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On the Structure of Quaternion Rings

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Abstract. Let $\mathcal{S}$ be a ring with identity in which 2 is invertible. In this paper we describe the structure of the quaternion ring $\mathcal{R} = H(\mathcal{S})$ which is a generalization of the Hamilton's division ring of real quaternions $\mathbb{H} = H(\mathbb{R})$.

1. Introduction

The ring of real quaternions $\mathbb{H} = \mathbb{R}\oplus\mathbb{R}i\oplus\mathbb{R}j\oplus\mathbb{R}k =: H(\mathbb{R})$ was discovered by William Rowan Hamilton (1805-1865) in 1843 as an extension of the complex numbers into four dimensions (see [6]). Algebraically speaking, $\mathbb{H}$ forms a division algebra over $\mathbb{R}$ of dimension four (see [6, Pages 195,196]). Hamilton expanded this detection to include applications in the area of physics and in 1853 published Lectures "On Quaternions" and in 1866 "Elements of Quaternions". Hamilton's quaternions have many applications other than in physics. They are extensively used in computer graphics to describe motion in 3-space, and more recently, they have been used in multiple antennae communications systems.

Since the quaternions were the first discovered noncommutative division ring, an investigation of their properties and construction became the basis of this study. The main purpose of this paper is to describe some algebraic structures of the quaternion ring $\mathcal{R} = H(\mathcal{S})$ for an arbitrary unital ring $\mathcal{S}$ with $2^{-1} \in \mathcal{S}$. Also, some properties of quaternion ring over the ring of integers $\mathbb{Z}$ and the ring $\mathbb{Z}_{p^n}$, where p is a prime number and $n \geq 1$, have been considered. It is shown that in most cases the structure of center and ideals of the quaternion rings and those of matrix rings are similar. The main motivation for our study actually comes from papers [1, 2, 7, 12]. Aristidou and Demetre [1] investigated the finite ring $H(\mathbb{Z}_p)$, where p is a prime, looking into its structure and some of its properties. They also established conditions for idempotency in $H(\mathbb{Z}_p)$ in [2]. Miguel and Serodio [12] found the number of idempotents and zero-divisors of $H(\mathbb{Z}_p)$ and provided a detailed structural description of the zero-divisor graph of $H(\mathbb{Z}_p)$. Ghahramani et al. [7] determined the structure of superderivations of the quaternion rings over a ring $\mathcal{S}$ as a $\mathbb{Z}_2$-graded ring. The first two authors proved in [3] that if $\mathcal{S}$ is a ring whose characteristic is an odd prime number, then the quaternion ring $H(\mathcal{S})$ is isomorphic to the 2×2 matrix ring $M_2(\mathcal{S})$. Moreover, recently Ghahramani et al. [8] described the form of some mappings on quaternion rings.

This paper is organized as follows. Some preliminaries are given in Section 2. The structure of ideals and some radicals of $H(\mathcal{S})$ are presented in Section 3, where its basic properties are given too. In Section 4,

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we show that under some mild conditions on $\mathcal{S}$, the rings $H(\mathcal{S})$ and $M_2(\mathcal{S})$ are isomorphic. Section 5 deals with some common and noncommon properties of $H(\mathcal{S})$ and $\mathcal{S}$.

2. Preliminaries

In this section we review some preliminary results which will be needed in the sequel.

Throughout, all rings are associative and possibly noncommutative with identity $1 \neq 0$. For any ring R, the center of R is denoted by $Z(R)$ and the Jacobson radical of R is denoted by $J(R)$. Also, we use $Nil_*(R)$ for lower nil radical, $Nil^*(R)$ for upper nilradical and $L\text{-}rad(R)$ for Levitzki radical of R. Recall that the ideal I of R is *nil* if for every $x \in I$ there exists $n \in \mathbb{N}$ such that $x^n = 0$, and I is *nilpotent* if there exists $n \in \mathbb{N}$ such that $I^n = 0$. Also, a subset S of R is *locally nilpotent* if for any finite subset $\{s_1, ..., s_n\} \subseteq S$ there exists an integer N such that any product of N elements from $\{s_1, ..., s_n\}$ is zero.

The ring of $n \times n$ matrices over a ring R is denoted by $M_n(R)$, I_n denotes the identity of this ring, and e_{ij} is the usual matrix unit.

Let $\mathcal{S}$ be a ring. Set

$$\mathcal{R} = H(\mathcal{S}) = \{s_0 + s_1 i + s_2 j + s_3 k : s_i \in \mathcal{S}\} = \mathcal{S} \oplus \mathcal{S}i \oplus \mathcal{S}j \oplus \mathcal{S}k.$$

Then, with the componentwise addition and multiplication subject to the relations $i^2 = j^2 = k^2 = ijk = -1$ and the convention that i, j, k commute with $\mathcal{S}$ elementwise, $\mathcal{R}$ is a ring called *quaternion ring* over $\mathcal{S}$.

For any quaternion $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathcal{R}$, the *conjugate* $\bar{q}$ of q is defined by $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$ and the *norm* $\eta(q)$ of q is defined by $\eta(q) = q_0^2 + q_1^2 + q_2^2 + q_3^2$. For more information the reader is referred to [10, 11].

We conclude this section with a lemma that will be useful in the sequel.

Lemma 2.1. *[8, Theorem 2.1] Let $\mathcal{S}$ be a ring. Then*

$$Z(H(\mathcal{S})) = Z(\mathcal{S}) \oplus (Z(\mathcal{S}) \cap B)i \oplus (Z(\mathcal{S}) \cap B)j \oplus (Z(\mathcal{S}) \cap B)k,$$

where $B = \{x \in \mathcal{S} : 2x = 0\}$ is an ideal of $\mathcal{S}$. Therefore, we have the following two special cases:

(i) *If $char(\mathcal{S}) = 2$, then $Z(H(\mathcal{S})) = H(Z(\mathcal{S}))$, from which it follows that if $\mathcal{S}$ is also commutative, so is $H(\mathcal{S})$,*
(ii) *If $\mathcal{S}$ is 2-torsion free, then $Z(H(\mathcal{S})) = Z(\mathcal{S})$.*

From now on, *we assume that $\mathcal{S}$ is a ring such that $2^{-1} \in \mathcal{S}$ and $\mathcal{R} = H(\mathcal{S})$.*

3. Description of ideals and some radicals of $\mathcal{R}$

This section is mainly devoted to the description of ideals and some radicals of the quaternion rings. For an analogous description for matrices see [11].

