Filomat 36:6 (2022), 1943–1954
https://doi.org/10.2298/FIL2206943I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Relation-Theoretic Fixed Point Results for Set-Valued Mappings via Simulation Functions with an Application

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Abstract. In this article, we adopt the idea of set-valued $(\mathcal{Z}, \mathfrak{R})$-contractions and establish some fixed point results for such contractions in metric spaces utilizing binary relation. To validate our newly obtained results, we also provide some illustrative examples. Finally, we apply our results to solve a family of nonlinear matrix equations under suitable assumptions.

1. Introduction

The classical Banach contraction principle [1] is one of the most fundamental, simple and natural results in nonlinear analysis. Due to its natural setting and growing applications, this theorem has been generalized and extended by several authors (e.g., see [2–5]) which also contains a multitude of noted generalizations as well. In 2015, a similar attempt was made by Khojasteh et al. [6] wherein the authors introduce the notion of $\mathcal{Z}$-contractions using a family of control functions now often referred as "simulation function" that unify several types of linear as well as nonlinear contractions of the existing literature. In 2018, Sawangsup and Sintunavarat [7] have introduced the concept of $(\mathcal{Z}, \mathfrak{R})$-contractions for single valued mapping and obtained some fixed point theorems in complete metric space endowed with a transitive binary relation.

Banach contraction principle was extended by Nadler [8] to set-valued contractions, often referred as Nadler's contraction principle which has attracted the attention of several mathematician and by now there exists a considerable literature on and around Nadler's contraction principle. Before presenting Nadler's theorem, we need to recall the following notations and terminologies in respect of set-valued mappings to make our exposition self-sustained.
In a metric space $(\mathcal{M}, d)$ let us adopt the following notations:

- $K(\mathcal{M}) := \{P \subset \mathcal{M};\ P \text{ is nonempty and compact}\}$;
- $CB(\mathcal{M}) := \{P \subset \mathcal{M};\ P \text{ is nonempty, closed and bounded}\}$;
- $P(\mathcal{M})$: the power set of $\mathcal{M}$.

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
Keywords. Fixed points, set-valued mapping, $(\mathcal{Z}, \mathfrak{R})$-contraction, simulation function, binary relation, nonlinear matrix equation
Received: 15 July 2019; Accepted: 18 October 2021
Communicated by Naseer Shahzad
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We shall adopt the Hausdorff distance $\mathcal{H} : CB(\mathcal{M}) \times CB(\mathcal{M}) \to [0, +\infty)$ defined as

$$\mathcal{H}(P, Q) := max\Big\{\sup_{p\in P} D(p, Q), \sup_{q\in Q} D(q, P)\Big\}, \quad P, Q \in CB(\mathcal{M})$$

where,

$$D(p, Q) := \inf\{d(p, q) : q \in Q\}.$$

Then, $\mathcal{H}$ is a metric on $CB(\mathcal{M})$. Moreover, $(CB(\mathcal{M}), \mathcal{H})$ is complete if $(\mathcal{M}, d)$ is complete. Let, $S : \mathcal{M} \to P(\mathcal{M})$, then an element $r \in \mathcal{M}$ is called fixed point of S if $r \in Sr$ (the collection of all such elements will be denoted by $Fix(S)$). For further details one can see [9].

Now, we state Nadler's contraction principle concerning the existence of fixed points for set-valued contractions in metric spaces.

Theorem 1.1. *[8] Let $(\mathcal{M}, d)$ a complete metric space, and $S : \mathcal{M} \to CB(\mathcal{M})$. Then S has a fixed point if there exists $\delta \in [0, 1)$ such that*

$$\mathcal{H}(Sr, Ss) \leq \delta d(r, s), \text{ for all } r, s \in \mathcal{M}. \tag{1}$$

A mapping S (as defined above) satisfying (1) is known as set-valued contraction.

Due to the applicability and usefulness of set-valued mappings in several domains namely optimization, economics, game theory, variational inequalities problem, etc., many researchers attempted to obtain further generalizations, extensions, and possible applications of Nadler's contraction principle (e.g., [10–17]) and the references cited therein. With similar quest, Sintunavarat et al. [18] proved some fixed point results for q-set-valued quasi-contractions wherein they introduced and utilized the idea of set-valued preserving mappings in b-metric space equipped with a binary relation.

In what follows, we adopt the concept of set-valued $(\mathcal{Z}, \mathfrak{R})$-contraction mappings and prove some existence fixed point results for such mappings via "simulation functions" employing a binary relation $\mathfrak{R}$. Thereafter, we present some explanatory examples to validate our main results. Consequently, we deduce some existence as well as uniqueness results for single-valued mappings. At last, we apply our results to ensure the existence as well as the uniqueness of a solution for nonlinear matrix equations.

2. Preliminaries

With a view to have a possibly self-contained presentation, we describe the following terminological and notational conventions. In what follows $\mathbb{N}, \mathbb{N}_0, \mathbb{R}$ denote the set of natural numbers, set of non-negative natural numbers and set of real numbers, respectively.

Now, we recall the notion of "simulation functions" due to Khojasteh et al. [6].

Definition 2.1. *Suppose $\xi : [0, \infty)^2 \to \mathbb{R}$ is a function satisfying:*

(ξ_1) *$\xi(0, 0) = 0$;*
(ξ_2) *$\xi(s, t) < t - s$ for all $s, t > 0$;*
(ξ_3) *if $\{s_n\}, \{t_n\}$ are sequences in $(0, \infty)$ satisfying $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$, then $\limsup_{n\to\infty} \xi(s_n, t_n) < 0$.*

Then, ξ is known as "simulation function" (where, $[0, \infty)^2 := [0, \infty) \times [0, \infty)$).

Thereafter, Argoubi et al. [19] refined the above definition by removing axiom (ξ_1) which was followed by yet another refinement due to [20, 21] wherein authors revised the axiom (ξ_3) by taking $s_n < t_n$. Henceforth, we call a function $\xi : [0, \infty)^2 \to \mathbb{R}$ to be a "simulation function" if it satisfy:

(ξ_2) $\xi(s,t) < t - s$ for all $s, t > 0$;
(ξ_3) if $\{s_n\}, \{t_n\}$ are sequences in $(0, \infty)$ satisfying $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n > 0$ and $s_n < t_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \xi(s_n, t_n) < 0$.

The family of all "simulation functions" will be denoted by $\mathcal{Z}$.

Remark 2.2. *Due to the condition* (ξ_2), *we have* $\xi(s,s) < 0$, *for all* $s > 0$.

