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On Category of T-rough Sets

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Abstract. We introduce three new categories in which their objects are T-approximation spaces and they are denoted by $\overline{\mathbf{NT}}\mathbf{AprS}$, $\overline{\mathbf{RNT}}\mathbf{AprS}$, and $\overline{\mathbf{LNT}}\mathbf{AprS}$. We verify the existence or nonexistence of products and coproducts in these three categories and characterized theirs epimorphisms and monomorphisms. We discuss equalizer and coequalizer of a pair of morphisms in the three categories. We introduce the notion of idempotent approximation space, and we show that idempotent approximation spaces and right upper natural transformations form a category, which is denoted by $\overline{\mathbf{RNT}}\mathbf{Apr^2S}$. Let $\underline{\mathrm{CS}}$ be the category of all closure spaces and closure preserving mappings. We define a functor F from $\overline{\mathbf{RNT}}\mathbf{Apr^2S}$ to $\underline{\mathrm{CS}}$ and show that F is a full functor and every object of $\underline{\mathrm{CS}}$ has a corefiection along F.

1. Introduction

In 1981, the concept of a rough set was originally proposed by Pawlak as a formal tool for modeling and processing incomplete information in information systems [24, 25]. Since then, this subject has been investigated in many papers, and subsequently the algebraic approach to rough sets has been studied by some authors. A key notion in the Pawlak rough set model is the equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. Let U be a set, and let θ be an equivalence relation on U. If $[x]_\theta$ denotes an equivalence class of θ containing x, then we define two operators $\underline{apr}_\theta$, $\overline{apr}_\theta \colon \mathcal{P}(U) \to \mathcal{P}(U)$ as $\underline{apr}_\theta(X) = \{x \in U | [x]_\theta \subseteq X\}$ and $\overline{apr}_\theta(X) = \{x \in U | [x]_\theta \cap X \neq \emptyset\}$ for every $X \in \mathcal{P}(U)$. The pair $(\underline{apr}_\theta(X), \overline{apr}_\theta(X))$ is called a rough set. However, equivalence relations are too respective for many applications; for instance, in existing databases, the values of attributes could be either symbolic or real-valued. Rough set theory would have difficulty in handling such values. It is a natural question to ask what happens if we substitute the universe set with an algebraic system. Some authors have studied the algebraic properties of rough sets. Biswas and Nanda [4] introduced the notion of a rough subgroup. Kuroki [21] introduced the notion of a rough ideal in a semigroup. Mordeson [22] used covers of the universal set to define an approximation operator on the power set of the given set. Estaji, Hooshmandasl, and Davva [14] considered the connection between a rough set and lattice theory and they

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introduced the concepts of upper and lower ideals (filters) in a lattice. Also, Estaji, Khodaii, and Bahram [15] introduced the notion of θ-upper and θ-lower approximation of a fuzzy subset of the lattice (also, see [13]). Davvaz [6, 7] concerned a relationship between a rough set and the ring theory and considered a ring as a universal set and introduced the notion of a rough ideal and a rough subring with respect to an ideal of a ring; see [9]. Kazanci and Davvaz [20] introduced the notions of a rough prime (primary) ideal and a rough fuzzy prime (primary) ideal in a ring and gave some properties of such ideals. Rough modules have been investigated by Davvaz and Mahdavipour [10].

The rough set theory is widely known as a reasonable and efficient soft computing method for handling several decision making situations via attribute selections and rule acquisitions; see [19, 27, 28]. Moreover, in the past decades, various generalized rough set models have been constructed in step with the actual demands of real-world situations; see [26, 30]. In [33], for the sake of enhancing, the applicability of generalized rough set models in handling the two challenges are mentioned in an HFL group decision making. They tried to explore a novel rough set model by means of the multi-granularity three-way decisions paradigm. Multi-granularity three-way decisions, which originate from the granular computing frame-work [32], construct multi-level problem solving methods by providing information analysis and information fusion rules for solution spaces in different granularity levels based on the three-way decisions theory; see [31].

In applied mathematics, we encounter many examples of mathematical objects that can be multiplied to each other. First of all, the real numbers themselves are such objects. Other examples are real-valued functions, the complex numbers, matrices, infinite series, vectors in n-dimensional spaces, and vector-valued functions. On the other hand, rough sets were originally proposed in the presence of an equivalence relation. An equivalence relation is sometimes difficult to be obtained in real-world problems due to the vagueness and incompleteness of human knowledge. Davvaz [**?**] in 2008 introduced the concept of a T-rough set, which is a generalization of rough set. It is obvious that by placing the equivalence function instead of T, we will have the same concept of rough set. Also a T-rough homomorphism in a group was defined in [8], which is a generalization of ordinary homomorphism such that $T\colon U \to \mathcal{P}^*(W)$ is a set valued mapping and U, W are two nonempty sets. It is a worth recall that the concept T-rough mentioned here with the concept T-rough defined in [23] that is a lower semicontinuous triangular norm are two different T-roughs. Then using the definitions of lower and upper inverses, he ? introduced the definition of uniform set-valued homomorphism and proved that every set-valued homomorphism is uniform. In [29], the concepts of a set-valued and a strong set-valued homomorphism of a ring were introduced and their related properties were investigated. Also, the notions of generalized lower and upper approximation operators, constructed by means of a set-valued mapping, which is a generalization of the notion of lower and upper approximations of a ring, were provided.

Also, Hosseini, Jafarzadeh, and Gholami [17, 18], defined the concept of a T-rough semigroup and a T-rough commutative ring by using the definitions of lower and upper approximations. Let U, W be two nonempty sets and let $X \subseteq W$. Also let $t\colon U \to \mathcal{P}^*(W)$ be a set-valued mapping, where $\mathcal{P}^*(W)$ denotes the set of all nonempty subsets of W. The set of all nonempty subsets of W is called a *t-approximation space*. The upper inverse and lower inverse of X under t are defined by

$$\overline{apr}_t(X) = \{\, u \in U\colon t(u) \cap X \neq \emptyset \,\},$$

and

$$\underline{apr}_t(X) = \{\, u \in U\colon t(u) \subseteq X \,\}.$$

For the sake of illustration, we consider the following example.

Example 1.1. Let $U = \{x, y, z, e\}$ and let $W = \{a, b, c\}$. Consider the set-valued function $t\colon U \to \mathcal{P}^*(W)$ defined by

$$t := \begin{pmatrix} x & y & z & e \\ \{b\} & \{a, c\} & \{b\} & \{a, b, c\} \end{pmatrix}.$$

Then we have the following table.

X	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	W
$\overline{apr}_t(X)$	$\{y,e\}$	$\{x,z,e\}$	$\{y,e\}$	U	$\{y,t\}$	U	U
$\underline{apr}_t(X)$	$\emptyset$	$\{x,z\}$	$\emptyset$	$\{x,z\}$	$\{y\}$	$\{x,z\}$	U

Category theory is not only a tool commonly used by many capable pure mathematicians but also a tie that can connect relatively easily fields of mathematics and theoretical computer science (see [1, 2]).

Many researchers are actually concerned with or interested in deeper and pure mathematical approaches (including categorical approach) to some application-driven issues (including fuzzy set theory, rough set theory, and soft set theory). For example, Banerjee and Chakraborty [3] defined the category ROUGH of Pawlak approximation spaces (with an individual subset as a related concept set) and proved that ROUGH is finitely complete but not a topos. Diker [11] proved that **R-APR**, power sets and pairs of rough set approximation operators, is isomorphic to a full subcategory of the category **cdrTex** whose objects are complemented textures and morphisms are complemented direlations. Also, he showed that **R-APR** and **cdrTex** are new examples of dagger symmetric monoidal categories.