Theorem 3.1. *Let $\mathcal{S}$ be a ring with $2^{-1} \in \mathcal{S}$ and $\mathcal{R} = H(\mathcal{S})$. There is a one-to-one correspondence between the set of all two-sided ideals of $\mathcal{S}$ and the set of all two-sided ideals of $\mathcal{R}$ that associates to any ideal A of $\mathcal{S}$ the ideal $H(A) = A \oplus Ai \oplus Aj \oplus Ak$ of $\mathcal{R}$. The inverse bijection associates to every ideal $\mathcal{I}$ of $\mathcal{R}$ the ideal $\mathcal{I} \cap \mathcal{S}$ of $\mathcal{S}$.*

Proof. The ideal $H(A)$ is the kernel of the canonical mapping $H(\mathcal{S}) \to H(\mathcal{S}/A)$ and the ideal $\mathcal{I} \cap \mathcal{S}$ is the inverse image of $\mathcal{I}$ via the canonical embedding $\mathcal{S} \to H(\mathcal{S})$. In order to show that these two assignments are mutually inverse, we must prove that $H(A) \cap \mathcal{S} = A$ and $H(\mathcal{I} \cap \mathcal{S}) = \mathcal{I}$ for every A and $\mathcal{I}$. The first equality is trivial, and so is the inclusion $H(\mathcal{I} \cap \mathcal{S}) \subseteq \mathcal{I}$. In order to show that $\mathcal{I} \subseteq H(\mathcal{I} \cap \mathcal{S})$, fix an element $p = a + bi + cj + dk \in \mathcal{I}$ with $a, b, c, d \in \mathcal{S}$. It suffices to prove that $a, b, c, d \in \mathcal{I}$. We may assume $a \neq 0$ (otherwise, one can multiply p by i, j or k, depending on whether b, c or d is nonzero). Since $pi + ip \in \mathcal{I}$, we have $-2b + 2ai \in \mathcal{I}$. Now put $r = -2b + 2ai$. Since $jr - rj \in \mathcal{I}$ and $2^{-1} \in \mathcal{S}$, we get $a \in \mathcal{I}$. Since $-pi = b - ai - dj + ck \in \mathcal{I}$, we have $b \in \mathcal{I}$. Similarly, we get $c, d \in \mathcal{I}$. $\square$

One notes that by the statement of Theorem 3.1 every ideal of $H(\mathcal{S})$ is exactly of the form $H(A)$ for some uniquely determined ideal A of $\mathcal{S}$.

Remark 3.2. *Theorem 3.1 may not hold if $2^{-1} \notin \mathcal{S}$. For example, the ring $\mathcal{R} = H(\mathbb{Z}_2)$ is commutative, because $ji = -ij = ij$ and similarly $jk = kj$, $ik = ki$. Now if $\mathcal{I}$ is the ideal $(1+i)$ in $\mathcal{R}$, then the set $A = \mathcal{I} \cap \mathbb{Z}_2$ as defined in Theorem 3.1 is easily seen to be $\mathbb{Z}_2$, while $\mathcal{I} \neq \mathcal{R}$.*

The following corollary is immediate:

Corollary 3.3. *Let $\mathcal{S}$ be as above. Then*

(i) *For every ideal A of $\mathcal{S}$ we have $H(\frac{\mathcal{S}}{A}) \cong \frac{H(\mathcal{S})}{H(A)}$.*
(ii) *For every ideals A, B of $\mathcal{S}$ we have $H(A)H(B) = H(AB)$.*
(iii) *$Nil_*(H(\mathcal{S})) = H(Nil_*(\mathcal{S}))$.*

Lemma 3.4. *Suppose the ring $\mathcal{S}$ is commutative. Then A is a nil ideal of $\mathcal{S}$ if and only if $H(A)$ is nil ideal of $H(\mathcal{S})$.*

Proof. Let A be a nil ideal of $\mathcal{S}$ and let $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \in H(A)$. Choose n such that ${\alpha_i}^n = 0$ for all i. Since any summand of α^{4n} contains ${\alpha_i}^n$ for some i, we have $\alpha^{4n} = 0$ concluding that $H(A)$ is nil. The converse is clear. □

Proposition 3.5. *Let $\mathcal{S}$ be a commutative ring. Then*

$$Nil^*(H(\mathcal{S})) = H(Nil^*(\mathcal{S})).$$

Proof. Since $Nil^*(\mathcal{S})$ is nil, by the lemma above $H(Nil^*(\mathcal{S}))$ is also nil. So $H(Nil^*(\mathcal{S})) \subseteq Nil^*(H(\mathcal{S}))$. On the other hand, using Theorem 3.1 assume $Nil^*(H(\mathcal{S})) = H(A)$ for a suitable ideal A of $\mathcal{S}$. By the lemma above A is nil, hence $A \subseteq Nil^*(\mathcal{S})$. So $Nil^*(H(\mathcal{S})) = H(A) \subseteq H(Nil^*(\mathcal{S}))$. □