Here, for the sake of completeness we enlist some well known examples of "simulation functions" from the existing literature ([6, 21–23]):

Example 2.3. *Consider the mappings* $\xi_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *(for* $i = 1, 2, \cdots, 5$*) as follows:*

- $\xi_1(s,t) = \Psi(t) - \Phi(s), \forall s,t \in [0,\infty)$, *wherein* $\Phi, \Psi : [0,\infty) \to [0,\infty)$ *are continuous,* $\Psi(s) = \Phi(s) = 0 \iff s = 0$ *and* $\Psi(s) < s \leq \Phi(s)$ *for all* $s > 0$;
- $\xi_2(s,t) = \alpha t - s, \forall s,t \in [0,\infty)$, *with* $\alpha \in [0,1)$;
- $\xi_3(s,t) = t - \eta(t) - s$ *for all* $s,t \in [0,\infty)$, *where* $\eta : [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function and* $\eta(s) = 0 \iff s = 0$;
- $\xi_4(s,t) = t - \int_0^s \varphi(s)ds, \forall s,t \in [0,\infty)$, *wherein* $\varphi : [0,\infty) \to [0,\infty)$ *s.t.* $\int_0^\epsilon \varphi(s)ds$ *exists and* $\int_0^\epsilon \varphi(s)ds > \epsilon, \forall \epsilon > 0$.

Here, we include another example in this regard.

- $\xi_5(s,t) = \frac{t}{\alpha t+1} - s$ *for all* $s,t \in [0,\infty)$ *and* $\alpha > 0$.

Then $\xi_i's$ are "simulation functions" (for $i = 1, 2, \cdots, 5$). For more examples and related results on "simulation functions", one can consult [6, 21–26] and the references cited therein. Now, we recollect the definition of $\mathcal{Z}$-contraction.

Definition 2.4. *[6] Let* $(\mathcal{M}, d)$ *be a metric space and* S *a self mapping on* $\mathcal{M}$. *Then* S *is called a* $\mathcal{Z}$*-contraction w.r.t* ξ *if*

$$\xi(d(Sr, Ss), d(r,s)) \geq 0 \quad \text{for all } r,s \in \mathcal{M}. \tag{2}$$

In the above definition, if we take $\xi(s,t) = \alpha t - s$ for all $s,t \in [0,\infty)$ and $\alpha \in [0,1)$, then $\mathcal{Z}$-contraction takes the form of Banach contraction. Also, in view of Remark 2.2, it is obvious that any isometry defined on a metric space can not be $\mathcal{Z}$-contraction and vise-versa .

By defining $\mathcal{Z}$-contraction, Khajasteh et al. [6] obtained the following theorem and deduced several existing as well as some new fixed point results by varying "simulation functions".

Theorem 2.5. *[6] Let* $(\mathcal{M}, d)$ *be a complete metric space and* S *a self mapping on* $\mathcal{M}$. *If* S *is* $\mathcal{Z}$*-contraction w.r.t some* ξ. *Then* S *has a unique fixed point.*

3. Relation-Theoretic Notions and Auxiliary Results

In this section, we discuss some basic definitions, notions and related allied results involving a binary relation.

A subset $\mathfrak{R}$ of $\mathcal{M}^2$ is said to be a binary relation on $\mathcal{M}$. From now on by writing $\mathfrak{R}$, we will always mean a nonempty binary relation acting upon $\mathcal{M}$. If $(r,s) \in \mathfrak{R}$ and $(s,t) \in \mathfrak{R}$ imply $(r,t) \in \mathfrak{R}$, for any $r,s,t \in \mathcal{M}$ then $\mathfrak{R}$ is said to be transitive relation on $\mathcal{M}$. Also, we define $\mathfrak{R}^{-1} := \{(r,s) \in \mathcal{M}^2 : (s,r) \in \mathfrak{R}\}$ and $\mathfrak{R}^s = \mathfrak{R} \cup \mathfrak{R}^{-1}$. Let, $r,s \in \mathcal{M}$ then they are said to be $\mathfrak{R}$-comparable (denoted by $[r,s] \in \mathfrak{R}$) if $(r,s) \in \mathfrak{R}$ or $(s,r) \in \mathfrak{R}$.

From now on the triplets $(\mathcal{M}, d, \mathfrak{R})$ denotes a relational metric space where M remains a nonempty set, d a metric on M and $\mathfrak{R}$ an arbitrary relation on M.

Proposition 3.1. *[5] Let $\mathcal{M}$ be a non-empty set. For a binary relation $\mathfrak{R}$ on $\mathcal{M}$, we have*

$$(r,s) \in \mathfrak{R}^s \Leftrightarrow [r,s] \in \mathfrak{R}.$$

Definition 3.2. *[5] A sequence $\{r_n\} \subseteq \mathcal{M}$ is called $\mathfrak{R}$-preserving if*

$$(r_n, r_{n+1}) \in \mathfrak{R}, \quad \forall n \in \mathbb{N}_0.$$

Definition 3.3. *[27] Let $\mathcal{M}$ be a non-empty set. We say a binary relation $\mathfrak{R}$ is locally transitive on $\mathcal{M}$, if for any $\mathfrak{R}$-preserving sequence $\{r_n\} \subset \mathcal{M}$ (with range $P := \{r_n : n \in \mathbb{N}_0\}$), $\mathfrak{R}|_P$ is transitive.*

Definition 3.4. *[27] Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space, then it is said to be $\mathfrak{R}$-complete if every $\mathfrak{R}$-preserving Cauchy sequence converges to a point in $\mathcal{M}$, where $\mathfrak{R}$ is a binary relation acting on $\mathcal{M}$.*

From the above definition, it is clear that every complete metric space is $\mathfrak{R}$-complete, for an amorphous binary relation $\mathfrak{R}$. However, $\mathfrak{R}$-completeness takes the form of usual completeness employing universal relation.

Definition 3.5. *[5] Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space. Then, a binary relation $\mathfrak{R}$ is said to be d-self-closed if for any $\mathfrak{R}$-preserving sequence $\{r_n\}$ converges to r, then there must exists a subsequence $\{r_{n_l}\}$ of $\{r_n\}$ satisfying $[r_{n_l}, r] \in \mathfrak{R}$, $\forall l \in \mathbb{N}_0$.*

Definition 3.6. *[5] Let $\mathfrak{R}$ be a binary relation on $\mathcal{M}$ and $S : \mathcal{M} \to \mathcal{M}$. Then, we say $\mathfrak{R}$ is S-closed if for any $r, s \in \mathcal{M}$,*

$$(r,s) \in \mathfrak{R} \Rightarrow (Sr, Ss) \in \mathfrak{R}.$$

Now, we introduce a relatively new definition to this effect.

