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Some Iterative Methods for Solving Operator Equations by Using Fusion Frames

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Abstract. In this paper, two iterative methods are constructed to solve the operator equation $Lu = f$ where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. By using the concept of fusion frames, which is a generalization of frame theory, we design some algorithms based on Chebyshev polynomials and adaptive one according to conjugate gradient iterative method, and accordingly, we then investigate their convergence via their correspond convergence rates.

1. Introduction

In the twenty-first century, scientists face massive amounts of data, which can typically no longer be handled with a single processing system. A seemingly unrelated problem arises in sensor networks when communication between any pair of sensors is not possible due to, for instance, low communication bandwidth. Yet another question is the design of erasure-resilient packet-based encoding when data is broken into packets for separate transmission.

All these problems can be regarded as belonging to the field of distributed processing. However, they have an even more special structure in common, since each can be regarded as a special case of the following mathematical framework: Given data and a collection of subspaces, project the data onto the subspaces, then process the data within each subspace, and finally "fuse" the locally computed objects. The decomposition of the given data into the subspaces coincides with the splitting into different processing systems, the local measurements of groups of close sensors, and the generation of packets. The distributed fusion models the reconstruction procedure, also enabling, for instance, an error analysis of resilience against erasures. This is however only possible if the data is decomposed in a *redundant* way, which forces the subspaces to be redundant.

Fusion frames provide a suitable mathematical framework to design and analyze such applications under distributed processing requirements. Interestingly, fusion frames are also a versatile tool for more theoretically oriented problems in mathematics.

The goal of this paper is to study the application of *fusion frames* in designing some algorithms, within which two newly defined iterative methods are used for solving operator equation $Lu = f$, where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. In [1, 6–8] some numerical algorithms for solving this system have been developed by using wavelets and frames.

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The authors view fusion frame theory as an area of mathematics with important practical applications in computation, and intend to provide here an introduction to the theoretical foundations which underlie two algorithms of everyday use. For this reason, for each method of iteration studied at least one algorithm leading to actual numerical approximations is described and, indeed, traced from its theoretical origins to its present formulation. Actual algorithms that are described in detail are intended more to illustrate one possibility than to suggest that they are the "best" available. In proof of convergence of both algorithms, we observe that the selection of suitable frame bounds is very efficient to converge faster the algorithms

First algorithm is designed on the basis of preconditioning the operator equation $Lu = f$, using fusion frames and then applying well-known Chebyshev polynomials lead to an iterative method with comparatively lower convergence rate with respect to mere Richardson iterative method derived by the same preconditioning. For detailed information we refer the reader to [10].

Second algorithm, in addition to employing the notions and foundations considered in preceding algorithm, we tend to define an vector space H_n from which we derive an approximate solution h_n adaptively, in the sense that the error estimation is bounded by a constant with power number which equals to $\dim H_n$. A typical adaptive algorithm uses information gained during a given stage of the computation to produce a new mesh for the next iteration. Thus, the adaptive procedure depends on the current numerical resolution of u. Accordingly, these methods produce a form of nonlinear approximation of the solution, in contrast with linear methods in which the numerical procedure is set in advance and does not depend on the solution to be resolved.

2. Preliminaries

We will now give a brief review about the concepts of frame and fusion frame. Throughout this paper H will be a separable Hilbert space and Λ a countable index set.

2.1. *Frames*

We begin defining the concept of frame.

Definition 2.1. *Let $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$. Then $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ is a frame for H, if there exist constants $0 < A_\Psi \leq B_\Psi < \infty$ such that*

$$A_\Psi \parallel f \parallel_H^2 \leq \sum_{\lambda\in\Lambda} |< f, \psi_\lambda >|^2 \leq B_\Psi \parallel f \parallel_H^2, \quad \forall f \in H.$$

The constants A_Ψ and B_Ψ are called the lower and upper frame bounds, respectively. If $A_\Psi = B_\Psi$, we call $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ an A_Ψ-tight frame, and if $A_\Psi = B_\Psi = 1$ it is a *Parseval frame*.

We associate to a frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ the *synthesis operator*

$$T : \ell_2(\Lambda) \to H, \; T((c_\lambda)_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

and the *analysis operator*

$$T^* : H \to \ell_2(\Lambda), \; T^*(f) = (< f, \psi_\lambda >)_{\lambda\in\Lambda}.$$

For a frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ the operator

$$S = TT^* : H \to H, \; S(f) = \sum_{\lambda\in\Lambda} < f, \psi_\lambda > \psi_\lambda,$$

is called *frame operator* which is positive, selfadjoint, invertible and satisfies $A_\Psi I_H \leq S \leq B_\Psi I_H$ and $B_\Psi^{-1} I_H \leq S^{-1} \leq A_\Psi^{-1} I_H$. In fact, the sequence

$$\widetilde{\Psi} = (\widetilde{\psi_\lambda})_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda},$$

is a frame (called the canonical dual frame) for H with the bounds B_Ψ^{-1} and A_Ψ^{-1}. Every $f \in H$ has the expansion

$$f = \sum_{\lambda\in\Lambda} \langle f, \psi_\lambda \rangle \widetilde{\psi}_\lambda = \sum_{\lambda\in\Lambda} \langle f, \widetilde{\psi}_\lambda \rangle \psi_\lambda.$$

Also for an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a frame sequence if it is a frame for its closed span. For more details we refer to [2, 5].