Theorem 3.6. *Let $\mathcal{S}$ be a ring with $2^{-1} \in \mathcal{S}$. Then*

$$J(H(\mathcal{S})) = H(J(\mathcal{S})).$$

Proof. In order to show that $J(H(\mathcal{S})) \subseteq H(J(\mathcal{S}))$, fix an element $p = a + bi + cj + dk \in J(H(\mathcal{S}))$ with $a, b, c, d \in \mathcal{S}$. It suffices to show that $a, b, c, d \in J(\mathcal{S})$. By a similar argument given in the proof of Theorem 3.1, a belongs to $J(H(\mathcal{S}))$. Hence $1 - sa$ is (left) invertible for any $s \in H(\mathcal{S})$. In particular for any $s \in \mathcal{S} \subseteq H(\mathcal{S})$, $1 - sa$ is (left) invertible in $H(\mathcal{S})$. So, there exists $p' = a' + b'i + c'j + d'k \in H(\mathcal{S})$ such that $p'(1 - sa) = 1$. That is, $a'(1 - sa) = 1$, concluding that $1 - sa$ is left invertible for all $s \in \mathcal{S}$. Therefore $a \in J(\mathcal{S})$. Similarly, we get $b, c, d \in J(\mathcal{S})$. For the converse inclusion it suffices to show that each $x \in J(\mathcal{S})$ belongs to $J(H(\mathcal{S}))$. Let $\alpha = a + bi + cj + dk$ be arbitrary in $H(\mathcal{S})$. We show that $1 - \alpha x = 1 - ax - bxi - cxj - dxk$ is invertible in $H(\mathcal{S})$. We prove the theorem by checking the following two claims.

Claim 1. For every $a \in \mathcal{S}$, $q = 1 - axi \in U(H(\mathcal{S}))$. We have $q\bar{q} = 1 + (ax)^2 = u \in U(\mathcal{S})$, since $x \in J(\mathcal{S})$. Hence, $q^{-1} = \bar{q}u^{-1} = b + ci \in U(H(\mathcal{S}))$ for some $b, c \in \mathcal{S}$.

Claim 2. For every $a, b \in \mathcal{S}$, $p = 1 - axj - bxk \in U(H(\mathcal{S}))$. We have

$$\begin{aligned} p\bar{p} &= (1 - axj - bxk)(1 + axj + bxk) \\ &= 1 + (ax)^2 + (bx)^2 + (bxax - axbx)i \\ &= u + (bxax - axbx)i \quad \text{for some } u \in U(\mathcal{S}) \\ &= u(1 + u^{-1}(bxax - axbx)i) \\ &= uv \quad \text{for some } v \in U(H(\mathcal{S})) \text{ by Claim 1.} \end{aligned}$$

Now we have

$$\begin{aligned}
1-\alpha x &= u_1 - bxi - cxj - dxk \quad \text{for some } u_1 \in U(\mathcal{S})\\
&= u_1(1-u_1^{-1}(bxi+cxj+dxk))\\
&= u_1(1-s_1xi-s_2xj-s_3xk) \quad \text{for some } s_i \in \mathcal{S}\\
&= u_1(u_2-s_2xj-s_3xk) \quad \text{for some } u_2 \in U(H(\mathcal{S})) \text{ by Claim 1}\\
&= u_1u_2(1-u_2^{-1}(s_2xj+s_3xk))\\
&= u_1u_2(1-(q_1+q_2i)(s_2xj+s_3xk)) \quad \text{for some } q_i \in \mathcal{S} \text{ by Claim 1}\\
&= u_1u_2(1-p_1xj-p_2xk) \quad \text{for some } p_i \in \mathcal{S}.
\end{aligned}$$

By Claim 2, the last term is invertible in $H(\mathcal{S})$, and this completes the proof. □

Note that in the theorem above the conclusion is not necessarily true if $2^{-1} \notin \mathcal{S}$: Let $\mathcal{S} = \mathbb{Z}_2$. Then it is easily observed that

$$J(H(\mathcal{S})) = \{0, 1+i, 1+j, 1+k, i+j, i+k, j+k, 1+i+j+k\}.$$

But $H(J(\mathcal{S})) = 0$.

Remark 3.7. *The case $\mathcal{S} = \mathbb{Z}$ shows however that the condition $2^{-1} \in \mathcal{S}$ is not necessary in the theorem above: obviously $H(J(\mathbb{Z})) = H(0) = 0$. Now, let $\alpha = a + bi + cj + dk \in J(H(\mathbb{Z}))$. Then $1 + a + bi + cj + dk \in U(H(\mathbb{Z})) = \{1, -1, i, -i, j, -j, k, -k\}$. An easy computation shows that the only possible case is $a = b = c = d = 0$, so that $\alpha = 0$. Therefore, $H(J(\mathbb{Z})) = J(H(\mathbb{Z})) = 0$.*

Theorem 3.8. *L-rad*$(H(\mathcal{S})) = H($*L-rad*$(\mathcal{S}))$.

Proof. Recall that an ideal $\mathcal{I}$ of $\mathcal{R}$ is of the form $H(A)$ for some ideal A of $\mathcal{S}$. Now, we claim that any ideal $\mathcal{I} = H(A)$ in $\mathcal{R}$ is locally nilpotent if and only if the ideal A in $\mathcal{S}$ is locally nilpotent: to see this, first assume that $\mathcal{I}$ is locally nilpotent. Then since $A \subseteq \mathcal{I}$, for any subset $\{a_1, ..., a_n\}$ of A, there exists N such that any product of N elements from $\{a_1, ..., a_n\}$ is zero. Hence, A is locally nilpotent.