Definition 3.7. *Let $\mathcal{M}$ be a nonempty, $\mathfrak{R}$ a binary relation on $\mathcal{M}$ and $S : \mathcal{M} \to \mathcal{M}$. Then we say $\mathfrak{R}$ is triangular S-closed if $\mathfrak{R}$ is S-closed and for any $r, s \in \mathcal{M}$,*

$$(r,s) \in \mathfrak{R} \Rightarrow (r, Ss) \in \mathfrak{R}.$$

Definition 3.8. *[28] Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to \mathcal{M}$. Then S is called $\mathfrak{R}$-continuous at $r \in \mathcal{M}$ if any $\mathfrak{R}$-preserving sequence $\{r_n\} \subseteq \mathcal{M}$ s.t. $r_n \xrightarrow{d} r$, implies $Sr_n \xrightarrow{d} Sr$. If S is $\mathfrak{R}$-continuous at every points of $\mathcal{M}$ then S is referred as $\mathfrak{R}$-continuous.*

Every continuous mapping can be treated as $\mathfrak{R}$-continuous mapping (irrespective of a binary relation $\mathfrak{R}$). However, $\mathfrak{R}$-continuity matches with the usual continuity only when $\mathfrak{R}$ is taken to be the universal relation.

Definition 3.9. *[29] Let $r, s \in \mathcal{M}$, then a path in $\mathfrak{R}$ from r to s of length n (where, $n \in \mathbb{N}$) is a sequence (finite) $\{r_0, r_1, r_2, ..., r_n\} \subseteq \mathcal{M}$ such that $r_0 = r$, $r_n = s$ with $(r_l, r_{l+1}) \in \mathfrak{R}$, for each $l \in \{0, 1, ..., n-1\}$.*

Definition 3.10. *[28] For each $r, s \in A(\subseteq \mathcal{M})$, if there always exists a path from r to s in A, then we say A is $\mathfrak{R}$-connected in $\mathcal{M}$.*

We use the following notations to this effect.

- (•) $\mathcal{M}(S; \mathfrak{R}) := \{r \in \mathcal{M} : (r, Sr) \in \mathfrak{R}\}$, where $S : \mathcal{M} \to \mathcal{M}$ be any given mapping;
- (•) $\Upsilon(r, s, \mathfrak{R})$:= the collection of all possible paths from r to s in $\mathfrak{R}$, where $r, s \in \mathcal{M}$.

In 2018, Sawangsup and Sintunavarat [7] introduced the notation of $(\mathcal{Z}, \mathfrak{R})$-contraction for single-valued mappings utilizing a binary relation in a metric space.

Definition 3.11. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to \mathcal{M}$. Then, S is called a $\mathcal{Z}_{\mathfrak{R}}$-contraction w.r.t $\xi \in \mathcal{Z}$ if the following holds:*

$$\xi(d(Sr, Ss), d(r,s)) \geq 0 \quad \forall r, s \in \mathcal{M} \ \text{with } (r,s) \in \mathfrak{R}. \tag{3}$$

The authors in [7] obtained the following result by using a transitive binary relation in a complete metric space.

Theorem 3.12. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to \mathcal{M}$. If the following hypotheses hold:*

(*i*) $\mathcal{M}(S; \mathfrak{R}) \neq \emptyset$;
(*ii*) $\mathfrak{R}$ *is S-closed;*
(*iii*) $\mathfrak{R}$ *is transitive;*
(*iv*) S *is* $(\mathcal{Z}, \mathfrak{R})$*-contraction w.r.t some* $\xi \in \mathcal{Z}$;
(*v*) $\mathcal{M}$ *is complete;*
(*vi*) S *is continuous (or* $\mathfrak{R}$ *is d-self-closed);*

Then S admits a fixed point. Furthermore, if $\Upsilon(r, s, \mathfrak{R})$ is nonempty for all $r, s \in \mathcal{M}$, then the fixed point of S is unique.

Now, we recall the definition of $\mathfrak{R}_{\mathcal{H}}$-continuity for set-valued mappings.

Definition 3.13. *[30] Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to CB(\mathcal{M})$. Then S is said to be $\mathfrak{R}_{\mathcal{H}}$-continuous at $r \in \mathcal{M}$ if for any $\mathfrak{R}$-preserving sequence $\{r_n\} \subseteq \mathcal{M}$ with $r_n \xrightarrow{d} r$, implies $Sr_n \xrightarrow{\mathcal{H}} Sr$ (as $n \to \infty$). If S is $\mathfrak{R}_{\mathcal{H}}$-continuous at each point of $\mathcal{M}$, then we say that S is $\mathfrak{R}_{\mathcal{H}}$-continuous.*

Remark 3.14. *Every continuous mapping can be treated as $\mathfrak{R}_{\mathcal{H}}$-continuous mapping (irrespective of a binary relation $\mathfrak{R}$). On the other side, $\mathfrak{R}_{\mathcal{H}}$-continuity turn into the usual continuity under the universal relation.*

We will make use of the following lemma while proving our main results.

Lemma 3.15. *[24] Let $(\mathcal{M}, d)$ be a metric space and a sequence $\{r_n\}$ in $\mathcal{M}$ such that*

$$\lim_{n\to\infty} d(r_n, r_{n+1}) = 0.$$

Suppose $\{r_n\}$ is not a Cauchy sequence, then there always exist an $\epsilon > 0$ and two subsequences $\{r_{m(l)}\}, \{r_{n(l)}\}$ of $\{r_n\}$ with $l < m(l) < n(l)$ and the following sequences tend to ϵ as $l \to \infty$:

$$\{d(r_{m(l)}, r_{n(l)})\}, \{d(r_{m(l)}, r_{n(l)+1})\}, \{d(r_{m(l)-1}, r_{n(l)})\}, \{d(r_{m(l)-1}, r_{n(l)+1})\}, \{d(r_{m(l)+1}, r_{n(l)+1})\}.$$

4. Main Results

In this section, firstly we define the notion of set-valued $(\mathcal{Z}, \mathfrak{R})$-contraction.