2.2. *Fusion Frames*

Let $\{W_\lambda\}_{\lambda\in\Lambda}$ be a family of closed subspaces of H, and let $\{\omega_\lambda\}_{\lambda\in\Lambda}$ be a family of weights, i.e., $\omega_\lambda > 0$ for all $\lambda \in \Lambda$. We will denote $\{W_\lambda\}_{\lambda\in\Lambda}$ by W, $\{\omega_\lambda\}_{\lambda\in\Lambda}$ by ω and $\{(W_\lambda, \omega_\lambda)\}_{\lambda\in\Lambda}$ by (W, ω). We consider the Hilbert space

$$K_W := \oplus_{\lambda\in\Lambda} W_\lambda = \left\{ \{f_\lambda\}_{\lambda\in\Lambda} : \ f_\lambda \in W_\lambda \ and \ \left\{ \left\| f_\lambda \right\| \right\}_{\lambda\in\Lambda} \in \ell_2(\Lambda) \right\},$$

with inner product $\langle \{f_\lambda\}_{\lambda\in\Lambda}, \{g_\lambda\}_{\lambda\in\Lambda} \rangle = \sum_{\lambda\in\Lambda} \langle f_\lambda, g_\lambda \rangle$.

For V a closed subspace of H, π_V is the orthogonal projection onto V.

Definition 2.2. *We say that (W, ω) is a fusion frame for H, if there exist constants $0 < A \leq B < \infty$ such that*

$$A \parallel f \parallel^2 \leq \sum_{\lambda\in\Lambda} \omega_\lambda^2 \parallel \pi_{W_\lambda}(f) \parallel^2 \leq B \parallel f \parallel^2 \quad \forall f \in H. \tag{1}$$

We call A and B the lower and upper fusion frame bounds, respectively. The family (W, ω) is called an A-tight fusion frame, if in (1) the constants A and B can be chosen so that $A = B$, and a Parseval fusion frame provided that $A = B = 1$.

We associate to a fusion frame (W, ω) the following bounded operators:

$$T_{W,\omega} : K_W \to H, \ T_{W,\omega}(\{f_\lambda\}_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} \omega_\lambda f_\lambda,$$

called the *synthesis operator* and

$$T^*_{W,\omega} : H \to K_W, \ T^*_{W,\omega}(f) = \{\omega_\lambda \pi_{W_\lambda}(f)\}_{\lambda\in\Lambda},$$

named the *analysis operator*. It is easy to check that the analysis operator is the adjoint of the synthesis operator. Let (W, ω) be a fusion frame, then the fusion frame operator is defined by

$$S_{W,\omega} = T_{W,\omega} T^*_{W,\omega} : H \to H, \ S_{W,\omega}(f) = \sum_{\lambda\in\Lambda} \omega^2 \pi_{W_\lambda}(f),$$

is positive definite, selfadjoint and invertible. For more details see [3].
The following theorem, shows how we able to string together frames for each of the subspaces W_λ to get a fusion frame for H. [3].

Theorem 2.3. *Let Λ be countable index set, $\omega_\lambda > 0$ for each $\lambda \in \Lambda$, I_λ be a countable set for each $\lambda \in \Lambda$ and $\{\psi_{\lambda_i}\}_{i\in I_\lambda}$ be a frame sequence in H with frame bounds A_λ and B_λ. Define $W_\lambda = \overline{span}_{i\in I_\lambda}\{\psi_{\lambda_i}\}$ for all $\lambda \in \Lambda$, and suppose that $0 < A = \inf_{\lambda\in\Lambda} A_\lambda \leq B = \sup_{\lambda\in\Lambda} B_\lambda < \infty$. Then $\{\omega_\lambda \psi_{i_\lambda}\}_{\lambda\in\Lambda, i\in I_\lambda}$ is a frame for H if and only if $\{W_\lambda\}_{\lambda\in\Lambda}$ is a fusion frame with respect to $\{\omega_\lambda\}_{\lambda\in\Lambda}$ for H.*

If L is a bounded operator on H and (W, ω) is a fusion frame for H, we will write (LW, ω) for $\{(LW_\lambda, \omega)\}_{\lambda\in\Lambda}$. It is easy to check that the following statement holds.