Conversely, assume that $A \subseteq \mathcal{S}$ is locally nilpotent, and let $s_t = a_{t0} + a_{t1}i + a_{t2}j + a_{t3}k$, $1 \le t \le n$, be n arbitrary elements in $\mathcal{I}$. Set

$$S := \{a_{tl} \mid 1 \le t \le n, 0 \le l \le 3\}.$$

By hypothesis there exists N such that the product of any N elements from S is zero. From this, one can easily observe that the product of any N elements from $\{s_1, ..., s_n\} \subseteq \mathcal{I}$ is zero (since any summand of such product is a product of N members in S). Now, noting that *L-rad*$(\mathcal{R}) = \sum \mathcal{I}$, where $\mathcal{I}$ ranges over all locally nilpotent ideals of $\mathcal{R}$, by the claim above we have *L-rad*$(\mathcal{R}) = \sum H(A) = H(\sum A)$, where A ranges over all locally nilpotent ideals of $\mathcal{S}$. That is, *L-rad*$(H(\mathcal{S})) = H($*L-rad*$(\mathcal{S}))$. □

According to Lemma 2.1 and Theorem 3.1, the structure of center and ideals of the quaternion rings is similar to those of matrix rings. But in some cases they have some differences. For example, if S is a principal right ideal domain, then by a well-known result [10, Theorem 17.20], for any $n \ge 1$, $M_n(S)$ is a principal right ideal ring. This result does not hold for $H(S)$ in general: Let S be the ring of integers $\mathbb{Z}$. It is easy to see that the (right) ideal $(1 + i + j + k, 2)$ is not principal in $H(\mathbb{Z})$. More generally, every right ideal of $H(\mathbb{Z})$ is of the form (α) or $(\alpha, \alpha\frac{1+i+j+k}{2})$ for $\alpha \in H(\mathbb{Z})$. Note that if $\mathcal{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z} \quad \text{or} \quad a, b, c, d \in \mathbb{Z} + \frac{1}{2}\}$ is the Hurwitz ring of integral quaternions, then by Corollary 2.4 in [4] every one sided ideal of $\mathcal{H}$ is principal.

We proceed with a yet interesting result on the ideals of $H(\mathbb{Z})$.

Theorem 3.9. *Let p be an odd prime number, and (using Four Squares Theorem) let*

$$p = a^2 + b^2 + c^2 + d^2, \quad a, b, c, d \in \mathbb{Z}.$$

Then

$$(p) = H((p)) \subsetneq (a + bi + cj + dk) = H(\mathbb{Z}). \tag{1}$$

Proof. If the first inclusion is not true, then

$$a + bi + cj + dk = pt$$

for some $0 \neq t \in H(\mathbb{Z})$. Hence, multiplying by their conjugates, we arrive at $p = p^2 t\bar{t}$, a contradiction. To prove the second conclusion, factoring (1) modulo (p), we find that

$$0 \subsetneq \frac{(a + bi + cj + dk)}{(p)} \subseteq \frac{H(\mathbb{Z})}{(p)} \cong H(\frac{\mathbb{Z}}{(p)}) \cong H(\mathbb{Z}_p).$$

Since by Theorem 3.1 and the fact that $\mathbb{Z}_p$ is a simple ring, $H(\mathbb{Z}_p)$ is a simple ring, we conclude that $(a + bi + cj + dk) = H(\mathbb{Z})$. □

Corollary 3.10. *The theorem above shows that*

(i) *if p is an odd prime integer, then the ideal $(p) \subsetneq H(\mathbb{Z})$ is maximal (hence prime),*
(ii) *if n is an odd integer, then there exists a one-to-one correspondence between the set of all ideals U of $H(\mathbb{Z})$ satisfying $(n) \subseteq U \subseteq H(\mathbb{Z})$ and the set of all ideals V of $\mathbb{Z}$ satisfying $(n) \subseteq V \subseteq \mathbb{Z}$.*

Proof. Statement (i) follows immediately from the theorem above. For (ii), by [3, Corollary 3.13], if n is an odd integer, then

$$H(\mathbb{Z}/n\mathbb{Z}) \cong M_2(\mathbb{Z}/n\mathbb{Z}).$$

Hence, the basic result on ideals of matrix rings confirms our claim. □

Remark 3.11. *The results above are not true if n or p is not odd: Let $n = p = 2$. Then*

$$(2) \subsetneq (1 + i) \subsetneq H(\mathbb{Z}).$$

This shows that:

(a) The ideal $(2) \subsetneq H(\mathbb{Z})$ is not prime. $(1 - i), (1 + i) \subsetneq (2)$, *while* $(1 - i)(1 + i) \subseteq (2)$.

(b) The ideal $(1 + i)$ is strictly between (2) and $H(\mathbb{Z})$, while there is no ideals strictly between (2) and $\mathbb{Z}$.

4. Matrix representation of $\mathcal{R}$

In this section, we are concerned with the question "Under which conditions on the ring $\mathcal{S}$, the quaternion ring $H(\mathcal{S})$ is isomorphic to a 2×2 matrix ring?" This paves the way to our study of the quaternion rings.

Theorem 4.1. *[3, Theorem 3.10] Let F be a (not necessarily finite) field whose characteristic is an odd prime number p and let $\mathcal{S}$ be an algebra over F. Then*

$$H(\mathcal{S}) \cong M_2(\mathcal{S}).$$

The proof of the above theorem in [3] shows that if there exist commuting $\alpha, \beta \in \mathcal{S}$ such that $\frac{1}{4} + \alpha^2 + \beta^2 = 0$, then the quaternion ring $H(\mathcal{S})$ is isomorphic to a 2×2 matrix ring. Hence the following results are immediate:

Corollary 4.2. ([3, Corollary 3.12], [9, Proposition 4]) *If p is an odd prime number and $t \geq 1$, then*

$$H(\mathbb{Z}_{p^t}) \cong M_2(\mathbb{Z}_{p^t}).$$

Recall that the ring R is semilocal if $R/J(R)$ is semisimple; that is, $R/J(R) \cong \prod_{i=1}^{r} M_{n_i}(D_i)$, where $r \geq 1$, $n_i \geq 1$ and each D_i is a division ring. The following theorem inserts some conditions on $\mathcal{S}$ such that the ring $\mathcal{R}$ is isomorphic to a 2×2 matrix ring.