We recall from [5] that if θ is an equivalence relation on U and γ is an equivalence relation on V, then a function $\varphi : U \to V$ is called an upper natural transformation from (U, θ) into (V, γ), provided that the diagram

$$\begin{tikzcd} \mathcal{P}(U) \arrow[r, "\overline{apr}_\theta"] \arrow[d, "\overline{\varphi}"'] & \mathcal{P}(U) \arrow[d, "\overline{\varphi}"] \\ \mathcal{P}(V) \arrow[r, "\overline{apr}_\gamma"'] & \mathcal{P}(V) \end{tikzcd}$$

commutes, where $\overline{\varphi} : \mathcal{P}(U) \to \mathcal{P}(V)$ is the forward powerset operator induced by the mapping φ, that is, $\overline{\varphi}(A) := \varphi(A)$ for every $A \in \mathcal{P}(U)$. In the continuation, we show $\overline{\varphi}$ with φ. Also, approximation spaces and upper natural transformations form a category, which is denoted by $\overline{\textbf{AprS}}$. A lower natural transformation is defined similarly. Also, approximation spaces and lower natural transformations form a category, which is denoted by $\underline{\textbf{AprS}}$. Borzooei, Estaji, and Mobini [5] verified the existence or nonexistence of limits and colimits in two categories and characterized several kinds of epimorphisms and monomorphisms. Estaji and Mobini [16] studied injective objects in $\overline{AprS}$ and $\underline{AprS}$. Many applications of T-rough set theory and category in the various sciences and two categories defined in [5] have provided our main motivation for studying category theory of T-rough set theory.

2. On approximation spaces with the upper natural transformations

In this section, we introduce the concept of an upper natural transformation, a right (left) upper natural transformation, rough morphism, and right(left) rough morphism on an approximation space and show that approximation spaces with upper natural transformations or with right (left) upper natural transformations form a category, which are denoted by $\overline{\textbf{NTAprS}}$ or $\overline{\textbf{RNTAprS}}$ ($\overline{\textbf{LNTAprS}}$). In the following, we study the properties and relations between upper natural transformation, right(left) upper natural transformation, rough morphism, and right(left) rough morphism.

Definition 2.1. Let (U, t) and (V, s) be two approximation spaces, where $t\colon U \to \mathcal{P}^*(U)$ and $s\colon V \to \mathcal{P}^*(V)$ are functions. We say a function $\varphi\colon U \to V$ is

(1) an upper the natural transformation from (U, t) into (V, s) if the following diagram commutes:

$$\begin{tikzcd} \mathcal{P}(U) \arrow[r, "\overline{apr}_t"] \arrow[d, "\overline{\varphi}"'] & \mathcal{P}(U) \arrow[d, "\overline{\varphi}"] \\ \mathcal{P}(V) \arrow[r, "\overline{apr}_s"'] & \mathcal{P}(V). \end{tikzcd}$$

(2) a right upper natural transformation from (U, t) into (V, s) if

$$\varphi\Big(\overline{apr}_t(A)\Big) \subseteq \overline{apr}_s\Big(\varphi(A)\Big)$$

for every $A \in \mathcal{P}(U)$.

(3) a left upper natural transformation from (U, t) into (V, s) if

$$\overline{apr}_s\Big(\varphi(A)\Big) \subseteq \varphi\Big(\overline{apr}_t(A)\Big)$$

for every $A \in \mathcal{P}(U)$.

Remark 2.2. Let U be a set and let θ be an equivalence relation on U. We define $\theta_{equ} \colon U \to \mathcal{P}^*(U)$ by $\theta_{equ}(x) = [x]_\theta$. Throughout this paper, this notation will be used. It is evident that $x \in \theta_{equ}(x)$ and $\theta^2_{equ}(x) = \{\, \theta_{equ}(x) \,\}$ for every $x \in U$.

Let (U, t) be an approximation space such that $x \in t(x)$ and $t^2(x) = \{\, t(x) \,\}$ for every $x \in U$. Then $t(U)$ is a portion of U, and if there is an equivalence relation θ on U whose partition is $t(U)$, then $t = \theta_{equ}$.

Remark 2.3. Let (U, t) and (V, s) be two approximation spaces. Then the following statements hold:

(1) If $\varphi \colon U \to V$ is an upper natural transformation from (U, t) into (V, s), then $u \in \overline{apr}_t(\{x\})$ if and only if $x \in t(u)$ for every $u, x \in U$.

(2) If $\varphi \colon U \to V$ is an upper natural transformation from (U, t) into (V, s), then $\{\, \varphi(u) \colon u \in U,\ x \in t(u) \,\} = \{\, v \in V \colon \varphi(x) \in s(v) \,\}$ for every $x \in U$.

Let $\varphi \colon U \to V$ be an upper natural transformation from (U, t) into (V, s). Since

$$\varphi\Big(\overline{apr}_t(\{x\})\Big) = \{\, \varphi(u) \colon u \in U,\ x \in t(u) \,\}$$

and

$$\overline{apr}_s\Big(\varphi(\{x\})\Big) = \{\, v \in V \colon \varphi(x) \in s(v) \,\},$$

we conclude that

$$\{\, \varphi(u) \colon u \in U,\ x \in t(u) \,\} = \{\, v \in V \colon \varphi(x) \in s(v) \,\}$$

for every $x \in U$.

Proposition 2.4. *The following statements hold:*

(1) *Approximation spaces and right upper natural transformations form a category, which is denoted by* $\overline{\textbf{\textit{RNT}}}$***AprS***.

(2) *Approximation spaces and left upper natural transformations form a category, which is denoted by* $\overline{\textbf{\textit{LNT}}}$***AprS***.

(3) *Approximation spaces and upper natural transformations form a category, which is denoted by* $\overline{\textbf{\textit{NT}}}$***AprS***.

Proposition 2.5. *Let (U, t) and (V, s) be two approximation spaces. Then $\varphi\colon U \to V$ is an upper natural transformation from (U, t) into (V, s) if and only if*

$$\varphi\Big(\overline{apr}_t(\{x\})\Big) = \overline{apr}_s\Big(\varphi(\{x\})\Big)$$

for every $x \in U$.

Proof. *Necessity.* It is clear.

Sufficiency. Let $X \subseteq U$ be given. Since

$$\begin{aligned} y \in \varphi\Big(\overline{apr}_t(X)\Big) &\Rightarrow y = \varphi(u)\,\&\, t(u) \cap X \neq \emptyset \text{ for some } u \in U \\ &\Rightarrow y = \varphi(u)\,\&\, x \in t(u) \text{ for some } (u, x) \in U \times X \\ &\Rightarrow y = \varphi(u)\,\&\, u \in \overline{apr}_t(\{x\}) \text{ for some } (u, x) \in U \times X \\ &\Rightarrow y = \varphi(u) \in \varphi\Big(\overline{apr}_t(\{x\})\Big) \text{ for some } (u, x) \in U \times X \\ &\Rightarrow y \in \overline{apr}_s\Big(\varphi(\{x\})\Big) \subseteq \overline{apr}_s\Big(\varphi(X)\Big) \text{ for some } x \in X, \end{aligned}$$

and

$$\begin{aligned} y \in \overline{apr}_s\Big(\varphi(X)\Big) &\Rightarrow \varphi(x) \in s(y) \text{ for some } x \in X \\ &\Rightarrow y \in \overline{apr}_s\Big(\{\varphi(x)\}\Big) \text{ for some } x \in X \\ &\Rightarrow y \in \varphi\Big(\overline{apr}_t(\{x\})\Big) \subseteq \varphi\Big(\overline{apr}_t(X)\Big) \text{ for some } x \in X, \end{aligned}$$

we conclude that $\varphi\Big(\overline{apr}_t(X)\Big) = \overline{apr}_s\Big(\varphi(X)\Big)$. □

Proposition 2.6. *Let (U, t) and (V, s) be two approximation spaces. For every function $\varphi\colon U \to V$, the following statements are equivalent:*

(1) *φ is a right upper natural transformation from (U, t) into (V, s).*
(2) *$\varphi\Big(\overline{apr}_t(\{x\})\Big) \subseteq \overline{apr}_s\Big(\varphi(\{x\})\Big)$ for every $x \in U$.*
(3) *$\varphi\Big(t(x)\Big) \subseteq s\Big(\varphi(x)\Big)$ for every $x \in U$.*

Proof. (1) $\Rightarrow$ (2). It is evident.