Proposition 2.4. *Let (W, ω) be a fusion frames, and let $L : H \to H$ be a bounded and invertible operator on H. Then (LW, ω) is a fusion frame for H.*

3. Chebyshev Iteration Method by Using Fusion Frames

Let H be a separable Hilbert space, and

$$Lu = f, \tag{2}$$

be an operator equation where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on H. In this case, there exist two positive constants c_1 and c_2 such that

$$c_1 \parallel u \parallel_H \leq \parallel Lu \parallel_H \leq c_2 \parallel u \parallel_H, \ \forall u \in H. \tag{3}$$

Suppose $(W, \omega) = ((W_\lambda, \omega_\lambda))_{\lambda\in\Lambda}$ is a fusion frame in H with fusion frame operator S. By proposition 2.4, $(L(W_\lambda), \omega_\lambda)_{\lambda\in\Lambda}$ is also a fusion frame, which we denote its fusion frame operator by S'.

We precondition (2) by multiplying both sides by the matrix

$$M := \frac{2}{c_1^2 A + c_2^2 B} LS',$$

to obtain

$$\frac{2}{c_1^2 A + c_2^2 B} LS'Lu = \frac{2}{c_1^2 A + c_2^2 B} LS'f, \tag{4}$$

where A and B are lower and upper bounds of the fusion frame $(L(W_\lambda), \omega_\lambda)_{\lambda\in\Lambda}$ respectively. Now, applying Richardson iteration method [11] on (4) yields the following iterative sequence

$$u_k = u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} LS' \left(f - Lu_{k-1}\right),$$

with initial guess $u_0 = 0$. For the sequence u_k, we define $h_n := \sum_{k=1}^{n} a_{n,k} u_k$ such that $\sum_{k=1}^{n} a_{n,k} = 1$. We have

$$u - h_n = \sum_{k=1}^{n} a_{n,k}(u - u_k) = \sum_{k=1}^{n} a_{n,k} \left(I - \frac{2}{c_1^2 A + c_2^2 B} LS'L\right)^k (u - u_0), \tag{5}$$

and by defining $Y := I - \frac{2}{c_1^2 A + c_2^2 B} LS'L$ and $Q_n(x) := \sum_{k=1}^{n} a_{n,k} x^k$, then one would rewrite (5) as

$$u - h_n = Q_n(Y)(u - u_0). \tag{6}$$

Remark 3.1. *For all $f \in H$, we have*

$$\begin{aligned}
\left\langle \left(I - \frac{2}{c_1^2 A + c_2^2 B} LS'L\right) f, f \right\rangle &= \left\|f\right\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \left\langle S'Lf, Lf \right\rangle \\
&= \left\|f\right\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \left\langle \sum_{\lambda\in\Lambda} \omega_\lambda^2 \pi_{LW_\lambda}(Lf), Lf \right\rangle \\
&= \left\|f\right\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \sum_{\lambda\in\Lambda} \omega_\lambda^2 \left\|\pi_{LW_\lambda}(Lf)\right\|_H^2 \\
&\leq \left\|f\right\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} \left\|Lf\right\|_H^2 \\
&\leq \left\|f\right\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} \left\|Lf\right\|_H^2 \\
&\leq \left\|f\right\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} c_1^2 \left\|f\right\|_H^2 \\
&\leq \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B} \left\|f\right\|_H^2,
\end{aligned}$$

and similarly

$$-\frac{c_2^2B-c_1^2A}{c_1^2A+c_2^2B}\left\|f\right\|_H^2 \le \left\langle \left(I-\frac{2}{c_1^2A+c_2^2B}LS'L\right)f,f\right\rangle.$$

Therefore, one could follow that the spectrum of Y is a subset of the interval $\left[\frac{C_1^2A-c_2^2B}{C_1^2A+c_2^2B}, \frac{c_2^2B-C_1^2A}{C_1^2A+c_2^2B}\right]$.