Definition 4.1. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to CB(\mathcal{M})$. Given $\xi \in \mathcal{Z}$, we say that S is set-valued $(\mathcal{Z}, \mathfrak{R})$-contraction if the following holds:*

$$\xi(\mathcal{H}(Sr, Ss), d(r, s)) \geq 0 \quad \forall r, s \in \mathcal{M} \text{ with } (r, s) \in \mathfrak{R}. \tag{4}$$

Remark 4.2. *From the preceding definition, it is easy to observe that if S satisfy Eq. (4.1) for $(r, s) \in \mathfrak{R}$, then S also satisfy the same equation for $(s, r) \in \mathfrak{R}$ as the metrics d and $\mathcal{H}$ are symmetric in both the variable, and hence the equation (4.1) is satisfied by the mapping S for $[r, s] \in \mathfrak{R}$.*

Remark 4.3. *If we choose $\mathfrak{R}$ to be the universal relation, then by taking $\alpha \in [0, 1)$ and $\xi(s, t) = \alpha t - s$ for all $s, t \in [0, \infty)$ in Definition 4.1, we obtain set-valued contraction defined in Theorem 1.1.*

Remark 4.4. *For single valued mapping, Definition 4.1 naturally reduces to Definition 3.11.*

Now, we adopt the concepts of preserving and triangular preserving set-valued mappings in metric spaces employing a binary relation.

Definition 4.5. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to CB(\mathcal{M})$. Then S is said to be a preserving mapping if for each $r \in \mathcal{M}$ and $s \in Sr$ with $(r, s) \in \mathfrak{R}$, we have $(s, t) \in \mathfrak{R}$ for all $t \in Ss$.*

Definition 4.6. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to CB(\mathcal{M})$. Then S is said to be a triangular preserving mapping if S is a preserving mapping and*

$$(r, s) \in \mathfrak{R} \text{ and } (s, t) \in \mathfrak{R} \Rightarrow (r, t) \in \mathfrak{R}, \text{ for all } t \in Ss.$$

Now, we deduce the following two lemmas which will be useful while proving our main results.

Lemma 4.7. *Let $S : \mathcal{M} \to CB(\mathcal{M})$ be a triangular preserving mapping. Assume that there exist $r_0 \in \mathcal{M}$ and $r_1 \in Sr_0$ such that $(r_0, r_1) \in \mathfrak{R}$. Then for a sequence $\{r_n\}$ with $r_{n+1} \in Sr_n$, we have $(r_m, r_n) \in \mathfrak{R}$ with $m < n$ for all $m, n \in \mathbb{N}$.*

Proof. By the supposition there exist $r_0 \in \mathcal{M}$ and $r_1 \in Sr_0$ with $(r_0, r_1) \in \mathfrak{R}$. Since S is a preserving mapping, then by Definition 4.5 we have $(r_1, r_2) \in \mathfrak{R}$. Continuing this process, we obtain $(r_n, r_{n+1}) \in \mathfrak{R}$ for all $n \in \mathbb{N}_0$. Now, as S is a triangular preserving mapping, then by Definition 4.6 we have $(r_n, r_{n+2}) \in \mathfrak{R}$ for all $n \in \mathbb{N}_0$. Again, As $(r_n, r_{n+2}) \in \mathfrak{R}$ and $(r_{n+2}, r_{n+3}) \in \mathfrak{R}$, therefore we deduce $(r_n, r_{n+3}) \in \mathfrak{R}$ for all $n \in \mathbb{N}_0$. Recursively, we obtain $(r_m, r_n) \in \mathfrak{R}$ with $m < n$ for all $m, n \in \mathbb{N}$. □

Lemma 4.8. *Let $S : \mathcal{M} \to CB(\mathcal{M})$ be a preserving mapping. Suppose that there exists $r_0 \in \mathcal{M}$ and $r_1 \in Sr_0$ such that $(r_0, r_1) \in \mathfrak{R}$, where $\mathfrak{R}$ is locally transitive binary relation on $\mathcal{M}$. Then for a sequence $\{r_n\}$ with $r_{n+1} \in Sr_n$, we have $(r_m, r_n) \in \mathfrak{R}$ with $m < n$ for all $m, n \in \mathbb{N}$.*

Proof. Since S is preserving mapping then by the same lines of the above Lemma 4.7 we obtain $(r_n, r_{n+1}) \in \mathfrak{R}$ for all $n \in \mathbb{N}_0$. Since, $\mathfrak{R}$ is locally transitive, we get $(r_n, r_{n+2}) \in \mathfrak{R}$ for all $n \in \mathbb{N}_0$. Now, $(r_n, r_{n+2}) \in \mathfrak{R}$ and $(r_{n+2}, r_{n+3}) \in \mathfrak{R}$ (where $\mathfrak{R}$ is locally transitive), therefore we deduce $(r_n, r_{n+3}) \in \mathfrak{R}$ for all $n \in \mathbb{N}_0$. Recursively, we obtain $(r_m, r_n) \in \mathfrak{R}$ for all $m, n \in \mathbb{N}$ with $m < n$. □

Now, we prove our first main result utilizing the idea of triangular preserving mappings.

Theorem 4.9. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to K(\mathcal{M})$. If the following hypotheses hold:*

(*i*) *S is triangular preserving mapping;*
(*ii*) *$\exists r_0 \in \mathcal{M}$ and $r_1 \in Sr_0$ with $(r_0, r_1) \in \mathfrak{R}$;*
(*iii*) *S is a set-valued $(\mathcal{Z}, \mathfrak{R})$-contraction w.r.t some $\xi \in \mathcal{Z}$;*
(*iv*) *$(\mathcal{M}, d)$ is $\mathfrak{R}$-complete;*
(*v*) *S is $\mathfrak{R}_{\mathcal{H}}$-continuous (or $\mathfrak{R}$ is d-self-closed).*

Then S admits a fixed point.