(2) $\Rightarrow$ (3). For every $x \in U$,

$$\begin{aligned} a \in \varphi\Big(t(x)\Big) &\Rightarrow a = \varphi(b) \text{ for some } b \in t(x) \\ &\Rightarrow a = \varphi(b) \text{ for some } b \in U \text{ such that } x \in \overline{apr}_t(\{b\}) \\ &\Rightarrow a = \varphi(b)\,\&\, \varphi(x) \in \varphi\Big(\overline{apr}_t(\{b\})\Big) \subseteq \overline{apr}_s\Big(\varphi(\{b\})\Big) \text{ for some } b \in U \\ &\Rightarrow a \in s\Big(\varphi(x)\Big). \end{aligned}$$

Therefore, $\varphi\Big(t(x)\Big) \subseteq s\Big(\varphi(x)\Big)$ for every $x \in U$.

(3) $\Rightarrow$ (1). For every $B \subseteq U$,

$$\begin{aligned} a \in \varphi\Big(\overline{apr}_t(B)\Big) &\Rightarrow a = \varphi(x) \text{ for some } x \in \overline{apr}_t(B) \\ &\Rightarrow a = \varphi(x)\,\&\, b \in t(x) \text{ for some } b \in B \\ &\Rightarrow a = \varphi(x)\,\&\, \varphi(b) \in \varphi\Big(t(x)\Big) \subseteq s\Big(\varphi(x)\Big) \text{ for some } b \in B \\ &\Rightarrow a \in \overline{apr}_s\Big(\varphi(B)\Big). \end{aligned}$$

Therefore, $\varphi\Big(\overline{apr}_t(B)\Big) \subseteq \overline{apr}_s\Big(\varphi(B)\Big)$ for every $B \subseteq U$. □

Proposition 2.7. *Let (U, t) be an approximation space such that $t^2(x) = \{t(x)\}$ for every $x \in U$. Then the following statements hold:*

(1) $\overline{apr}_t(\{x\}) \neq \emptyset$ *if and only if* $x \in t(x)$ *for every* $x \in U$.
(2) *If* $\overline{apr}_t(\{x\}) \neq \emptyset$, *for every* $x \in U$, *then* $t(U)$ *is a partition of* U.
(3) *If* $\overline{apr}_t(\{x\}) \neq \emptyset$, *for every* $x \in U$, *then* $\overline{apr}_t(\{x\}) = t(x)$ *for every* $x \in U$.

Proof. (1). Let $x \in U$ be given. Then

$$\begin{aligned} \overline{apr}_t(\{x\}) \neq \emptyset &\Rightarrow x \in t(y) \text{ for some } y \in U \\ &\Rightarrow x \in t(y) = t(x) \text{ for some } y \in U, \text{ by our hypothesis} \end{aligned}$$

The rest is evident. □

Corollary 2.8. *Let (U, t) and (V, s) be two approximation spaces and let $\varphi : U \to V$ be an upper natural transformation from (U, t) into (V, s). If $\varphi(a) = \varphi(b)$, then $\varphi\big(\overline{apr}_t(\{a\})\big) = \varphi\big(\overline{apr}_t(\{b\})\big)$.*

Proof. Let $a, b \in U$ with $\varphi(a) = \varphi(b)$ be given. Then Proposition 2.5 implies

$$\varphi\big(\overline{apr}_t(\{a\})\big) = \overline{apr}_s\big(\varphi(\{a\})\big) = \overline{apr}_s\big(\varphi(\{b\})\big) = \varphi\big(\overline{apr}_t(\{b\})\big).$$

□

Proposition 2.9. *Let (U, t) and (V, s) be two approximation spaces. For every bijection function $\varphi\colon U \to V$, the following statements are equivalent:*

(1) φ *is a left upper natural transformation from* (U, t) *into* (V, s).
(2) $\overline{apr}_s\big(\varphi(\{x\})\big) \subseteq \varphi\big(\overline{apr}_t(\{x\})\big)$ *for every* $x \in U$.
(3) $s\big(\varphi(x)\big) \subseteq \varphi\big(t(x)\big)$ *for every* $x \in U$.

Proof. (1) ⇒ (2). It is evident.

(2) ⇒ (3). By way of contradiction, assume that there is an element $u \in U$ such that $s\big(\varphi(u)\big) \nsubseteq \varphi\big(t(u)\big)$, which implies that there is an element $v \in V$ such that $v \in s\big(\varphi(u)\big) \setminus \varphi\big(t(u)\big)$, that is, $\varphi(u) \in \overline{apr}_s(\{v\})$. On the other hand, there is an element $u' \in U \setminus t(u)$ such that $v = \varphi(u')$, and this implies that $\varphi(u) \in \overline{apr}_s\big(\{\varphi(u')\}\big) \subseteq \varphi\big(\overline{apr}_t(\{u'\})\big)$, which implies that $u \in \overline{apr}_t(\{u'\})$, that is, $u' \in t(u)$, a contradiction.

(3) ⇒ (1). For every $B \subseteq U$,

$$\begin{aligned} a \in \overline{apr}_s\big(\varphi(B)\big) &\Rightarrow a = \varphi(x) \,\&\, \varphi(b) \in s\big(\varphi(x)\big) \subseteq \varphi\big(t(x)\big) \text{ for some } (b, x) \in B \times U \\ &\Rightarrow a = \varphi(x) \,\&\, b \in t(x) \text{ for some } (b, x) \in B \times U \\ &\Rightarrow a = \varphi(x) \text{ for some } x \in \overline{apr}_t(B) \\ &\Rightarrow a \in \varphi\big(\overline{apr}_t(B)\big). \end{aligned}$$

Therefore, φ is a left upper natural transformation from (U, t) into (V, s). □

The following two examples show that the bijections in Proposition 2.9, cannot be removed.

Example 2.10. Consider $U := \{x, y, z\}$ and $V := \{a, b, c, d\}$. Define $t\colon U \to \mathcal{P}^*(U)$ and $s\colon V \to \mathcal{P}^*(V)$ by

$$t := \begin{pmatrix} x & y & z \\ \{y, z\} & \{x, z\} & \{x\} \end{pmatrix} \text{ and } s := \begin{pmatrix} a & b & c & d \\ \{b, d\} & \{a, d\} & \{c\} & \{c\} \end{pmatrix}.$$

If $\varphi\colon U \to V$ is given by $\varphi := \begin{pmatrix} x & y & z \\ a & b & d \end{pmatrix}$, then it is clear that φ is a left upper natural transformation from (U, t) to (V, s), but $\varphi\big(t(z)\big) = \{a\} \neq \{c\} = s\big(\varphi(z)\big)$.

Example 2.11. Consider $U := \{w, x, y, z\}$ and $V := \{a, b\}$. Define $t\colon U \to \mathcal{P}^*(U)$ and $s\colon V \to \mathcal{P}^*(V)$ by

$$t := \begin{pmatrix} w & x & y & z \\ \{x, w\} & \{x, y, z\} & \{y, z\} & \{x\} \end{pmatrix} \text{ and } s := \begin{pmatrix} a & b \\ \{a\} & \{a, b\} \end{pmatrix}.$$

If $\varphi\colon U \to V$ is given by $\varphi := \begin{pmatrix} w & x & y & z \\ b & a & b & b \end{pmatrix}$, then

$$\overline{apr}_s\big(\varphi(\{w\})\big) = \{b\} = \varphi\big(\overline{apr}_t(\{w\})\big) \text{ and } \overline{apr}_s\big(\varphi(A)\big) \subseteq \{a, b\} = \varphi\big(\overline{apr}_t(A)\big)$$

for every $A \in \mathcal{P}^*(U) \setminus \{\{w\}\}$. Hence, φ is a left upper natural transformation from (U, t) to (V, s), but $s\big(\varphi(y)\big) = \{a, b\} \nsubseteq \{b\} = \varphi\big(t(y)\big)$.