Now, since $LS'L$ is a positive definite operator, by preceding remark and (6), as well as applying spectral theorem [9], one may implicate

$$\|u-h_n\|_H \le \|Q_n(Y)\|_{H\to H}\,\|u-u_0\|_H \le \max_{|x|\le \frac{c_2^2B-C_1^2A}{C_1^2A+c_2^2B}} |Q_n(x)|\,\|u-u_0\|_H\,. \tag{7}$$

Thus, our problem then is to minimize the error estimation $\|u-h_n\|_H$. To this end, we need only find

$$\min_{P_n\in\mathbb{P}_n}\ \max_{|x|\le \frac{c_2^2B-C_1^2A}{C_1^2A+c_2^2B}} \{P_n(x)\}\,, \tag{8}$$

where the minimum is taken over all polynomials

$$\mathbb{P}_n = \{P_n(x) : \deg(P_n) \le n, P_n(1) = 1\}\,.$$

An appropriate answer to this minimization problem is given by Chebyshev polynomial [4, 10] which is defined by

$$c_n(x) = \begin{cases} \cos(n\cos^{-1}x), & |x|\le 1 \\ \cosh(n\cosh^{-1}x) = \dfrac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x+\sqrt{x^2-1}\right)^{-n}\right), & |x|>1 \end{cases},$$

and satisfying the following recurrence relation

$$c_n(x) = 2xc_{n-1}(x) - c_{n-2}(x),\quad c_1(x) = x, c_0(x) = 1. \tag{9}$$

Lemma 3.2 ([4]). *Let $a < b < 1$ be given constants, set*

$$p_n(x) = \frac{c_n\left(\frac{2x-a-b}{b-a}\right)}{c_n\left(\frac{2-a-b}{b-a}\right)} \tag{10}$$

Then for all polynomials Q_n of degree n satisfying $Q_n(1) = 1$, we have

$$\max_{a\le x\le b}\left|p_n(x)\right| \le \max_{a\le x\le b}|Q_n(x)|\,.$$

Moreover,

$$\max_{a\le x\le b}\left|p_n(x)\right| = \frac{1}{c_n\left(\frac{2-a-b}{b-a}\right)}. \tag{11}$$

In the sequel, we prepare the needed context and notions to provide with an algorithm which includes an iterative method for solving (2), based on Chebyshev polynomials, and then, we investigate its convergence in view of related convergence rate.

First of all, for $n \ge 1$, we introduce the recurrence sequence of $\lambda_n := \left(1+\frac{\rho^2}{4}\lambda_{n-1}\right)^{-1}$ where $\rho = \frac{c_2^2B-c_1^2A}{c_1^2A+c_2^2B}$ with $\lambda_1 = 2$. It will also turn out that, by means of this recurrence sequence, we can rewrite h_n as

$$h_n = \lambda_n\left(h_{n-1}-h_{n-2}+\frac{2}{c_1^2A+c_2^2B}LS'\left(f-Lh_{n-1}\right)\right)+h_{n-2}, \tag{12}$$

with $h_1 = \frac{2}{c_1^2A+c_2^2B}LS'f$. By using this newly found formula for h_n, we state the following property.

Theorem 3.3. *For any $n \geq 1$, the sequence h_n satisfies in (12), and consequently in the following inequality.*

$$\| u - h_n \| \leq \frac{2\sigma^n}{1+\sigma^{2n}} \frac{\| f \|}{c_1}, \tag{13}$$

where $\sigma := \frac{c_2\sqrt{B}-c_1\sqrt{A}}{c_2\sqrt{B}+c_1\sqrt{A}}$ *and u is the exact solution of (2).*

Proof. By setting $a = -\rho$ and $b = \rho$ in lemma 3.2, we obtain

$$p_n(x) = \frac{c_n\left(\frac{2x+\rho-\rho}{2\rho}\right)}{c_n\left(\frac{2+\rho-\rho}{2\rho}\right)} = \frac{c_n\left(\frac{x}{\rho}\right)}{c_n\left(\frac{1}{\rho}\right)}, \tag{14}$$

which solves the problem (8) and minimizes the error estimation $\|u - h_n\|_H$ in (7). To verify the recurrence relation (12), we first apply the relation (9) to (14), to obtain

$$\begin{aligned} c_n\left(\frac{1}{\rho}\right) p_n(x) &= c_n\left(\frac{x}{\rho}\right) \\ &= \frac{2x}{\rho} c_{n-1}\left(\frac{x}{\rho}\right) - c_{n-1}\left(\frac{x}{\rho}\right) \\ &= \frac{2x}{\rho} c_{n-1}\left(\frac{1}{\rho}\right) p_{n-1}(x) - c_{n-2}\left(\frac{1}{\rho}\right) p_{n-2}(x). \end{aligned}$$

Here, if x replaced by Y, and apply the resulting operator identity to $(u - u_0)$, we then get

$$c_n\left(\frac{1}{\rho}\right) p_n(Y)(u-u_0) = \frac{2Y}{\rho} c_{n-1}\left(\frac{1}{\rho}\right) p_{n-1}(Y)(u-u_0) - c_{n-2}\left(\frac{1}{\rho}\right) p_{n-2}(Y)(u-u_0).$$

Combining this equation with the relation (6) yield

$$c_n\left(\frac{1}{\rho}\right)(u-h_n) = \frac{2}{\rho} c_{n-1}\left(\frac{1}{\rho}\right) Y(u-h_{n-1}) - c_{n-2}\left(\frac{1}{\rho}\right)(u-h_{n-2}). \tag{15}$$