Proof. By assumption (*ii*), there exists $r_0 \in \mathcal{M}$ and $r_1 \in Sr_0$ such that $(r_0, r_1) \in \mathfrak{R}$. If $r_0 = r_1$ or $r_1 \in Sr_1$, then r_1 is a fixed point of S and the proof is over. Therefore, let us assume that $r_1 \notin Sr_1$, then $Sr_0 \neq Sr_1$, i.e., $\mathcal{H}(Sr_0, Sr_1) > 0$. Since S is mapping from $\mathcal{M}$ to $K(\mathcal{M})$, so we can choose $r_2 \in Sr_1$ with $(r_1, r_2) \in \mathfrak{R}$ (as S is preserving map) such that

$$d(r_1, r_2) \leq \mathcal{H}(Sr_0, Sr_1). \tag{5}$$

Now, using the condition (*iii*), we have

$$\xi(\mathcal{H}(Sr_0, Sr_1), d(r_0, r_1)) \geq 0, \quad (\text{as } (r_0, r_1) \in \mathfrak{R})$$

by (ξ_2), we obtain

$$d(r_0, r_1) > \mathcal{H}(Sr_0, Sr_1). \tag{6}$$

From (5) and (6), we get

$$d(r_1, r_2) \leq \mathcal{H}(Sr_0, Sr_1) < d(r_0, r_1). \tag{7}$$

Similarly, we have $r_3 \in Sr_2$ with $(r_2, r_3) \in \mathfrak{R}$ such that

$$d(r_2, r_3) \leq \mathcal{H}(Sr_1, Sr_2) < d(r_1, r_2).$$

Recursively, we get a sequence $\{r_n\}$ in $\mathcal{M}$ with $r_{n+1} \in Sr_n$ such that $(r_n, r_{n+1}) \in \mathfrak{R}$ and

$$d(r_{n+1}, r_{n+2}) \leq \mathcal{H}(Sr_n, Sr_{n+1}) < d(r_n, r_{n+1}) \quad \text{for all } n \in \mathbb{N}_0. \tag{8}$$

Therefore, $\{d(r_n, r_{n+1})\}_{n=0}^{\infty}$ is a monotonically decreasing sequence of non-negative real numbers, and hence there exists $l \geq 0$ such that $\lim_{n\to\infty} d(r_n, r_{n+1}) = l$. We show that $l = 0$. On contrary, let us assume that $l > 0$ then from (8), we have

$$\lim_{n\to\infty} \mathcal{H}(Sr_n, Sr_{n+1}) = l. \tag{9}$$

Using (4) and (ξ_3), we obtain

$$0 \leq \limsup_{n\to\infty} \xi\Big(\mathcal{H}(Sr_n, Sr_{n+1}), d(r_n, r_{n+1})\Big) < 0,$$

which is a contradiction and hence $l = 0$, i.e., $\lim_{n\to\infty} d(r_n, r_{n+1}) = 0$.

With a veiw to prove Cauchy-ness of the sequence $\{r_n\}$. Let $\{r_n\}$ is not Cauchy, then in view of Lemma 3.15, there exist $\epsilon > 0$ and two subsequences $\{r_{m(l)}\}$ and $\{r_{n(l)}\}$ of $\{r_n\}$ such that $l < m(l) < n(l)$ and

$$\lim_{l\to\infty} d(r_{m(l)}, r_{n(l)}) = \lim_{l\to\infty} d(r_{m(l)+1}, r_{n(l)+1}) = \epsilon. \tag{10}$$

Since S is triangular-preserving mapping, then due to Lemma 4.7, we have $(r_{m(l)}, r_{n(l)}) \in \mathfrak{R}$. Now, by equation (4), we have

$$0 \leq \xi(\mathcal{H}(Sr_{m(l)}, Sr_{n(l)}), d(r_{m(l)}, r_{n(l)})).$$

Making use of the condition (ξ_2), we get

$$d(r_{m(l)}, r_{n(l)}) > \mathcal{H}(Sr_{m(l)}, Sr_{n(l)}).$$

Again, since S is a mapping from $\mathcal{M}$ to $K(\mathcal{M})$ and $r_{m(l)+1} \in Sr_{m(l)}$, $r_{n(l)+1} \in Sr_{n(l)}$, we deduce

$$d(r_{m(l)+1}, r_{n(l)+1}) \leq \mathcal{H}(Sr_{m(l)}, Sr_{n(l)}) < d(r_{m(l)}, r_{n(l)}). \tag{11}$$

Using (10) in (11), we obtain

$$\lim_{l\to\infty} \mathcal{H}(Sr_{m(l)}, Sr_{n(l)}) = \epsilon.$$

Therefore, using Definition 4.1 and (ξ_3), we get

$$0 \leq \limsup_{l\to\infty} \xi\Big(\mathcal{H}(Sr_{m(l)}, Sr_{n(l)}), d(r_{m(l)}, r_{n(l)})\Big) < 0,$$

which is a contradiction. Which shows that, $\{r_n\}$ is $\mathfrak{R}$-preserving Cauchy sequence in $\mathcal{M}$. Also, due to the assumption (iv), there exists $r^* \in \mathcal{M}$ such that $\lim_{n\to\infty} r_n = r^*$ (as $(\mathcal{M}, d)$ is $\mathfrak{R}$-complete).
Now, we have two alternative cases for the condition (v). Firstly, if S is $\mathfrak{R}_{\mathcal{H}}$-continuous, then we must have $\mathcal{H}(Sr_n, Sr^*) \to 0$ as $n \to \infty$ (due to $\mathfrak{R}_{\mathcal{H}}$-continuity of S). Now, as $r_{n+1} \in Sr_n$, we get

$$0 \leq D(r_{n+1}, Sr^*) \leq \mathcal{H}(Sr_n, Sr^*),$$

and therefore

$$0 \leq \lim_{n\to\infty} D(r_{n+1}, Sr^*) \leq \lim_{n\to\infty} \mathcal{H}(Sr_n, Sr^*) = 0.$$

Thus, we have $\lim_{n\to\infty} D(r_{n+1}, Sr^*) = 0$ which implies that $r_{n+1} \in \overline{Sr^*}$ (as $n \to \infty$). Since Sr^* is closed and $r_{n+1} \to r^*$(as $n \to \infty$) then $r^* \in Sr^*$. Therefore, r^* is a fixed point of S.

Alternatively, if $\mathfrak{R}$ is d-self-closed. Then, there exists a subsequence $\{r_{n_l}\}$ of $\{r_n\}$ with $[r_{n_l}, r^*] \in \mathfrak{R}$, $\forall l \in \mathbb{N}_0$. Also, from the condition (ξ_2) of Definition 2.1, we have

$$\mathcal{H}(Sr, Ss) < d(r, s), \quad \text{for all } r, s \in \mathcal{M} \text{ with } (r, s) \in \mathfrak{R} \text{ such that } Sr \neq Ss,$$

On using condition (*iii*), we obtain

$$D(r_{n(l)+1}, Sr^*) \leq \mathcal{H}(Sr_{n(l)}, Sr^*) < d(r_{n(l)}, r^*), \text{ as } (r_{n(l)}, r^*) \in \mathfrak{R},\ \forall l \in \mathbb{N}_0.$$

Taking limit as $n \to \infty$, we have $D(r_{n(l)+1}, Sr^*) = 0$, which implies that $r_{n(l)+1} \in \overline{Sr^*}$ (as $n \to \infty$). Since Sr^* is closed and $r_{n(l)+1} \to r^*$(as $n \to \infty$), we have $r^* \in Sr^*$. Hence, S enjoys a fixed point. This finishes the proof. □

Now, we prove our next main result for preserving mappings utilizing a locally transitive binary relation.