Remark 2.12. Let (U, t) be an approximation space such that for every $X \subseteq \mathcal{P}^*(U)$, there exists a subset Y of U such that $X \subseteq t(Y)$. Then for every $X \in \mathcal{P}^*(U)$, there exists an element $y \in U$ such that $t(y) = A$, which implies that t is a surjective function, and so $|U| = |\mathcal{P}^*(U)|$, that is, $|U| = 1$.

It is well known that the pair (X, f) is called a closure space if $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a function such that the following conditions hold for every $M, N \in \mathcal{P}(X)$:

(1) $M \subseteq f(M)$;
(2) if $M \subseteq N$, then $f(M) \subseteq f(N)$;
(3) $f\big(f(M)\big) = f(M)$;
(4) $f(\emptyset) = \emptyset$.

If (X, f) is a closure space, then the subsets $M \in \mathcal{P}(X)$ such that $f(M) = M$ are called f-closed sets.

Let (X, f) and (Y, g) be closure spaces. A function $\varphi\colon X \to Y$ is said to be a closure preserving map if $\varphi\big(f(M)\big) \subseteq g\big(\varphi(M)\big)$ holds for every $M \in \mathcal{P}(X)$.

Recall from [12] that the category of all closure spaces and closure preserving mapping form a category denoted by $\underline{\mathrm{CS}}$.

Example 2.13. Consider $U := \{a, b, c, d\}$. Define $t\colon U \to \mathcal{P}^*(U)$ by

$$t = \begin{pmatrix} a & b & c & d \\ \{a, b\} & \{c\} & \{a, c\} & \{d\} \end{pmatrix}.$$

Since $\overline{apr}_t(\{a\}) = \{a, c\} \neq \{a, b, c\} = \overline{apr}_t\big(\overline{apr}_t(\{b\})\big)$, we conclude that $(U, \overline{apr}_t)$ is not a closure space.

Definition 2.14. A approximation space of (U, t) is called idempotent if $\overline{apr}_t = \overline{apr}_t^2$.

It is evident that idempotent approximation spaces and right upper natural transformations form a category, which is denoted by $\overline{\mathbf{RNT}}\mathbf{Apr}^2\mathbf{S}$, and also, idempotent approximation spaces and upper natural transformations form a category, which is denoted by $\overline{\mathbf{NT}}\mathbf{Apr}^2\mathbf{S}$.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and let B be an object of $\mathcal{B}$. Recall that a coreflection of B along F is a pair (R_B, ε_B), where

1. R_B is an object of $\mathcal{A}$ and $\varepsilon_B : F(R_B) \to B$ is a morphism of $\mathcal{B}$.
2. If A is an object of $\mathcal{A}$ and $b : F(A) \to B$ is a morphism of $\mathcal{B}$, then there exists a unique morphism $a : A \to R_B$ in $\mathcal{A}$ such that $\varepsilon_B \circ F(a) = b$. See the following diagram:

$$\begin{tikzcd} R_B & F(R_B) \arrow[r, "\varepsilon_B"] & B \\ A \arrow[u, "a"'] & F(A) \arrow[u, "F(a)"] \arrow[ur, "b"'] & \end{tikzcd}$$

Proposition 2.15. *Let* $F\colon \overline{\boldsymbol{RNTApr}^2\boldsymbol{S}} \to \underline{\mathrm{CS}}$ *be given by*

$$\begin{tikzcd} (U,t) \arrow[r, mapsto, "F"] \arrow[d, "f"'] & (U,\overline{apr}_t) \arrow[d, "F(f)=f"] \\ (V,s) \arrow[r, mapsto, "F"'] & (V,\overline{apr}_s) \end{tikzcd}$$

Then F is a full functor and the corefiection of (X,f) along F exists for every closure space (X,f).

Proof. It is evident that F is a full functor. Let (X,f) be a closure space. Define $t_f\colon X \to \mathcal{P}^*(X)$ by $t_f(x) = f(\{x\})$. We set $R_{(X,f)} := (X,t_f)$ and $\varepsilon_{(X,f)} := \mathrm{id}_X\colon X \to X$. It is evident that the pair $\left(R_{(X,f)}, \varepsilon_{(X,f)}\right)$ is a coreflection of (X,f) along F. $\square$

Remark 2.16. Define $F\colon \overline{\mathbf{NT}\mathbf{Apr}^2\mathbf{S}} \to \underline{\mathrm{CS}}$ by

$$\begin{tikzcd} (U,t) \arrow[r, mapsto, "F"] \arrow[d, "f"'] & (U,\overline{apr}_t) \arrow[d, "F(f)=f"] \\ (V,s) \arrow[r, mapsto, "F"'] & (V,\overline{apr}_s) \end{tikzcd}$$

Then F is a functor.

Consider $U := \{a,b,c\}$ and $V := \{x,y,z\}$. Define $t\colon U \to \mathcal{P}^*(U)$ and $s\colon V \to \mathcal{P}^*(V)$ by

$$t = \begin{pmatrix} a & b & c \\ \{a,b\} & \{b\} & \{c\} \end{pmatrix} \text{ and } s = \begin{pmatrix} x & y & z \\ \{x,y\} & \{x,y\} & \{z\} \end{pmatrix}.$$

It is evident that $\overline{apr}_t = \overline{apr}_t^2$ and $\overline{apr}_s = \overline{apr}_s^2$. If $\phi\colon U \to V$ is given by $\phi = \begin{pmatrix} a & b & c \\ x & x & z \end{pmatrix}$, then ϕ is a right upper natural transformation from (U,t) to (V,s). Since $\phi\big(\overline{apr}_t(\{b\})\big) = \{x\} \neq \{x,y\} = \overline{apr}_s\big(\phi(\{b\})\big)$, we conclude that ϕ is not an upper natural transformation from (U,t) to (V,s). Therefore, F is not a full functor.

Question 2.17. Is the category $\overline{\mathbf{NT}\mathbf{Apr}^2\mathbf{S}}$ a coreflection of the category $\underline{\mathrm{CS}}$?

3. Monomorphisms and epimorphisms

We recall from [1] that a morphism $\varphi\colon A \to B$ in a category $\mathfrak{D}$ is said to be

(1) a monomorphism provided that for all pairs $\psi,\phi\colon C \to A$ of morphisms in the category $\mathfrak{D}$ such that $\varphi\psi = \varphi\phi$, it follows that $\psi = \phi$.
(2) an epimorphism provided that for all pairs $\psi,\phi\colon C \to A$ of morphisms in the category $\mathfrak{D}$ such that $\psi\varphi = \phi\varphi$, it follows that $\psi = \phi$.

Although a function is an epimorphism in **Set** if and only if it is surjective and a function is a monomorphism in **Set** if and only if it is injective but it is not evident in $\overline{\mathbf{NTAprS}}$, $\overline{\mathbf{RNTAprS}}$, and $\overline{\mathbf{LNTAprS}}$ in general.