If we write $Y(u) = u - \frac{2}{c_1^2 A + c_2^2 B} LS'Lu$, then by (15), we obtain

$$c_n\left(\frac{1}{\rho}\right)(u-h_n) = \frac{2}{\rho}\left[c_{n-1}\left(\frac{1}{\rho}\right)\left(I - \frac{2}{c_1^2 A + c_2^2 B} LS'L\right)\right](u-h_{n-1}) - c_{n-2}\left(\frac{1}{\rho}\right)(u-h_{n-2}).$$

The previous relation together with (9), induce

$$c_n\left(\frac{1}{\rho}\right) h_n = \frac{2}{\rho} c_{n-1}\left(\frac{1}{\rho}\right)\left(h_{n-1} + \frac{2}{c_1^2 A + c_2^2 B} LS'L(u-h_{n-1})\right) - c_{n-2}\left(\frac{1}{\rho}\right) h_{n-2}.$$

To continue, we set

$$\lambda_n = \frac{2}{\rho} \frac{c_{n-1}\left(\frac{1}{\rho}\right)}{c_n\left(\frac{1}{\rho}\right)},$$

which together with (9), yields

$$1 - \lambda_n = -\frac{2}{\rho} \frac{c_{n-2}\left(\frac{1}{\rho}\right)}{c_n\left(\frac{1}{\rho}\right)},$$

and again by (9), two proceeding relations provide us with the following recurrence formula

$$\lambda_n = \left(\frac{\rho}{2}\frac{c_n\left(\frac{1}{\rho}\right)}{c_n\left(\frac{1}{\rho}\right)}\right)^{-1} = \left(1 - \frac{\rho^2}{4}\lambda_{n-1}\right)^{-1}.$$

This proves the assertion made by (12). The remainder of the theorem is a consequence of combining (3), (7) and (11), which under our assumption $u_0 = h_0 = 0$ give

$$\begin{aligned}\|u - h_n\| &\leq \frac{1}{c_n\left(\frac{1}{\rho}\right)}\|u - u_0\| \\ &= \frac{1}{c_n\left(\frac{1}{\rho}\right)}\|u\| \\ &\leq \frac{1}{c_n\left(\frac{1}{\rho}\right)}\frac{\|f\|}{c_1}.\end{aligned}$$

On the other hand, by definition of c_n, we have

$$\begin{aligned}c_n\left(\frac{1}{\rho}\right) &= c_n\left(\frac{c_2^2B + c_1^2A}{c_2^2B - c_1^2A}\right) \\ &= \frac{1}{2}\left(\left(\frac{c_2^2B + c_1^2A}{c_2^2B - c_1^2A} + \sqrt{\frac{c_2^2B + c_1^2A}{c_2^2B - c_1^2A} - 1}\right)^n + \frac{1}{\left(\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A} + \sqrt{\frac{c_2^2B+c_1^2A}{c_2^2B-c_1^2A} - 1}\right)^n}\right) \\ &= \frac{1}{2}\left(\left(\frac{\left(\sqrt{c_1^2B} + \sqrt{c_1^2A}\right)^2}{c_2^2B + c_1^2A}\right)^n + \frac{1}{\left(\frac{\left(\sqrt{c_1^2B}+\sqrt{c_1^2A}\right)^2}{c_2^2B+c_1^2A}\right)^n}\right) \\ &= \frac{1}{2}\left(\left(\frac{c_2\sqrt{B} + c_1\sqrt{A}}{c_2\sqrt{B} + c_1\sqrt{A}}\right)^n + \frac{1}{\left(\frac{c_2\sqrt{B}+c_1\sqrt{A}}{c_2\sqrt{B}+c_1\sqrt{A}}\right)^n}\right) \\ &= \frac{1}{2}\left(\frac{1}{\sigma^n} + \sigma^n\right) = \frac{1 + \sigma^{2n}}{1 + 2\sigma^n},\end{aligned}$$

and therefore

$$c_n\left(\frac{1}{\rho}\right)^{-1} = \frac{2\sigma^n}{1 + \sigma^{2n}}.$$

This completes the proof. □

Now, we are ready to design an algorithm, by using fusion frames and based on the Chebyshev polynomials, that gives an approximate solution to the equation (2). Accordingly, we suppose that $(L(W_\lambda), \omega_\lambda)_{\lambda\in\Lambda}$ is the fusion frame with bounds A, B, and with fusion frame operator S', associated to a prime fusion frame $(W_\lambda, \omega_\lambda)_{\lambda\in\Lambda}$, based on proposition 2.4.

