021
KLF	1.1867	0.0003	4.5764e-05	0.02912679	0.4223	0.00052

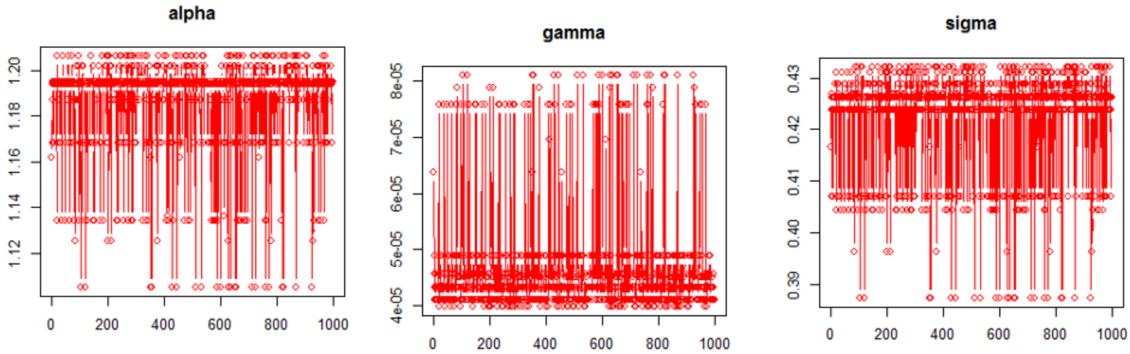

Figure 8: Plots of Bayesian analysis and performance of Gibbs sampling for Medical Care Insurance data set. Trace plots of each parameter of NEC-W distribution.

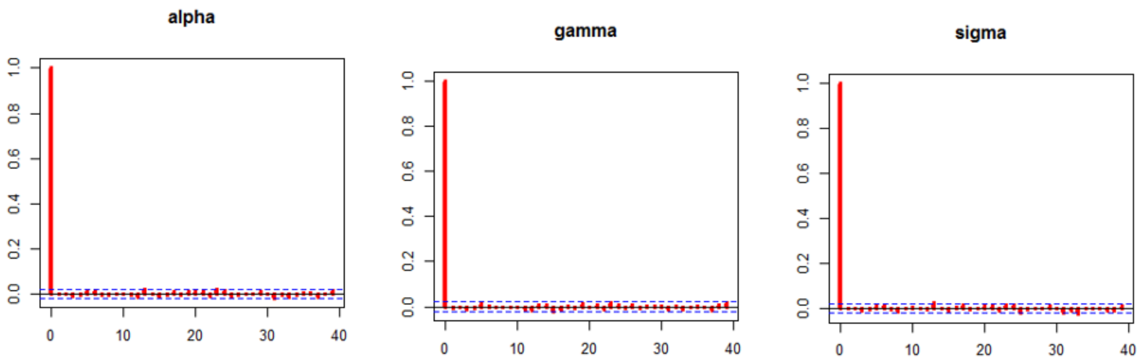

Figure 9: Plots of Bayesian analysis and performance of Gibbs sampling for Medical Care Insurance data set. Autocorrelation plots of each parameter of NEC-W distribution.

Table 7: Credible and HPD intervals of the parameters α, γ and σ for Medical Care Insurance data.

Parameters	Credible interval	HPD interval
α	(1.187, 1.195)	(1.134, 1.206)
γ	(0.00004, 0.00005)	(0.00003, 0.00007)
σ	(0.4237, 0.4269)	(0.4044, 0.4321)

9. Concluding Remarks

In this study, a new exponential cosine-X family with baseline Weibull distribution is introduced. The proposed distribution is very flexible and possesses heavy tails. Some mathematical properties along with characterizations of the NEC-X family are presented. The maximum likelihood and Bayesian estimation methods are employed to estimates of the model parameters. Furthermore, a simulation study is provided to evaluate the behavior of the estimators. The proposed new exponential cosine Weibull distribution is illustrated via analyzing a heavy tailed medical care insurance data set and the comparison is made with some well-known distributions. From the real application, we observe that the proposed model provides a better fit to the heavy tailed medical care insurance data than the other distributions.

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Dedication

The First author would like to dedicate this article to the memory of his late parents.

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