Example 3.1. Consider $U := \{a,b\}$ and $V := \{x\}$. Let $t\colon U \to \mathcal{P}^*(U)$ and $s\colon V \to \mathcal{P}^*(V)$ be given by $t := \begin{pmatrix} a & b \\ \{a\} & \{a\} \end{pmatrix}$ and $s := \begin{pmatrix} x \\ \{x\} \end{pmatrix}$. If $\varphi\colon U \to V$ is given by $\varphi := \begin{pmatrix} a & b \\ x & x \end{pmatrix}$, then φ is a right upper natural transformation. Let (W,r) be an approximation space, and assume that $\alpha,\beta\colon W \to U$ are two morphisms in the category $\overline{\mathbf{NT}}\mathbf{AprS}$ such that $\varphi\alpha = \varphi\beta$ and $\alpha \neq \beta$. Then there exists an element $w \in W$ such that $\alpha(w) \neq \beta(w)$. Hence we can assume that $\alpha(w) = a$ and $\beta(w) = b$. Therefore,

$$\begin{aligned}\{x\} &= \varphi\Big(\overline{apr}_t(\{\alpha(w)\})\Big)\\ &= \varphi\alpha\Big(\overline{apr}_r(\{w\})\Big)\\ &= \varphi\beta\Big(\overline{apr}_r(\{w\})\Big)\\ &= \varphi\Big(\overline{apr}_t(\{\beta(w)\})\Big)\\ &= \emptyset,\end{aligned}$$

and this is a contradiction. Hence α is a monomorphism in the category $\overline{\mathbf{NT}}\mathbf{AprS}$, but it is not an injective function.

In the following two propositions, under the stronger conditions, in he categories $\overline{\mathbf{NT}}\mathbf{AprS}$, $\overline{\mathbf{RNT}}\mathbf{AprS}$, and $\overline{\mathbf{LNT}}\mathbf{AprS}$, we show that a function is an epimorphism if and only if it is surjective and that a function is a monomorphism if and only if it is injective

Proposition 3.2. *Let (U, t) and (V, s) be approximation spaces and let $\varphi : U \to V$ be a function. Then the following statements hold:*

(1) *Let φ be an upper natural transformation from (U, t) into (V, s) with the following properties:*
  (a) $\overline{apr}_s(\{v\}) \neq \emptyset$ *for every* $v \in V$,
  (b) $v \in \varphi(U)$ *if and only if* $\overline{apr}_s(\{v\}) \subseteq \varphi(U)$, *and*
  (c) $v \in V \setminus \varphi(U)$ *if and only if* $\overline{apr}_s(\{v\}) \subseteq V \setminus \varphi(U)$.

  Then φ is an epimorphism in $\overline{\mathbf{NT}}\mathbf{AprS}$ if and only if $\varphi : U \to V$ is a surjective function.

(2) *If φ is a right upper natural transformation from (U, t) into (V, s) such that*

$$v \in \varphi(U) \Leftrightarrow \overline{apr}_s(\{v\}) \subseteq \varphi(U)$$

  for every $v \in V$, then φ is an epimorphism in $\overline{\mathbf{RNT}}\mathbf{AprS}$ if and only if $\varphi : U \to V$ is a surjective function.

(3) *If φ is a left upper natural transformation from (U, t) into (V, s) such that $\overline{apr}_s(A) \neq \emptyset$ and*

$$\overline{apr}_s(A) \subseteq \varphi(U) \Rightarrow A \subseteq \varphi(U)$$

  for every $\emptyset \neq A \subseteq V$, then φ is an epimorphism in $\overline{\mathbf{LNT}}\mathbf{AprS}$ if and only if $\varphi : U \to V$ is a surjective function.

Proof. (1). *Necessity.* We proceed by contradiction. Assume that $\varphi(U) \neq V$. Let $W = \{1, 2\}$ and let $r\colon W \to \mathcal{P}^*(W)$ be given by $r(1) = \{1\}$ and $r(2) = \{2\}$. Define $\alpha, \beta : V \to W$ by $\alpha(v) = 1$ and

$$\beta(v) = \begin{cases} 1 & \text{if } v \in \varphi(U), \\ 2 & \text{if } v \notin \varphi(U). \end{cases}$$

It is clear that $\alpha\Big(\overline{apr}_s(\{v\})\Big) = \{1\} = \overline{apr}_r\Big(\alpha(\{v\})\Big)$. for every $v \in V$. Since $\overline{apr}_s(\{v\}) \neq \emptyset$, we have

$$\beta\Big(\overline{apr}_s(\{v\})\Big) = \begin{cases} \{1\}, & \overline{apr}_s(\{v\}) \subseteq \varphi(U), \\ \{2\}, & \overline{apr}_s(\{v\}) \subseteq V \setminus \varphi(U), \\ \{1, 2\} & \text{otherwise,} \end{cases}$$

and

$$\overline{apr}_r\Big(\beta(\{v\})\Big) = \begin{cases} \{1\}, & v \in \varphi(U), \\ \{2\}, & \text{otherwise.} \end{cases}$$

Hence, by the hypothesis, α and β are upper natural transformations and $\alpha\varphi = \beta\varphi$. Since the upper natural transformation φ is an epimorphism, then $\alpha = \beta$, which is a contradiction.

Sufficiency. The proof is clear.

The proof of the rest of the statements is similar to the proof of the first statement. □

Proposition 3.3. *Let (U, t) and (V, s) be approximation spaces and let $\varphi : U \to V$ be a function. Then the following statements hold:*

(1) *Let φ be a right upper natural transformation from (U, t) into (V, s), and suppose that $x \in t(x)$ for every $x \in U$. Then φ is a monomorphism in $\overline{\textbf{RNT}}\textbf{AprS}$ if and only if $\varphi : U \to V$ is an injection function.*

(2) *Let φ be a left upper natural transformation from (U, t) into (V, s) such that the following statements hold:*
   (a) *$x \in t(x)$ and $t^2(x) = \{t(x)\}$ for every $x \in U$, and*
   (b) *$\overline{apr}_t(\varphi^{-1}(\varphi(x))) = \overline{apr}_t(\{x\})$ for every $x \in U$.*

   Then φ is a monomorphism in $\overline{\textbf{LNT}}\textbf{AprS}$ if and only if $\varphi : U \to V$ is an injection function.

(3) *let φ be an upper natural transformation from (U, t) into (V, s) such that the following statements hold:*
   (a) *$t(u) = \{u\}$ for every $u \in U$, and*
   (b) *$s(\varphi(u)) \cap \varphi(U) \neq \emptyset$ for every $u \in U$.*

   Then φ is a monomorphism in $\overline{\textbf{NT}}\textbf{AprS}$ if and only if $\varphi : U \to V$ is an injection function.

Proof. (1). *Necessity.* By way of contradiction, assume that there exist $a, b \in U$ such that $\varphi(a) = \varphi(b)$ with $a \neq b$. Let $W := \{a, b\}$ and let $r : W \to \mathcal{P}^*(W)$ be given by $r(a) = \{a\}$ and $r(b) = \{b\}$. Define $\alpha, \beta : W \to U$ by $\alpha := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, and $\beta := \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. One can see immediately that α and β are right upper natural transformations and $\varphi\alpha = \varphi\beta$. The hypothesis implies $\alpha = \beta$, which is a contradiction.

Sufficiency. The proof is clear.

(2). *Necessity.* We argue by contradiction. Assume that there exist $a, b \in U$ such that $\varphi(a) = \varphi(b)$ with $a \neq b$. Let $W = \overline{apr}_t(\{a\})$ and let $r : W \to \mathcal{P}^*(W)$ be given by $r(x) = W$. Let $h \in \prod_{x \in \varphi\left(W\right)} \varphi^{-1}(x)$ such that $h(x) \in \overline{apr}_t(\{b\})$ and $h(\varphi(a)) = b$. Define $\alpha, \beta : W \to U$ by $\alpha(x) = x$ and $\beta(x) = h\left(\varphi(x)\right)$ for every $x \in W$. Since $\alpha\left(\overline{apr}_r(A)\right) = W \subseteq \overline{apr}_t\left(\alpha(A)\right)$ for every $A \subseteq W$, we conclude that α is a left upper natural transformation. Let $y \in h\left(\varphi(W)\right)$ be given. Then there exists an element $z \in W$ such that $y = h\left(\varphi(z)\right) \in \varphi^{-1}\left(\varphi(z)\right)$, which implies that

$$y \in \overline{apr}_t(\{y\}) \subseteq \overline{apr}_t(\varphi^{-1}\left(\varphi(z)\right)) = \overline{apr}_t(\{z\}) = \overline{apr}_t(\{a\}) = W.$$

Hence $h\left(\varphi(W)\right) \subseteq W$. Let $\emptyset \neq A \subseteq W$ be given. Since

$$\overline{apr}_t\left(\beta(A)\right) \subseteq \overline{apr}_t(\varphi^{-1}\left(\varphi(A)\right)) = W,$$

we conclude that $\overline{apr}_t\left(\beta(A)\right) = W$. Therefore,

$$\beta\left(\overline{apr}_r(A)\right) = \beta(W) = h\left(\varphi(W)\right) \subseteq W = \overline{apr}_t\left(\beta(A)\right).$$

We infer that β is a left upper natural transformation. Also we have $\varphi\alpha = \varphi\beta$. The hypothesis implies $\alpha = \beta$, which is a contradiction.