FFCHEBYSHEV $[L, \epsilon, A, B, c_1, c_2] \to h_\epsilon$

(i) Let $\rho = \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}$, $\sigma = \frac{c_2\sqrt{B} - c_1\sqrt{A}}{c_2\sqrt{B} + c_1\sqrt{A}}$, $h_0 = 0$

(ii) $k := 1$, $h_1 = \frac{2}{c_1^2 A + c_2^2 B} LS' f$, $\lambda_1 = 2$

(iii) $k := k + 1$

(iv) $\lambda_k = \left(1 + \frac{\rho^2}{4}\lambda_{k-1}\right)^{-1}$

(v) $h_k = \lambda_k \left(h_{k-1} + h_{k-2} + \frac{2}{c_1^2 A + c_2^2 B} LS'(f - Lh_{k-1})\right) + h_{k-2}$

(vi) If $\frac{2\sigma^n}{1+\sigma^{2n}} \frac{\|f\|}{c_1} \leq \epsilon$ stop and set $h_\epsilon := h_k$, if else go to ***(iii)***.

4. Conjugate Gradient Method

In the previous section, for Chebyshev method to be effective, a knowledge of an interval $[a, b]$ enclosing the spectrum of L is required. If this interval is too crude, the process loses its efficiency. An important advantage of *Conjugate Gradient Method* is that no priori information about the location of the spectrum is required. In this section, we introduce an iterative method by using fusion frames and based on the conjugate gradient method for solving operator equation (2), and then, we strive to design an algorithm, via this method, to drive an approximate solution. Furthermore, contrary to the case of Chebyshev iteration method, this implies a considerable advantage of adaptivity. The hidden polynomial Q_n in (6), depends nonlinearly on u, and arises from a minimization problem. First of all, we note that since $LS'L$ is a positive definite operator, we can define the following $LS'L$-norm by

$$\|f\|_{LS'L} = \langle LS'Lf, f\rangle^{\frac{1}{2}} = \left\|(LS'L)^{\frac{1}{2}} f\right\|, \quad \forall f \in H,$$

with corresponding inner product

$$\langle f, g\rangle_{LS'L} = \langle LS'Lf, g\rangle, \quad \forall f, g \in H.$$

To continue, we define the recurrence sequence

$$P_{n+1} = LS'LP_n - \frac{\langle LS'LP_n, LS'LP_n\rangle}{\langle P_n, LS'LP_n\rangle} P_n - \frac{\langle LS'LP_n, LS'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle} P_{n-1} \ n \geq 0, \tag{16}$$

with $P_{-1} = 0$, $P_0 = \frac{2}{c_1^2 A + c_2^2 B} LS'Lu$. For this sequence, we have some pleasant properties exhibited in the two following lemmas.

Lemma 4.1. *Let $H_n = span\left\{(LS'L)^i u : 1 \leq i \leq n\right\}$, then for vectors P_i defined by (16), we have*

$$\{P_0, P_1, \cdots, P_{n-1}\} \subset H_n. \tag{17}$$

Proof. We verify the claim by induction. It is obvious for $n = 1$. Assume that the theorem holds true for all $k \leq n$. For $k = n + 1$, by (16) and the definition of H_n we get

$$P_n = LS'LP_{n-1} - \frac{\langle LS'LP_{n-1}, LS'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle} P_{n-1} - \frac{\langle LS'LP_{n-1}, LS'LP_{n-2}\rangle}{\langle P_{n-2}, LS'LP_{n-2}\rangle} P_{n-2}, \tag{18}$$

where the right-hand side of (18) belongs to $LS'LH_n + H_n \subset H_{n+1}$. From here, the result follows as desired. □

Actually, there is more to say about the set introduced in (17), which will be discussed as follows.

Lemma 4.2. *The system $\{P_0, P_1, \cdots, P_{n-1}\}$, forms an orthogonal basis for H_n with respect to the inner product $\langle\cdot,\cdot\rangle_{LS'L}$.*

Proof. By virtue of (16), the theorem follows obviously for $n = 1, 2$. Now, we assume that the theorem holds for $k = n$, namely, $\langle P_n, LS'LP_i\rangle = 0$ for all $i = 0, \cdots, n-1$, and that $\{P_0, P_1, \cdots, P_n\}$ is an $LS'L$-orthogonal basis for H_{n+1}. The step of $k = n+1$ can be followed immediately for $i = n-1, n$ via (16). For $i < n-1$, since $LS'LP_i \in LS'LH_{n-1}$, induction hypothesis yields $LS'LP_i = \sum_{j=0}^{n-1} c_j P_j$ for some coefficients $c_j \in \mathbb{C}$. Therefore, using the $LS'L$-orthogonality of P_i, $i \leq n$, we obtain for $i < n-1$,

$$\begin{aligned}\langle P_{n+1}, LS'LP_i\rangle &= \left\langle LS'LP_n - \frac{\langle LS'LP_n, LS'LP_n\rangle}{\langle P_n, LS'LP_n\rangle}P_n - \frac{\langle LS'LP_n, LS'LP_{n-1}\rangle}{\langle P_{n-1}, LS'LP_{n-1}\rangle}P_{n-1}, LS'LP_i\right\rangle\\ &= \langle LS'LP_n, LS'LP_i\rangle\\ &= \left\langle LS'LP_n, \sum_{j=0}^{n-1} c_j P_j\right\rangle = 0.\end{aligned}$$