Theorem 4.3. *Let $\mathcal{S}$ be a semilocal ring such that $J(\mathcal{S})$ is nil, and assume that the division rings D_i in the decomposition $\frac{\mathcal{S}}{J(\mathcal{S})} \cong \prod_{i=1}^{r} M_{n_i}(D_i)$ have nonzero odd characteristics. Then $\mathcal{R} = H(\mathcal{S})$ is a 2×2 matrix ring.*

Proof. Identify $\bar{\mathcal{S}} := \frac{\mathcal{S}}{J(\mathcal{S})}$ with $\prod_{i=1}^{r} M_{n_i}(D_i)$, and assume that $char(D_i) = p_i$, where by assumption, p_i are (not necessarily distinct) odd prime numbers. For each i, identify the prime subfield $\mathbb{Z}_{p_i}$ of D_i with the subring $\{aI_{n_i} : a \in \mathbb{Z}_{p_i}\}$ of $M_{n_i}(D_i)$, and choose $a_i, b_i \in \mathbb{Z}$ such that in $\mathbb{Z}_{p_i}$, $1 + a_i^2 + b_i^2 = 0$. Noting that a_i, b_i are not both zero, we may assume that for each i, $a_i \neq 0$. Now, for the elements $\bar{a} = (a_1, ..., a_r)$ and $\bar{b} = (b_1, ..., b_r)$ in $\bar{\mathcal{S}}$, we have

$$\begin{aligned} 1 + \bar{a}^2 + \bar{b}^2 &= (1, ..., 1) + (a_1, ..., a_r)^2 + (b_1, ..., b_r)^2 \\ &= (1 + a_1^2 + b_1^2, ..., 1 + a_r^2 + b_r^2) = 0 = J(\mathcal{S}). \end{aligned}$$

Therefore, we have $1 + \bar{a}^2 + \bar{b}^2 = j$ for some $j \in J(\mathcal{S}) \subseteq \mathcal{S} \subseteq \mathcal{R}$. Now, we claim that for every integer $t \geq 0$, there exists $r_t \in U(\mathcal{R}) \cap \mathbb{Z}$ such that

$$1 + r_t^2 + b^2 = \alpha_t j^{2^t} \text{ for some } \alpha_t \in Z(\mathcal{R}). \tag{2}$$

For the time being, assume that the claim is true, and assume $j^n = 0$. Choose t such that $2^t \geq n$. In this case, $1 + r_t^2 + b^2 = 0$ for some $r_t, b \in \mathcal{R} \cap \mathbb{Z}$. Since 2 is invertible in $\mathcal{R}$, we conclude that $\frac{1}{4} + c^2 + d^2 = 0$ for some $c, d \in \mathcal{R}$.

Now, an easy computation shows that the elements $e_{11} := \frac{1}{2} + ci + dj$, $e_{22} := \frac{1}{2} - ci - dj$, $e_{12} := -e_{11}k = -di + cj - \frac{k}{2}$ and $e_{21} := ke_{11} = -di + cj + \frac{k}{2}$ make a full set of matrix units for $\mathcal{R}$ (in the sense that $e_{ij}e_{rs} = \delta_{jr}e_{is}$, where δ_{jr} is the δ-Kronecker function, and $e_{11} + e_{22} = 1$) so that, by [10, Theorem 17.5], $\mathcal{R}$ is a 2×2 full matrix ring.

To prove the claim, note that $r_0 := a \in \mathcal{R}$ is invertible (since $\bar{a} = (a_1, ..., a_r) \in U(\bar{\mathcal{S}})$) and that r_0 satisfies (2). Assume that we have chosen $r_t \in U(\mathcal{R}) \cap \mathbb{Z}$ in such way that $1 + r_t^2 + b^2 = \alpha_t j^{2^t} = j'$ for some $\alpha_t \in Z(\mathcal{R})$. Define $r_{t+1} = r_t - \frac{j'}{2r_t}$. Since $\bar{r}_{t+1} = \bar{r}_t \in U(\bar{\mathcal{S}})$, we conclude that $r_{t+1} \in U(\mathcal{R})$. We have

$$1 + r_{t+1}^2 + b^2 = 1 + (r_t - \frac{j'}{2r_t})^2 + b^2 = \frac{j'^2}{4r_t{}^2} = \alpha_{t+1} j^{2^{t+1}},$$

where $\alpha_{t+1} = \frac{\alpha_t^2}{4r_t^2} \in Z(\mathcal{R})$, as claimed. This completes the proof. □

In the theorem above,

(a) if we remove the assumption that $J(\mathcal{S})$ is nil, the conclusion may be false: Let p be any odd prime number and consider the localization $\mathcal{S} : = \mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at (p). Let $\mathcal{R} = H(\mathcal{S})$. We have $J(\mathcal{S}) = p\mathbb{Z}_{(p)}$, the unique maximal ideal of $\mathcal{S}$. Since $2^{-1} \in \mathcal{S}$, by Theorem 3.6 we have

$$J(\mathcal{R}) = H(J(\mathcal{S})) = H(p\mathbb{Z}_{(p)}).$$

Thus, from Corollary 3.3 it follows that

$$\frac{\mathcal{R}}{J(\mathcal{R})} \cong H(\frac{\mathbb{Z}_{(p)}}{p\mathbb{Z}_{(p)}}) \cong H(\mathbb{Z}_p) \cong M_2(\mathbb{Z}_p).$$

I.e, $\mathcal{R}$ is a similocal ring, but it is not a 2×2 matrix ring: for otherwise, there would exist a nonzero element

$$\alpha = \frac{1}{s}(a + bi + cj + dk) \in H(\mathcal{S}) \qquad a, b, c, d \in \mathbb{Z}, s \in \mathbb{Z}_{(p)}$$

with $\alpha^2 = 0$, so that

$$a^2 - b^2 - c^2 - d^2 + 2abi + 2acj + 2adk = 0,$$

from which it follows that $a = b = c = d = 0$, a contradiction. Note that $p\mathbb{Z}_{(p)} = J(\mathcal{S})$ is not nil.

(b) if $char(D_i) = 0$ for some i, the conclusion may not hold: Let $\mathcal{S} = \mathbb{Q}$. Then the rational quaternions $H(\mathbb{Q})$ is a division ring.