Sufficiency. The proof is clear.

(3). *Necessity.* We proceed by contradiction. Assume that there exist $a, b \in U$ such that $\varphi(a) = \varphi(b)$ with $a \neq b$. Let $W := \varphi(U)$ and let $r\colon W \to \mathcal{P}^*(W)$ be given by $r(w) = s(w) \cap \varphi(U)$. Define $\alpha, \beta : W \to U$ by $\alpha(w) = a$ and $\beta(w) = b$. For every $x \in U$, we have

$$\overline{apr}_t\Big(\alpha\Big(\varphi(u)\Big)\Big) = \overline{apr}_t(a) = \{a\} = \alpha\Big(\overline{apr}_r(\{\varphi(u)\})\Big).$$

Hence α is an upper natural transformation, and similarly, β is an upper natural transformation. It is clear $\varphi\alpha = \varphi\beta$, but $\alpha \neq \beta$.

Sufficiency. The proof is clear.

□

4. Product and coproduct

In this section, we study the ***coproduct*** and ***product*** on approximation spaces. First, we see that $\overline{\textbf{RNTAprS}}$, $\overline{\textbf{LNTAprS}}$, and $\overline{\textbf{NTAprS}}$ have all coproducts and then we see that under the stronger conditions $\overline{\textbf{LNTAprS}}$ and $\overline{\textbf{NTAprS}}$ have all products.

Let $\mathbb{C}$ be a category and let $(A_\alpha)_{\alpha\in I}$ be a family of objects in $\mathbb{C}$. Then a ***coproduct*** of this family is an object A, denoted by $\coprod_{\alpha\in I} A_\alpha$, together with a family of morphisms $(\iota_\alpha : A_\alpha \to A)_{\alpha\in I}$, called **injections**, such that for each object C and family of morphisms $(f_\alpha : A_\alpha \to C)_{\alpha\in I}$, there exists a unique morphism $f : A \to C$ such that $f\iota_\alpha = f_\alpha$ for each $\alpha \in I$. Hence for every $\alpha \in I$, the following diagram is commutative.

$$\begin{tikzcd}
A \arrow[dr, dotted, "f"'] & A_\alpha \arrow[l, "\iota_\alpha"'] \arrow[d, "f_\alpha"] \\
 & C
\end{tikzcd}$$

Proposition 4.1. *The following statements hold:*

(1) $\overline{\textbf{\textit{RNTAprS}}}$ *has all coproducts.*

(2) $\overline{\textbf{\textit{LNTAprS}}}$ *has all coproducts.*

(3) $\overline{\textbf{\textit{NTAprS}}}$ *has all coproducts.*

Proof. (1). Let $\{(U_j, t_j)\}_{j\in J}$ be a family of approximation spaces and let $U = \bigcup_{j\in J} U_j \times \{j\}$. For every $x \in U$, there exists a unique element $j \in J$ such that $x \in U_j \times \{j\}$, which implies that $t\colon U \to \mathcal{P}^*(U)$ given by $t(u) = t_j(u) \times \{j\}$ for every $u \in U_j \times \{j\}$ is a function. For every $j \in J$, we define $\iota_j\colon U_j \to U$ by $\iota_j(x) = (x, j)$. Since ι_j is an upper natural transformation, then ι_j is a right upper natural transformation. We claim that (U, t) together with $\{\iota_j\}_{j\in J}$, is a coproduct of the family $\{(U_j, t_j)\}_{j\in J}$. Let φ_j be a right upper natural transformation from (U_j, t_j) into (W, k) for any $j \in J$. Then, $\varphi_j\Big(\overline{apr}_{t_j}(\{x\})\Big) \subseteq \overline{apr}_k\Big((\varphi_j\{x\})\Big)$ for every $x \in U_j$, and also by the universal property of coproduct in the sets category, there exists a unique map $\varphi : U \to W$ such that $\varphi\iota_j = \varphi_j$. It is sufficient to show that φ is a unique a right upper natural transformation from (U, t) into (W, k). Let $a \in U$ be given; then there exist $j \in J$ and $a_j \in U_j$ such that $a = \iota_j(a_j)$, which implies that

$$\begin{aligned}
\varphi\Big(\overline{apr}_t(\{a\})\Big) &= \varphi\Big(\overline{apr}_t(\{\iota_j(a_j)\})\Big) \\
&= \varphi\iota_j\Big(\overline{apr}_{t_j}(\{a_j\})\Big) \\
&= \varphi_j\Big(\overline{apr}_{t_j}(\{a_j\})\Big) \\
&\subseteq \overline{apr}_k\Big(\varphi_j(\{a_j\})\Big) \\
&= \overline{apr}_k\Big(\varphi\iota_j(\{a_j\})\Big) \\
&= \overline{apr}_k\Big(\varphi(\{a\})\Big).
\end{aligned}$$

It implies that φ is a right upper natural transformation. It is clear that φ is unique.

(2). Let $\{(U_j, t_j)\}_{j\in J}$ be a family of approximation spaces and let $U = \bigcup_{j\in J} U_j \times \{j\}$. For every $x \in U$, there exists a unique element $j \in J$ such that $x \in U_j \times \{j\}$, which implies that $t\colon U \to \mathcal{P}^*(U)$ given by $t(u, j) = t_j(u)\times\{j\}$ for every $(u, j) \in U_j \times \{j\}$ is a function. For every $j \in J$, we define $\iota_j\colon U_j \to U$ by $\iota_j(x) = (x, j)$. For every $A \subseteq U_j$, we have

$$\begin{aligned}(u, j) \in \overline{apr}_t(\iota_j(A)) &\Rightarrow (a, j) \in t\big((u, j)\big) = t_j(u) \times \{j\} \text{ for some } a \in A\\ &\Rightarrow u \in \overline{apr}_{t_j}(a) \subseteq apr_{t_j}(A) \text{ for some } a \in A\\ &\Rightarrow (u, j) \in \iota_j\big(apr_{t_j}(A)\big).\end{aligned}$$

Therefore ι_j is a left upper natural transformation for every $j \in J$. We claim that (U, t) together with $\{\iota_j\}_{j\in J}$, is a coproduct of the family $\{(U_j, t_j)\}_{j\in J}$. Let φ_j be a left upper natural transformation from (U_j, t_j) into (W, k) for any $j \in J$. Then, $\overline{apr}_k\big((\varphi_j\{x\})\big) \subseteq \varphi_j\big(\overline{apr}_{t_j}(\{x\})\big)$ for every $(x, j) \in U_j \times J$, and also by the universal property of coproduct in the sets category, there exists a unique map $\varphi : U \to W$ such that $\varphi\iota_j = \varphi_j$. It is sufficient to show that φ is a unique a left upper natural transformation from (U, t) into (W, k). Let $A \subseteq U$ be given. Hence we have

$$\begin{aligned}u \in \overline{apr}_k\big(\varphi(A)\big) &\Rightarrow \varphi(a, j) \in k(u) \quad \text{for some } (a, j) \in A\\ &\Rightarrow \varphi(a, j) = \varphi\big(\iota_j(a)\big) = \varphi_j(a) \in k(u) \quad \text{for some } (a, j) \in A\\ &\Rightarrow u \in \overline{apr}_k\big(\varphi_j(\{a\})\big) \subseteq \varphi_j\big(\overline{apr}_{t_j}(\{a\})\big) = \varphi\iota_j\big(\overline{apr}_{t_j}(\{a\})\big) \quad \text{for some } (a, j) \in A\\ &\Rightarrow u \in \varphi\big(\overline{apr}_{t_j}(\{a\}) \times \{j\}\big) = \varphi\big(\overline{apr}_t\big(\{(a, j)\}\big)\big) \subseteq \varphi\big(\overline{apr}_t(A)\big) \quad \text{for some } (a, j) \in A.\end{aligned}$$

It implies that φ is a left upper natural transformation. It is clear that φ is unique.