It turns out that $\{P_0, P_1, \cdots, P_{n-1}\}$ is indeed a basis for H_{n+1} since

$$n+1 = \dim\{P_0, P_1, \cdots, P_{n-1}\} \leq \dim H_{n+1} = n+1.$$

As we desired. □

Here, we intend to design an algorithm based on conjugate gradient method. Right after that, within two theorems, we attempt to investigate its convergence via its convergence rate. In this direction, we again suppose that $(L(W_\lambda), \omega_\lambda)_{\lambda\in\Lambda}$ is the fusion frame with bounds A, B, and with fusion frame operator S', associated to a prime fusion frame $(W_\lambda, \omega_\lambda)_{\lambda\in\Lambda}$, based on proposition 2.4.

FFCG $[L, \epsilon, A, B, c_1, c_2] \to h_\epsilon$

(i) Put $\sigma = \frac{c_2\sqrt{B} - c_1\sqrt{A}}{c_2\sqrt{B} + c_1\sqrt{A}}$, $P_{-1} = 0$

(ii) $k := 0$, $h_0 = 0$, $r_0 = P_0 = \frac{2}{c_1^2 A + c_2^2 B} LS'f$

(iii) $k := k+1$

(iv) $\lambda_{k-1} = \frac{\langle r_{k-1}, P_{k-1}\rangle}{\langle P_{k-1}, LS'LP_{k-1}\rangle}$

(v) $h_k = h_{k-1} + \lambda_{k-1} P_{k-1}$

(vi) $r_k = r_{k-1} - \lambda_{k-1} LS'LP_{k-1}$

(vii) $P_k = LS'LP_{k-1} - \frac{\langle LS'LP_n, LS'LP_{k-1}\rangle}{\langle P_{k-1}, LS'LP_{k-1}\rangle} P_{k-1} - \frac{\langle LS'LP_{k-1}, LS'LP_{k-2}\rangle}{\langle P_{k-2}, LS'LP_{k-2}\rangle} P_{k-2}$

(viii) If $\frac{2\sigma^k}{1+\sigma^{2k}} \frac{\sqrt{B}\|f\|}{c_1} \leq \epsilon$ stop and set $h_\epsilon := h_k$, if else go to ***(iii)***.

Theorem 4.3. *The term h_n in algorithm **FFCG** is the orthogonal projection of the exact solution u of (2) onto H_n with respect to $\langle\cdot,\cdot\rangle_{LS'L}$. That is,*

$$\|u - h_n\|_{LS'L} \leq \|u - g\|_{LS'L}, \quad \forall g \in H.$$

Proof. Since $h_n = \sum_{j=0}^{n-1} c_j P_j \in H_n$, it suffices to show that $\langle u - h_n, h_n\rangle_{LS'L} = 0$. To this end, by lemma 4.2, we observe that

$$\langle h_i, P_i\rangle = \left\langle \sum_{j=0}^{i-1} c_j P_j, P_i\right\rangle = 0. \tag{19}$$

Here, according to the algorithm **FFCG**, we can rewrite r_i as

$$\begin{aligned} r_i &= r_{i-1} - \lambda_i LS'LP_{i-1} \\ &= \cdots = r_0 - \sum_{j=0}^{i-1} \lambda_j LS'LP_j \\ &= r_0 - LS'Lh_i = LS'L(u - h_i). \end{aligned} \tag{20}$$

Therefore

$$\lambda_i = \frac{\langle r_i, P_i \rangle}{\langle P_i, LS'LP_i \rangle} = \frac{\langle LS'L(u - h_i), P_i \rangle}{\langle P_i, P_i \rangle_{LS'L}}.$$