Theorem 4.4. *Let $\mathcal{S} = \mathbb{Z}_{2^n}$ $(n \geq 1)$ and $J = J(H(\mathbb{Z}_{2^n}))$. We have that:*

(i) $J = \{\, a + bi + cj + dk \in \mathcal{R} \,|\, a + b + c + d \equiv 0 \pmod 2 \,\}$

(ii) *$\mathcal{R}$ is a local ring with* $\dfrac{\mathcal{R}}{J(\mathcal{R})} \cong \mathbb{Z}_2$.

Proof. (i) Put $A = \{\, a + bi + cj + dk \in \mathcal{R} \,|\, a + b + c + d \equiv 0 \pmod 2 \,\}$. It is easy to see that A is an ideal of $\mathcal{R}$. Let $\alpha = a + bi + cj + dk \in A$. We have

$$\alpha^2 = a^2 - b^2 - c^2 - d^2 + 2abi + 2acj + 2adk.$$

On the other hand, $-t^2 \equiv t^2 \equiv t \pmod 2$ for each $t \in \mathbb{Z}$. So

$$a^2 - b^2 - c^2 - d^2 \equiv a + b + c + d \equiv 0 \pmod 2.$$

Hence, $\alpha^2 = 2q$ for some $q \in \mathcal{R}$ and so α is nilpotent (since $(\alpha^2)^{2^n} = 0$). That is, $A \subseteq J$. If there exists $\alpha = a + bi + cj + dk \in J - A$ then $\beta = 1 + \alpha \in A$ and so $\beta \in J$, hence $1 = \beta - \alpha \in J$, a contradiction. That is, $J \subseteq A$.

(ii) Since $\mathcal{I} = H(2\mathbb{Z}_{2^n})$ is a nilpotent ideal of $\mathcal{R}$, we have $\mathcal{I} \subseteq J$. On the other hand, we have

$$J(\frac{\mathcal{R}}{\mathcal{I}}) = J(\frac{H(\mathbb{Z}_{2^n})}{H(2\mathbb{Z}_{2^n})}) \cong J(H(\mathbb{Z}_2)).$$

Now, since $|\mathcal{I}| = (2^{n-1})^4 = 2^{4n-4}$ and $J(\frac{\mathcal{R}}{\mathcal{I}}) = \frac{J(\mathcal{R})}{\mathcal{I}}$, we have $|J(\mathcal{R})| = 2^{4n-4} \times 2^3 = 2^{4n-1}$, and so $\dfrac{\mathcal{R}}{J(\mathcal{R})} \cong \mathbb{Z}_2$. $\square$

Theorem 4.5. *Assume that the rings S_1, S_2 satisfy the hypotheses of Theorem 4.3. Let M be a unitary (S_1, S_2)-bimodule, and consider the triangular matrix ring $\mathcal{T} = \begin{pmatrix} S_1 & M \\ 0 & S_2 \end{pmatrix}$. Then*

$$H(\mathcal{T}) \cong M_2(\mathcal{T}).$$

Proof. It is easy to verify that $J := J(\mathcal{T}) = \begin{pmatrix} J(S_1) & M \\ 0 & J(S_2) \end{pmatrix}$. Let $t = \begin{pmatrix} s_1 & m \\ 0 & s_2 \end{pmatrix} \in J$ with $s_1^{n_1} = 0 = s_2^{n_2}$. Then $t^{n_1+n_2} = \begin{pmatrix} 0 & m' \\ 0 & 0 \end{pmatrix}$ for some $m' \in M$, so $t^{2(n_1+n_2)} = 0$, concluding that J is nil. Noting that

$$\mathcal{T}/J \cong S_1/J(S_1) \times S_2/J(S_2),$$

Theorem 4.3 can now be applied to $\mathcal{T}$. This completes the proof. $\square$

Let M be a unitary (R, S)-bimodule, and $\mathcal{T}$ be the triangular matrix ring $\mathcal{T} = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. Then

$$H(M) = M \oplus Mi \oplus Mj \oplus Mk$$

is a unitary $(H(R), H(S))$-bimodule with scalar multiplications given by:

$$(r_0 + r_1 i + r_2 j + r_3 k)(m_0 + m_1 i + m_2 j + m_3 k) = r_0 m_0 + r_1 m_1 i + r_2 m_2 j + r_3 m_3 k,$$

$$(m_0 + m_1 i + m_2 j + m_3 k)(s_0 + s_1 i + s_2 j + s_3 k) = m_0 s_0 + m_1 s_1 i + m_2 s_2 j + m_3 s_3 k.$$

In this case we have the following interesting theorem:

Theorem 4.6. *Let $\mathcal{T}$ and $H(M)$ be as above. Then*

$$H(\mathcal{T}) \cong \begin{pmatrix} H(R) & H(M) \\ 0 & H(S) \end{pmatrix}$$

.

Proof. Define $\varphi : H(\mathcal{T}) \to \begin{pmatrix} H(R) & H(M) \\ 0 & H(S) \end{pmatrix}$ naturally by

$$\begin{aligned}
&\varphi\Big(\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} + \begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} i + \begin{pmatrix} r_3 & m_3 \\ 0 & s_3 \end{pmatrix} j + \begin{pmatrix} r_4 & m_4 \\ 0 & s_4 \end{pmatrix} k\Big) \\
&= \begin{pmatrix} r_1 + r_2 i + r_3 j + r_4 k & m_1 + m_2 i + m_3 j + m_4 k \\ 0 & s_1 + s_2 i + s_3 j + s_4 k \end{pmatrix}.
\end{aligned}$$