Theorem 4.10. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space, where $\mathfrak{R}$ is locally transitive and $S : \mathcal{M} \to K(\mathcal{M})$. If the following hypotheses hold:*

(*i*) *S is preserving mapping;*
(*ii*) *$\exists r_0 \in \mathcal{M}$ and $r_1 \in Sr_0$ such that $(r_0, r_1) \in \mathfrak{R}$;*
(*iii*) *S is a set-valued $(\mathcal{Z}, \mathfrak{R})$-contraction w.r.t some $\xi \in \mathcal{Z}$;*
(*iv*) *$(\mathcal{M}, d)$ is $\mathfrak{R}$-complete;*
(*v*) *S is $\mathfrak{R}_{\mathcal{H}}$-continuous (or $\mathfrak{R}$ is d-self-closed).*

Then S admits a fixed point.

Proof. The proof of this theorem follows the same lines as in Theorem 4.9 up to equation (10). Thereafter, as S is preserving mapping and $\mathfrak{R}$ is locally transitive, then due to Lemma 4.8, we have $(r_m, r_n) \in \mathfrak{R}$ for all $m, n \in \mathbb{N}$ with $m < n$. Consequently, rest of the proof is also same as Theorem 4.9. □

Next, we adopt the have the following explanatory example in support of Theorem 4.9 (also, of Theorem 4.10).

Example 4.11. *Consider the metric space $(\mathcal{M} = [-20, 20), d)$, where d is the usual metric. Let the binary relation $\mathfrak{R}$ be defined over $\mathcal{M}$ by*

$$(r, s) \in \mathfrak{R} \iff r > s \text{ and } r, s \in [0, 1].$$

If we consider $S : \mathcal{M} \to K(\mathcal{M})$ defined by

$$Sr = \begin{cases} [-20, -\frac{|r|}{3}], & if\ -20 \leq r < 0; \\ [0, \frac{3r}{4}], & if\ \ 0 \leq r \leq 1; \\ \{r, e^r\}, & if\ 1 < r < 20. \end{cases}$$

Clearly, S is not continuous but $\mathfrak{R}_{\mathcal{H}}$-continuous. Observe that $(\mathcal{M}, d)$ is not complete but it is $\mathfrak{R}$-complete. Now, for any $r, s \in \mathcal{M}$ such that $s \in Sr$ and if $(r, s) \in \mathfrak{R}$ then we have $r, s \in [0, 1]$ with $r > s$, gives rise $Ss \subset Sr \subseteq [0, \frac{3}{4}]$. Clearly, for any $t \in Ss$ we get $s > t$, by the definition of relation $\mathfrak{R}$, we obtain $(s, t) \in \mathfrak{R}$. Thus, S is $\mathfrak{R}$-preserving mapping. Moreover, $r > s$ and $s > t$, yields $r > t$, therefore $(r, t) \in \mathfrak{R}$ and hence S is triangular preserving mapping. Choose any $r_0 \in (0, 1]$ then we always have $r_1 \in Sr_0$ such that $(r_0, r_1) \in \mathfrak{R}$, i.e., the condition (ii) of Theorem 4.9 (also, Theorem 4.10) is satisfied. Also, $\mathfrak{R}$ is locally transitive (being transitive). Now, if we take $\xi^*(s, t) = \frac{3}{4}t - s$, then S is set-valued $(\mathcal{Z}, \mathfrak{R})$-contraction w.r.t ξ^*. Therefore, all the conditions of Theorem 4.9 (also, Theorem 4.10) are satisfied. Accordingly, S has fixed points in $\mathcal{M}$. $(Fix(S) = [-20, 0] \cup (1, 20))$.

Since every single valued mappings can be viewed as set-valued mappings by setting $Sr = \{Sr\}$ (for all $r \in \mathcal{M}$), therefore from Theorems 4.9 and 4.10, we get the following fixed point results:

Theorem 4.12. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space and $S : \mathcal{M} \to \mathcal{M}$. If the following hypotheses hold:*

(*i*) *$\mathfrak{R}$ is triangular S-closed;*
(*ii*) $\exists r_0 \in \mathcal{M}$ *with* $(r_0, Sr_0) \in \mathfrak{R}$;
(*iii*) *S is $(\mathcal{Z}, \mathfrak{R})$-contraction w.r.t some $\xi \in \mathcal{Z}$;*
(*iv*) *$(\mathcal{M}, d)$ is $\mathfrak{R}$-complete;*
(*v*) *S is $\mathfrak{R}$-continuous (or $\mathfrak{R}$ is d-self-closed).*

Then S admits a fixed point.

Theorem 4.13. *Let $(\mathcal{M}, d, \mathfrak{R})$ be a relational metric space, where $\mathfrak{R}$ is locally transitive and $S : \mathcal{M} \to \mathcal{M}$. If the following hypotheses hold:*

(*i*) *$\mathfrak{R}$ is S-closed;*
(*ii*) $\exists r_0 \in \mathcal{M}$ *with* $(r_0, Sr_0) \in \mathfrak{R}$;
(*iii*) *S is $(\mathcal{Z}, \mathfrak{R})$-contraction w.r.t some $\xi \in \mathcal{Z}$;*
(*iv*) *$(\mathcal{M}, d)$ is $\mathfrak{R}$-complete;*
(*v*) *S is $\mathfrak{R}$-continuous (or $\mathfrak{R}$ is d-self-closed).*

Then S admits a fixed point.

Next, we present a result for the uniqueness of fixed point.