This equality, together with (19) and (20), yields

$$\begin{aligned} \langle u - h_n, h_n \rangle_{LS'L} &= \left\langle u - \sum_{j=0}^{n-1} \lambda_j P_j, \sum_{j=0}^{n-1} \lambda_j P_j \right\rangle_{LS'L} \\ &= \sum_{j=0}^{n-1} \left(\overline{\lambda}_j \left\langle u, P_j \right\rangle_{LS'L} - \left| \lambda_j \right|^2 \langle P_i, P_i \rangle_{LS'L} \right) \\ &= \sum_{j=0}^{n-1} \overline{\lambda}_j \left(\left\langle u, P_j \right\rangle_{LS'L} - \lambda_j \left\langle P_j, P_j \right\rangle_{LS'L} \right) \\ &= \sum_{j=0}^{n-1} \overline{\lambda}_j \left(\left\langle LS'Lu, P_j \right\rangle - \frac{\left\langle LS'L(u - h_j), P_j \right\rangle}{\left\langle P_j, P_j \right\rangle_{LS'L}} \left\langle P_j, P_j \right\rangle_{LS'L} \right) \\ &= \sum_{j=0}^{n-1} \overline{\lambda}_j \left\langle LS'Lu - LS'L(u - h_j), P_j \right\rangle = \sum_{j=0}^{n-1} \overline{\lambda}_j \left\langle LS'Lh_j, P_j \right\rangle = 0, \end{aligned}$$

as we desired. □

The following theorem indicates the convergence of the **FFCG** algorithm.

Theorem 4.4. *The sequence h_n, defined in algorithm **FFCG**, satisfies the following inequality*

$$\|u - h_n\|_{LS'L} \leq \frac{2\sigma^n}{1 + \sigma^{2n}} \frac{\sqrt{B} \, \| f \|_H}{c_1}, \tag{21}$$

where $\sigma = \frac{c_2\sqrt{B} - c_1\sqrt{A}}{c_2\sqrt{B} + c_1\sqrt{A}}$.

Proof. First, we note that by definition of H_n the vector $h_n \in H_n$ has the representation as

$$h_n = q_{n-1}(LS'L) \frac{2}{c_1^2 A + c_2^2 B} LS'Lu,$$

where q_{n-1} is a polynomial of degree $n - 1$. Hence, for the error term $u - h_n$, we obtain

$$u - h_n = (I - q_{n-1}(LS'L) LS'L) u = Q_n \left(I - \frac{2}{c_1^2 A + c_2^2 B} LS'L \right) u,$$

where $Q_n(x) = 1-(1-x)q_{n-1}\left(1 - x/\frac{2}{c_1^2A+c_2^2B}\right)$ is a polynomial of degree n and with the property that $Q_n(1) = 1$. Thus, the immediately preceding theorem shows

$$\|u - h_n\|_{LS'L} = \left\|Q_n\left(I - \frac{2}{c_1^2A + c_2^2B}LS'L\right)u\right\|_{LS'L} \leq \left\|P_n\left(I - \frac{2}{c_1^2A + c_2^2B}LS'L\right)u\right\|_{LS'L},$$

for all polynomials P_n of degree n with $P_n(1) = 1$. Therefore, by using Remark 3.1 and lemma 3.2, we obtain

$$\begin{aligned}
\|u - h_n\|_{LS'L} &\leq \left\|P_n\left(I - \frac{2}{c_1^2A + c_2^2B}LS'L\right)u\right\|_{LS'L} \\
&= \left\|(LS'L)^{\frac{1}{2}}\, P_n\left(I - \frac{2}{c_1^2A + c_2^2B}LS'L\right)(LS'L)^{-\frac{1}{2}}\,(LS'L)^{\frac{1}{2}}\,u\right\|_H \\
&\leq \left\|(LS'L)^{\frac{1}{2}}\, P_n\left(I - \frac{2}{c_1^2A + c_2^2B}LS'L\right)(LS'L)^{-\frac{1}{2}}\right\|_{H\to H} \left\|(LS'L)^{\frac{1}{2}}\,u\right\|_H \\
&\leq \left\|P_n\left(I - \frac{2}{c_1^2A + c_2^2B}LS'L\right)\right\|_{H\to H} \|u\|_{LS'L} \\
&\leq \max_{\frac{c_2^2B-c_1^2A}{c_1^2A+c_2^2B}\leq x\leq\frac{c_2^2B-c_1^2A}{c_1^2A+c_2^2B}} |P_n(x)|\,\|u\|_{LS'L} \\
&\leq \frac{1}{c_n\left(\frac{b+a}{b-a}\right)}\frac{\sqrt{B}\,\|c\|}{c_1},
\end{aligned}$$

where c_n is the chebyshev polynomial. On the other hand, as we proved in Theorem 3.3

$$\frac{1}{c_n\left(\frac{b+a}{b-a}\right)} = \frac{2\sigma^n}{1+\sigma^{2n}}.$$

This completes the proof. □

The concept of adaptivity, here, refers to the power variable n appeared in (21), which at the same time equals to the dimension of the space H_n, from where the approximate solution h_n is obtained.

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