(3). Let $\{(U_j, t_j)\}_{j\in J}$ be a family of approximation spaces and let $U = \bigcup_{j\in J} U_j \times \{j\}$. For every $x \in U$, there exists a unique element $j \in J$ such that $x \in U_j \times \{j\}$, which implies that $t\colon U \to \mathcal{P}^*(U)$ given by $t(u) = t_j(u)\times\{j\}$ for every $u \in U_j \times \{j\}$ is a function. For every $j \in J$, we define $\iota_j\colon U_j \to U$ by $\iota_j(x) = (x, j)$. For every $x \in U_j$,we have

$$\iota_j\big(\overline{apr}_{t_j}(\{x\})\big) = \overline{apr}_{t_j}(\{x\}) \times \{j\} = \overline{apr}_t\big(\iota_j(\{x\})\big).$$

Therefore, by Proposition 2.5, ι_j is an upper natural transformation. We claim that (U, t) together with $\{\iota_j\}_{j\in J}$, is a coproduct of the family $\{(U_j, t_j)\}_{j\in J}$. Let φ_j be an upper natural transformation from (U_j, t_j) into (W, k) for any $j \in J$. Then, $\varphi_j\big(\overline{apr}_{t_j}(\{x\})\big) = \overline{apr}_t\big((\varphi_j\{x\})\big)$ for every $x \in U_j$, and also by the universal property of coproduct in the sets category, there exists a unique map $\varphi : U \to W$ such that $\varphi\iota_j = \varphi_j$. It is sufficient to show that φ is a unique upper natural transformation from (U, t) into (W, k). Let $a \in U$ be given; then there exist $j \in J$ and $a_j \in U_j$ such that $a = \iota_j(a_j)$. Hence we have

$$\begin{aligned}\varphi\big(\overline{apr}_t(\{a\})\big) &= \varphi\big(\overline{apr}_t(\{\iota_j(a_j)\})\big)\\ &= \varphi\iota_j\big(\overline{apr}_{t_j}(\{a_j\})\big)\\ &= \varphi_j\big(\overline{apr}_{t_j}(\{a_j\})\big)\\ &= \overline{apr}_k\big(\varphi_j(\{a_j\})\big)\\ &= \overline{apr}_k\big(\varphi\iota_j(\{a_j\})\big)\\ &= \overline{apr}_k\big(\varphi(\{a\})\big).\end{aligned}$$

It implies that φ is an upper natural transformation. It is clear that φ is unique. $\square$

Let $\mathbb{C}$ be a category and let $(A_\alpha)_{\alpha\in I}$ be a family of objects in $\mathbb{C}$. Then a ***product*** of this family is an object A, denoted by $\prod_{\alpha\in I} A_\alpha$, together with a family of morphisms $(p_\alpha : A \to A_\alpha)_{\alpha\in I}$, called **projections**, such that for each object C and family of morphisms $(f_\alpha : C \to A_\alpha)_{\alpha\in I}$, there exists a unique morphism $f : C \to A$ such that $p_\alpha f = f_\alpha$ for each $\alpha \in I$. Hence for every $\alpha \in I$, the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "p_\alpha"] & A_\alpha \\ & C \arrow[u, "f_\alpha"'] \arrow[ul, dotted, "f"] \end{tikzcd}$$

Proposition 4.2. *The category* $\overline{\textbf{\textit{RNTAprS}}}$ *has all products.*

Proof. Let $\{(U_j, t_j)\}_{j\in J}$ be a family of approximation spaces. Consider the set theoretic Cartesian product $U = \prod_{j\in J} U_j$ and the projection map $\pi_j\colon U \to U_j$, that is, $\pi_j((a_j)_{j\in J}) = a_j$ for every $(a_j)_{j\in J} \in U$. Define $t\colon U \to \mathcal{P}^*(U)$ by $t\Big((a_j)_{j\in J}\Big) = \prod_{j\in J} t_j(a_j)$. We claim that (U, t) together with $\{\pi_j\}_{j\in J}$ is a product of $\{(U_j, t_j)\}_{j\in J}$ in the category $\overline{\textbf{RNTAprS}}$. First we show that π_j is a right upper natural transformation from (U, t) into (U_j, t_j) for every $j \in J$. It is easy to see that

$$\begin{aligned}
\pi_j\Big(\overline{apr}_t\Big(\{(a_j)_{j\in J}\}\Big)\Big) &= \Big\{\pi_j\Big((u_j)_{j\in J}\Big)\colon (u_j)_{j\in J} \in \overline{apr}_t\Big((a_j)_{j\in J}\Big)\Big\} \\
&= \Big\{\pi_j\Big((u_j)_{j\in J}\Big)\colon u_j \in \overline{apr}_{t_j}(a_j) \text{ for every } j \in J\Big\} \\
&\subseteq \Big\{u \in U_j\colon u \in \overline{apr}_{t_j}(\{a_j\})\Big\} \\
&= \Big\{u \in U_j\colon u \in \overline{apr}_{t_j}\Big(\pi_j\Big(\{(a_j)_{j\in J}\}\Big)\Big)\Big\} \\
&= \overline{apr}_{t_j}\Big(\pi_j\Big(\{(a_j)_{j\in J}\}\Big)\Big)
\end{aligned}$$

for every $(a_j)_{j\in J} \in U$, which implies that π_j is a right upper natural transformation for every $j \in J$. Now let $\varphi_j\colon V \to U_j$ be a right upper natural transformation from (V, s) into (U_j, t_j) for every $j \in J$. Then $\varphi_j\Big(\overline{apr}_s(\{x\})\Big) \subseteq \overline{apr}_{t_j}\Big(\varphi_j(\{x\})\Big)$ for every $(x, j) \in V \times J$. Define $\varphi\colon V \to U$ by $\varphi(x) = \Big(\varphi_j(x)\Big)_{j\in J}$. We show that φ is a right upper natural transformation from (V, s) into (U, t). Let $x \in V$ and let $y \in \varphi\Big(\overline{apr}_s(\{x\})\Big)$ be given. Then $y = \varphi(a)$ for some $a \in \overline{apr}_s(\{x\})$, which implies that

$$y = \Big(\varphi_j(a)\Big)_{j\in J} \in \prod_{j\in J} \varphi_j\Big(\overline{apr}_s(\{x\})\Big) \subseteq \prod_{j\in J} \overline{apr}_{t_j}\Big(\varphi_j(\{x\})\Big) = \overline{apr}_t\Big(\Big(\varphi_j(\{x\})\Big)_{j\in J}\Big)$$

for some $a \in \overline{apr}_s(\{x\})$. Thus, φ is a right upper natural transformation. The rest is evident. □

Remark 4.3. Let $\{(U_j, t_j)\}_{j\in J}$ be a family of approximation spaces. Consider the set theoretic Cartesian product $U = \prod_{j\in J} U_j$ and the projection map $\pi_j\colon U \to U_j$, that is, $\pi_j((a_j)_{j\in J}) = a_j$. Define $t\colon U \to \mathcal{P}^*(U)$ by $t\Big((a_j)_{j\in J}\Big) = \prod_{j\in J} t_j(a_j)$.