Theorem 4.14. *In Theorem 4.12 (or Theorem 4.13), additionally if the $Fix(S)$ is $\mathfrak{R}^s$-connected then S enjoys a unique fixed point.*

Proof. On contrary, let us consider $r, s \in Fix(S)$ with $r \neq s$. Then, due to the $\mathfrak{R}^s$-connectedness of $Fix(S)$, we have a path of length n from r to s in $\mathfrak{R}^s$ say $\{r = r_0, r_1, r_2, ..., r_n = s\} \subseteq Fix(S)$ (where $r_l \neq r_{l+1}$ for every l, $(0 \leq l \leq n-1)$) with $[r_l, r_{l+1}] \in \mathfrak{R}$ for every l, $(0 \leq l \leq n-1)$. Since $r_l \in Fix(S)$, then $Sr_l = r_l$, for each $l \in \{0, 1, 2, ..., n\}$. As S is $(\mathcal{Z}, \mathfrak{R})$-contraction, then using (4.1) and (ξ_3), we obtain (for all l, $(0 \leq l \leq n-1)$)

$$0 \leq \xi(d(Sr_l, Sr_{l+1}), d(r_l, r_{l+1})) = \xi(d(r_l, r_{l+1}), d(r_l, r_{l+1})) < 0,$$

a contradiction. This finishes the proof. □

Example 4.15. *Let $\mathcal{M} = \{0, \frac{2}{3}, 1\} \cup \{\frac{1}{3^n} | \, n \in \mathbb{N}\}$ be a metric space equipped with usual metric d. Also, define a binary relation $\mathfrak{R}$ on $\mathcal{M}$ by*

$$(r, s) \in \mathfrak{R} \iff \Big(\frac{1}{3} \geq r > s \text{ or } (r, s) \in \{(0, 0), (0, \frac{2}{3}), (\frac{2}{3}, 1)\}\Big).$$

Clearly, $(\mathcal{M}, d)$ is $\mathfrak{R}$-complete metric space. Now, we consider a mapping $S : \mathcal{M} \to \mathcal{M}$ by

$$Sr = \begin{cases} \{\frac{r}{3}\}, & if\ r = \frac{1}{3^n},\ n \in \mathbb{N}; \\ \{0\}, & otherwise. \end{cases}$$

Then $\mathfrak{R}$ is triangular S-closed. Notice that $\mathfrak{R}$ is not transitive as $(0, \frac{2}{3}), (\frac{2}{3}, 1) \in \mathfrak{R}$ but $(0, 1) \notin \mathfrak{R}$. Therefore, Theorem 3.12 is not applicable. Whereas, on applying our newly obtained result, i.e., Theorem 4.14 with $\xi(s, t) = \frac{t}{3} - s$ for all $s, t \in [0, \infty)$, we conclude that the fixed point S is unique (namely $r = 0$).

Remark 4.16. *The preceding example (i.e., Example 4.15) demonstrates that Theorem 4.14 remains a sharpened and improved version of Theorem 3.12 due to Sawangsup and Sintunavarat [7] in the context of involved binary relation, contractive condition and underlying space for a self-mapping S on $\mathcal{M}$.*

5. Application

Consider the nonlinear matrix equations

$$\mathcal{X} = G + \sum_{k=1}^{m} \mathcal{P}_k^* \mathcal{Q}(\mathcal{X}) \mathcal{P}_k, \tag{12}$$

where G be a "positive definite Hermitian matrix" and $\mathcal{Q}$ is a strict order preserving[1)] continuous mapping from "the set of Hermitian matrix" to "the set of positive definite Hermitian matrix" with $\mathcal{Q}(0) = 0$, $\mathcal{P}_k$ are arbitrary $n \times n$ matrices and $\mathcal{P}_k^*$ their conjugates.

The purpose of this section is to establish the existence as well as uniqueness of the solution for the equation (12) using our results.

By $\mathfrak{M}(n), \mathfrak{H}(n), \mathfrak{P}(n), \mathfrak{H}^+(n)$, we denote "the family of all complex matrices", "the family of all Hermitian matrices" in $\mathfrak{M}(n)$, "the family of all positive definite matrices" in $\mathfrak{M}(n)$ and "the family of all positive semi-definite matrices" in $\mathfrak{M}(n)$ of order n respectively. Also, if $\mathcal{S} \in \mathfrak{P}(n)$ ($\mathcal{S} \in \mathfrak{H}^+(n)$), we denote it by $\mathcal{S} \succ 0$ ($\mathcal{S} \succeq 0$). Moreover, $\mathcal{S} \succ \mathcal{T}$ ($\mathcal{S} \succeq \mathcal{T}$) is equivalent to saying $\mathcal{S} - \mathcal{T} \succ 0$ ($\mathcal{S} - \mathcal{T} \succeq 0$). By the symbol $\|\cdot\|$, we denote the "spectral norm" of a matrix $\mathcal{P}$ which is defined by $\|\mathcal{P}\| = \sqrt{\lambda^+(\mathcal{P}^*\mathcal{P})}$, where $\lambda^+(\mathcal{P}^*\mathcal{P})$ is the largest eigenvalue of $\mathcal{P}^*\mathcal{P}$, where $\mathcal{P}^*$ is the "conjugate transpose" of $\mathcal{P}$. We utilize the metric d induced by the "trace norm" $\|\cdot\|_{tr}$, defined as $\|\mathcal{P}\|_{tr} = \sum_{k=1}^{n} s_k(\mathcal{P})$, where $s_k(\mathcal{P})$ ($1 \leq k \leq n$) are the "singular values" of $\mathcal{P} \in \mathfrak{M}(n)$. Then, the induced metric space $(\mathfrak{H}(n), d)$ is complete (for more details see [31–34]).

The next two lemmas will be useful in our forthcoming discussion.

Lemma 5.1. *[32] If $\mathcal{P} \succeq 0$ and $\mathcal{R} \succeq 0$ are matrices of order n, then $0 \leq tr(\mathcal{P}\mathcal{R}) \leq \|\mathcal{P}\| tr(\mathcal{R})$.*

Lemma 5.2. *[33] If $\mathcal{P} \in \mathfrak{H}(n)$ satisfies $\mathcal{P} \prec I_n$, then $\|\mathcal{P}\| < 1$.*

Theorem 5.3. *Consider the problem described by (12). Let us suppose that there exist $\tau > 0$ and $h > 0$ such that*

(*i*) $\forall \mathcal{S}, \mathcal{T} \in \mathfrak{H}(n)$ *with* $\mathcal{S} \prec \mathcal{T}$, *we have* $\left| tr\Big(\mathcal{Q}(\mathcal{T}) - \mathcal{Q}(\mathcal{S})\Big)\right| \leq \frac{|tr(\mathcal{T}-\mathcal{S})|}{h\Big(1+\tau|tr(\mathcal{T}-\mathcal{S})|\Big)}$;

(*ii*) $\sum_{k=1}^{m} \mathcal{P}_k \mathcal{P}_k^* \prec h I_n$;

(*iii*) $\exists\, G$ *such that* $\sum_{k=1}^{m} \mathcal{P}_k^* \mathcal{Q}(G) \mathcal{P}_k \succ 0$.