A straightforward computation shows that φ is a ring isomorphism. □

5. Some common and noncommon properties of $\mathcal{S}$ and $H(\mathcal{S})$

Let $\mathcal{S}$ and $\mathcal{R}$ be as above. By Theorem 3.1, $\mathcal{S}$ is simple if and only if so is $\mathcal{R}$. Also, as $H(\mathcal{S})$ is a finitely generated $\mathcal{S}$-module, $\mathcal{S}$ is a left Noetherian (resp. left Artinian) ring if and only if so is $\mathcal{R}$. Moreover, the following results are consequences of Theorems 3.1, 3.6 and Corollary 3.3:

Corollary 5.1. *$\mathcal{S}$ is semisimple if and only if so is $\mathcal{R}$.*

Proof. $\mathcal{S}$ is semisimple if and only if $\mathcal{S}$ is left Artinian and $J(\mathcal{S}) = 0$ if and only if $\mathcal{R}$ is left Artinian and $J(\mathcal{R}) = 0$ (since $\mathcal{R}$ is a finitely generated $\mathcal{S}$-module and also, by Theorem 3.6 we get $J(\mathcal{R}) = 0$) if and only if $\mathcal{R}$ is semisimple. □

Corollary 5.2. *$\mathcal{S}$ is semilocal if and only if so is $H(\mathcal{S})$.*

Proof. Immediate from Theorem 3.6 and the above corollary. □

If $\mathcal{S}$ is a local ring, $H(\mathcal{S})$ is not necessarily local: let $\mathcal{S}$ be the local ring $\mathbb{Z}_9$. Then by Theorem 3.6 and previous results, we have

$$\frac{H(\mathcal{S})}{J(H(\mathcal{S}))} = \frac{H(\mathcal{S})}{H(J(\mathcal{S}))} \cong H(\frac{\mathcal{S}}{J(\mathcal{S})}) = H(\frac{\mathbb{Z}_9}{3\mathbb{Z}_9}) \cong H(\mathbb{Z}_3),$$

which is not a division ring.

The following is a purely ring theoretic consequence of Theorem 3.1.

Corollary 5.3. *Let $\mathcal{S}$ and $\mathcal{R}$ be as above. Then $\mathcal{R}$ is a prime (resp. semiprime) ring if and only if so is $\mathcal{S}$.*

Proof. We prove the prime case. Let $\mathcal{R}$ be a prime ring and let $AB = 0$ for some ideals A, B of $\mathcal{S}$. A straightforward computation shows that $H(A)H(B) = H(AB) = 0$. Since $\mathcal{R}$ is prime, we have $H(A) = 0$ or $H(B) = 0$. Hence, $A = 0$ or $B = 0$. Conversely, let $\mathcal{I}\mathcal{J} = 0$ for some ideals $\mathcal{I}, \mathcal{J}$ of $\mathcal{R}$. By Theorem 3.1, $\mathcal{I} = H(A), \mathcal{J} = H(B)$ for some ideals A, B of $\mathcal{S}$, hence $H(AB) = H(A)H(B) = 0$ forces $AB = 0$. Now, since $\mathcal{S}$ is a prime ring, we have $A = 0$ or $B = 0$, and hence $\mathcal{I} = H(A) = 0$ or $\mathcal{J} = H(B) = 0$. This completes the proof. □

Corollary 5.4. *$\mathcal{S}$ is a semiprimary ring if and only if so is $\mathcal{R}$.*

Proof. $\mathcal{S}$ is semiprimary if and only if $J(\mathcal{S})$ is nilpotent and $\frac{\mathcal{S}}{J(\mathcal{S})}$ is semisimple and by Corollary 3.3 and Theorem 3.6, these happen if and only if $J(H(\mathcal{S}))$ is nilpotent and $\frac{H(\mathcal{S})}{J(H(\mathcal{S}))}$ is semisimple. □

Remark 5.5. *Following a construction of Shepherson [13], if R is an algebra over a field F generated by* $\{s, t, u, v, w, x, y, z\}$ *with relations dictated by the matrix equation* $AB = I_2$ *where* $A = \begin{pmatrix} s & u \\ t & v \end{pmatrix}$ *and* $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, *then R is a domain, but* $BA \neq I_2$, *so that* $M_2(R)$ *is not Dedekind-finite. For the special case when char(F) is an odd prime number, by Theorem 4.1 we get* $H(\mathcal{S}) \cong M_2(\mathcal{S})$, *and thus* $H(\mathcal{S})$ *is not Dedekind-finite.*

The following proposition shows however that if the ring $\mathcal{S}$ is commutative, the conclusion differs.

Proposition 5.6. *If* $\mathcal{S}$ *is a commutative ring, then* $H(\mathcal{S})$ *is Dedekind-finite.*

Proof. Let $A = a + bi + cj + dk \in H(\mathcal{S})$. Since $\mathcal{S}$ is commutative, $A\bar{A} = \bar{A}A = \eta(A)$ and $\eta(AB) = \eta(A)\eta(B) = \eta(B)\eta(A) = \eta(BA)$ for any $A, B \in \mathcal{R}$. Let $AB = 1$ for some $B \in H(\mathcal{S})$. By multiplying $\bar{A}$ from the left we have $\eta(A)B = \bar{A}$. Now, since $\eta(A)\eta(B) = \eta(B)\eta(A) = 1$, $\eta(A)$ is invertible and $B = (\eta(A))^{-1}\bar{A}$. That is,

$$BA = (\eta(A))^{-1}\bar{A}A = 1.$$

□

Now we return to the noncommutative setting:

Theorem 5.7. $\mathcal{R} = H(\mathcal{S})$ *is left primitive if and only if so is* $\mathcal{S}$.

Proof. Assume that $\mathcal{S}$ is left primitive. Then there exists a left ideal I of $\mathcal{S}$ such that I is comaximal with any nonzero ideal J of $\mathcal{S}$. (I.e. $I + J = \mathcal{S}$.) Let $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$. Then by Theorem 3.1, $\mathcal{I} = H(A)$ for some nonzero ideal A of $\mathcal{S}$. By hypothesis we have $I + A = \mathcal{S}$. So $\mathcal{R} = H(\mathcal{S}) = H(I + A) = H(I) + H(A)$. Hence $\mathcal{R}$ is left primitive.