Then the matrix equation (12) has a solution.

Proof. We define a map $\mathcal{I} : \mathfrak{H}(n) \to \mathfrak{H}(n)$ by

$$\mathcal{I}(\mathcal{S}) = G + \sum_{k=1}^{n} \mathcal{P}_k^* \mathcal{Q}(\mathcal{S}) \mathcal{P}_k, \quad \text{for all } \mathcal{S} \in \mathfrak{H}(n), \tag{13}$$

and a binary relation

$$\mathfrak{R} := \{(\mathcal{S}, \mathcal{T}) \in \mathfrak{H}(n) \times \mathfrak{H}(n) : \mathcal{S} \prec \mathcal{T}\}.$$

Clearly, fixed point of $\mathcal{I}$ remains the solution of the matrix equation (12). The mapping $\mathcal{I}$ is well defined, $\mathfrak{R}$-continuous and $\mathfrak{R}$ is $\mathcal{I}$-closed. To accomplish this, it is enough to show that $\mathcal{I}$ is $(\mathcal{Z}, \mathfrak{R})$-contraction (here $\mathfrak{R} :=$ " $\prec$ ") w.r.t the "simulation function" given by

$$\xi(s,t) = \frac{t}{\tau t + 1} - s \quad \text{for all } s, t \in [0, \infty) \text{ and } \tau > 0.$$

[1)] $\mathcal{Q}$ is strict order preserving if $S, T \in \mathfrak{H}(n)$ with $\mathcal{S} \prec \mathcal{T}$ implies that $\mathcal{Q}(\mathcal{S}) \prec \mathcal{Q}(\mathcal{T})$.

Take any $\mathcal{S},\mathcal{T} \in \mathfrak{H}(n)$ with $\mathcal{S} \prec \mathcal{T}$. Since $\mathcal{Q}$ is order preserving, therefore we obtain $\mathcal{Q}(\mathcal{S}) \prec \mathcal{Q}(\mathcal{T})$. Thus, we have

$$\begin{aligned}
\|\mathcal{I}(\mathcal{T}) - \mathcal{I}(\mathcal{S})\|_{tr} &= tr\Big(\mathcal{I}(\mathcal{T}) - \mathcal{I}(\mathcal{S})\Big)\\
&= tr\Big(\sum_{k=1}^{m} \mathcal{P}_k^*\Big(\mathcal{Q}(\mathcal{T}) - \mathcal{Q}(\mathcal{S})\Big)\mathcal{P}_k\Big)\\
&= \sum_{k=1}^{m} tr\Big(\mathcal{P}_k^*\Big(\mathcal{Q}(\mathcal{T}) - \mathcal{Q}(\mathcal{S})\Big)\mathcal{P}_k\Big)\\
&= \sum_{k=1}^{m} tr\Big(\mathcal{P}_k^*\mathcal{P}_k\Big(\mathcal{Q}(\mathcal{T}) - \mathcal{Q}(\mathcal{S})\Big)\Big)\\
&= tr\Big(\Big(\sum_{k=1}^{m} \mathcal{P}_k^*\mathcal{P}_k\Big)\Big(\mathcal{Q}(\mathcal{T}) - \mathcal{Q}(\mathcal{S})\Big)\Big)\\
&\leq \Big\|\sum_{k=1}^{m} \mathcal{P}_k^*\mathcal{P}_k\Big\| \|\mathcal{Q}(\mathcal{T}) - \mathcal{Q}(\mathcal{S})\|_{tr}\\
&\leq \frac{1}{h}\Big\|\sum_{k=1}^{m} \mathcal{P}_k^*\mathcal{P}_k\Big\|\Big(\frac{\|\mathcal{T} - \mathcal{S}\|_{tr}}{1 + \tau\|\mathcal{T} - \mathcal{S}\|_{tr}}\Big)\\
&< \frac{\|\mathcal{T} - \mathcal{S}\|_{tr}}{1 + \tau\|\mathcal{T} - \mathcal{S}\|_{tr}}
\end{aligned}$$

or

$$\frac{\|\mathcal{T} - \mathcal{S}\|_{tr}}{1 + \tau\|\mathcal{T} - \mathcal{S}\|_{tr}} - \|\mathcal{I}(\mathcal{T}) - \mathcal{I}(\mathcal{S})\|_{tr} > 0$$

so,

$$\xi(\|\mathcal{I}(\mathcal{T}) - \mathcal{I}(\mathcal{S})\|_{tr}, \|\mathcal{T} - \mathcal{S}\|_{tr}) \geq 0.$$

Hence, $\mathcal{I}$ is a $(\mathcal{Z}, \mathfrak{R})$-contraction w.r.t the given $\xi \in \mathcal{Z}$. As $\sum_{k=1}^{m} \mathcal{P}_k^*\mathcal{Q}(G)\mathcal{P}_k \succ 0$, we have $G \prec \mathcal{I}(G)$. Therefore, all the required conditions of Theorem 4.12 (also, of Theorem 4.13) are fulfilled. Consequently, there exists $\hat{\mathrm{S}} \in \mathfrak{H}(n)$ such that $\mathcal{I}(\hat{\mathrm{S}})=\hat{\mathrm{S}}$, which shows that the equation (12) has a solution in $\mathfrak{H}(n)$. □

Theorem 5.4. *In view of the assumptions of Theorem 5.3, the solution of the given matrix equation (12) is unique.*

Proof. Due to Theorem 5.3, the set $Fix(\mathcal{I})$ is nonempty. Also, in view of [32], for every $\mathcal{S},\mathcal{T} \in \mathfrak{H}(n)$, there always exist a least upper bound and a greatest lower bound. So, the set $Fix(\mathcal{I})$ is $\mathfrak{R}^s$-connected (where $\mathfrak{R}$:= " $\prec$ "). Therefore, on using Theorem 4.14, we conclude that $\mathcal{I}$ admits a unique fixed point, and consequently the matrix equation (12) admits a unique solution in $\mathfrak{H}(n)$. This finishes the proof. □

We conclude with the following possible question:

Question: Can Theorem 4.9 and Theorem 4.10 be extended to the class CB($\mathcal{M}$) from K($\mathcal{M}$)?

Competing interest
The authors declares that they have no competing interest.

Acknowledgments
All the authors are grateful to the learned referee(s) for their fruitful and constructive suggestions. Also, the second author sincerely acknowledges the financial assistance from CSIR-India, for providing financial assistance in the form of JRF & SRF during his Ph.D. tenure.

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