Conversely, let $\mathcal{R} = H(\mathcal{S})$ be left primitive. Hence there exists a left ideal $\mathcal{I}$ of $\mathcal{R}$ that is comaximal with any nonzero ideal of $\mathcal{R}$. Define

$$I = \{a \in \mathcal{S}; a + bi + cj + dk \in \mathcal{I} \text{ for some } b, c, d \in \mathcal{S}\}.$$

Then I is a nonzero left ideal of $\mathcal{S}$ and $0 \neq \mathcal{I} \subseteq H(I)$. Now let A be any nonzero ideal of $\mathcal{S}$. We claim that $I + A = \mathcal{S}$. We have $\mathcal{I} + H(A) = \mathcal{R}$. So $H(I + A) = H(I) + H(A) = \mathcal{R}$. That is, $I + A = \mathcal{S}$. □

To prove the next theorem, we need the following lemma:

Lemma 5.8. *[11, Exercise 6.19] For any von Neumann regular ring k, any finitely generated submodule M of a projective k-module P is a direct summand of P.*

Theorem 5.9. *Let* $\mathcal{S}$ *and* $\mathcal{R}$ *be as above.* $\mathcal{S}$ *is a von Neumann regular ring if and only if so is* $\mathcal{R}$.

Proof. Let $\mathcal{S}$ be von Neumann regular. In the lemma above, set $k = \mathcal{S}$, $P = \mathcal{R}$ (as a projective left $\mathcal{S}$-module) and let $M = \mathcal{I}$ be any finitely generated left ideal in $\mathcal{R}$. By the lemma above, $\mathcal{I}$ is a direct summand of the left $\mathcal{S}$-module $\mathcal{R}$. Hence, by [11, Theorem 4.23] $\mathcal{R}$ is a von Neumann regular ring. The converse is obvious. □

Theorem 5.10. *A ring* $\mathcal{S}$ (*in which* 2 *is not necessarily invertible*) *is indecomposable if and only if so is* $H(\mathcal{S})$.

Proof. Since $\mathcal{S}$ is indecomposable, $\mathcal{S}$ has no nontrivial central idempotents. Let $\alpha = a + bi + cj + dk$ be a central idempotent in $H(\mathcal{S})$, i.e.

$$\alpha^2 = (a + bi + cj + dk)^2 = a + bi + cj + dk \in Z(H(\mathcal{S})).$$

By Lemma 2.1 we have

$$Z(H(\mathcal{S})) = Z(\mathcal{S}) \oplus (Z(\mathcal{S}) \cap B)i \oplus (Z(\mathcal{S}) \cap B)j \oplus (Z(\mathcal{S}) \cap B)k,$$

such that $B = \{x \in \mathcal{S} : 2x = 0\}$. Therefore

$$a^2 - b^2 - c^2 - d^2 = a \in Z(\mathcal{S}),$$

$$ab + ba + cd - dc = b \in Z(\mathcal{S}) \cap B,$$

$$ac + ca - bd + db = c \in Z(\mathcal{S}) \cap B,$$

$$ad + da + bc - cb = d \in Z(\mathcal{S}) \cap B.$$

Thus, $2ac = c$, $2ab = b$ and $2ad = d$. Since $b, c, d \in B$, we have $b = c = d = 0$. So $a^2 = a$ is a central idempotent in $Z(\mathcal{S})$. By hypothesis $\mathcal{S}$ has no nontrivial central idempotents. Consequently, $H(\mathcal{S})$ is indecomposable.

Conversely, let $H(\mathcal{S})$ be indecomposable and let e be a central idempotent in $\mathcal{S}$. Since e is also a central idempotent in $H(\mathcal{S})$, e is a trivial idempotent. □

There is a one-to-one correspondence between the two-sided ideals of $M_n(R)$ and the two-sided ideals of R. This ideal correspondence actually arises from the fact that the rings R and $M_n(R)$ are Morita equivalent. This means that the categories of left R-modules and left $M_n(R)$-modules are similar. (For a full account on the Morita theory the reader is referred to the books [5] and [10]). Due to this observation, there exists a natural bijective correspondence between the isomorphism classes of the left R-modules and the left $M_n(R)$-modules, and between the isomorphism classes of the left ideals of R and $M_n(R)$. Analogous statements hold for right modules and right ideals. Through Morita equivalence, $M_n(R)$ can inherit any property of R which is Morita invariant: For example, simplicity, Artinianness, Noetherianness, primeness and numerous other properties. The correspondence between the two-sided ideals of $H(\mathcal{S})$ and those of $\mathcal{S}$ and some common properties between $H(\mathcal{S})$ and $\mathcal{S}$ discussed above suggests a question: Are the two rings $H(\mathcal{S})$ and $\mathcal{S}$ Morita equivalent? The answer is "no", because, for example, the two rings of $\mathbb{H}$ and $\mathbb{R}$ cannot be Morita equivalent. Otherwise, for $\mathcal{S} = \mathbb{R}$, according to [10, Theorem 17.25], there exists a progenerator $\mathcal{S}$-module P such that $H(\mathcal{S}) \cong End_{\mathcal{S}}(P)$. Since $\mathcal{S}$ is a division ring, P is a free $\mathcal{S}$-module; that is, $P \cong \oplus_{x \in X}\mathcal{S}$. If X is a singleton, we get a contradiction since $H(\mathcal{S}) \cong End_{\mathcal{S}}(P) \cong End_{\mathcal{S}}(\mathcal{S}) \cong \mathcal{S}$. If $|X| \geq 2$, then

$$H(\mathcal{S}) \cong End_{\mathcal{S}}(P) \cong End_{\mathcal{S}}(\mathcal{S} \oplus (\oplus_{x \in X - \{x_0\}}\mathcal{S})),$$

for some $x_0 \in X$ which is again a contradiction because the division ring $H(\mathbb{R})$ does not have nontrivial idempotents.

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