(1) Suppose that $U_j = \bigcup_{x\in U_j} t_j(x)$ for every $j \in J$. Let $A \in \mathcal{P}(U)$ and $z \in \overline{apr}_{t_j}\Big(\pi_j(A)\Big)$ be given. Then there exists an element $(a_j)_{j\in J} \in A$ such that $a_j = \pi_j\Big((a_j)_{j\in J}\Big) \in t_j(z)$, and also, by our hypothesis, there exists an element $z_i \in U_i$ such that $a_i \in t_i(z_i)$ for every $i \in J \setminus \{j\}$. We put $u_j := z$ and $u_i := z_i$ for every $i \in J \setminus \{j\}$, then $u := (u_j)_{j\in J} \in U$, which implies that

$$(a_j)_{j\in J} \in \prod_{j\in J} t_j(u_j) = t\Big((u_j)_{j\in J}\Big),$$

and so

$$z = u_j = \pi_j\Big((u_j)_{j\in J}\Big) \in \pi_j\Big(\overline{apr}_t(A)\Big).$$

Hence, π_j is a left upper natural transformation from (U, t) into (U_j, t_j) for every $j \in J$.

(2) Let $k \in J$ such that $U_k \neq \bigcup_{x\in U_k} t_k(x)$, then there exists an element $u \in U_k \setminus \bigcup_{x\in U_k} t_k(x)$. If $(a_j)_{j\in J} \in U$ such that $a_k = u$ and $\overline{apr}_{t_j}(\{a_j\}) \neq \emptyset$ for some $k \neq j \in J$, then

$$\overline{apr}_{t_j}\Big(\pi_j(\{(a_j)_{j\in J}\})\Big) = \overline{apr}_{t_j}(\{a_j\}) \neq \emptyset = \pi_j\Big(\overline{apr}_t\Big(\{(a_j)_{j\in J}\}\Big)\Big).$$

Therefore, if there exists an element $k \in J$ such that $U_k \neq \bigcup_{x\in U_k} t_k(x)$, then π_j is not a left upper transformation for every $k \neq j \in J$.

(3) Let $\overline{apr}_{t_j}(\{x\}) \neq \emptyset$ for every $(j, x) \in J \times U_j$. If there exists an element j in J such that $U_j \neq \bigcup_{x\in U_j} t_j(x)$ with $u \in U_j \setminus \bigcup_{x\in U_j} t_j(x)$, then there exists an element $z \in \overline{apr}_{t_j}(\{u\})$, which follows that $u \in t_j(z) \subseteq \bigcup_{x\in U_j} t_j(x)$, a contradiction. Hence, $U_j = \bigcup_{x\in U_j} t_j(x)$ for every $j \in J$. Therefore, π_j is a left upper natural transformation from (U, t) into (U_j, t_j) for every $j \in J$.

(4) Let $(u_j)_{j\in J} \in U$ with $\overline{apr}_t\Big(\{(u_j)_{j\in J}\}\Big) \neq \emptyset$ be given. By Proposition 4.5, we have

$$\emptyset \neq \pi_j\Big(\overline{apr}_t\Big(\{(u_j)_{j\in J}\}\Big)\Big) \subseteq \overline{apr}_{t_j}\Big(\pi_j(\{(u_j)_{j\in J}\})\Big) = \overline{apr}_{t_j}(u_j)$$

for every $j \in J$. Hence, the following statements are equivalent:
(1) $\overline{apr}_t(\{u\}) \neq \emptyset$ for every $u \in U$.
(2) $\overline{apr}_{t_j}(\{x\}) \neq \emptyset$ for every $(j, x) \in J \times U_j$.

Example 4.4. Consider $U_1 := \{a_1, b_1\}$, $U_2 := \{a_2, b_2\}$, and $V := \{v, v'\}$. Let (U_1, t_1), (U_2, t_2), and (V, s) be approximation spaces such that

$$t_1 := \begin{pmatrix} a_1 & b_1 \\ \{a_1, b_1\} & \{b_1\} \end{pmatrix},\ t_2 := \begin{pmatrix} a_2 & b_2 \\ \{a_2\} & \{b_2\} \end{pmatrix},\ \text{and}\ s := \begin{pmatrix} v & v' \\ \{v, v'\} & \{v'\} \end{pmatrix}.$$

We define $\varphi_i \colon V \to U_i$ by $\varphi_i := \begin{pmatrix} v & v' \\ a_i & b_i \end{pmatrix}$ for every $i \in \{1, 2\}$. One can easily see that φ_i is a left upper natural transformation from (V, s) to (U_i, t_i). Consider $U := U_1 \times U_2$, and define $t \colon U \to \mathcal{P}^*(U)$ by $t(x, y) = t_1(x) \times t_2(y)$. If (U, t) together with $\{\pi_i\}_{i\in\{1,2\}}$ is a product of the family $\{(U_i, t_i)\}_{i\in\{1,2\}}$, then there exists a left upper natural transformation φ from (V, s) to (U, t) such that $\pi_1\varphi = \varphi_1$ and $\pi_2\varphi = \varphi_2$, which implies that $\varphi = \begin{pmatrix} v & v' \\ (a_1, a_2) & (b_1, b_2) \end{pmatrix}$. On the other hand, we have

$$\begin{aligned} \overline{apr}_t\Big(\varphi(\{v'\})\Big) &= \overline{apr}_t\Big(\{(b_1, b_2)\}\Big) \\ &= \{(a_1, b_2), (b_1, b_2)\} \\ &\nsubseteq \{(a_1, a_2), (b_1, b_2)\} \\ &= \varphi\Big(\{v, v'\}\Big) \\ &= \varphi\Big(\overline{apr}_s(v')\Big), \end{aligned}$$

which is a contradiction. Hence, the family $\{(U_i, t_i)\}_{i\in\{1,2\}}$ does not have product in the category $\overline{\textbf{LNTAprS}}$.

In the following proposition under the same conditions, we show that $\overline{\textbf{LNTAprS}}$ and $\overline{\textbf{NTAprS}}$ have all products.

Proposition 4.5. *Let $\{(U_j, t_j)\}_{j\in J}$ be a family of approximation spaces such that $t_j\colon U_j \to \mathcal{P}^*(U_j)$ is given by $t_j(x) = \{x\}$ for every $(j, x) \in J \times U_j$. Then the following statements hold:*

(1) *$\{(U_j, t_j)\}_{j\in J}$ has the product in the category $\overline{\boldsymbol{LNT}}\boldsymbol{AprS}$.*
(2) *$\{(U_j, t_j)\}_{j\in J}$ has the product in the category $\overline{\boldsymbol{NT}}\boldsymbol{AprS}$.*

Proof. Consider the set theoretic Cartesian product $U = \prod_{j\in J} U_j$ and the projection map $\pi_j\colon U \to U_j$. Define $t\colon U \to \mathcal{P}^*(U)$ by $t\Big((a_j)_{j\in J}\Big) = \prod_{j\in J} t_j(a_j)$.

(1). By Remark 4.3, π_j is a left upper natural transformation from (U, t) into (U_j, t_j) for every $j \in J$. Now let $\varphi_j\colon V \to U_j$ be a left upper natural transformation from (V, s) into (U_j, t_j) for every $j \in J$. Then $\overline{apr}_{t_j}\Big(\varphi_j(A)\Big) \subseteq \varphi_j\Big(\overline{apr}_s(A)\Big)$ for every $A \subseteq V$ and every $j \in J$. Define $\varphi\colon V \to U$ by $\varphi(x) = \Big(\varphi_j(x)\Big)_{j\in J}$. We show that φ is a left upper natural transformation from (V, s) into (U, t). Let $A \subseteq V$ be given. Then

$$\begin{aligned}
(u_j)_{j\in J} \in \overline{apr}_t\Big(\varphi(A)\Big) &\Rightarrow \varphi(a) \in t\Big(\{(u_j)_{j\in J}\}\Big) \quad \text{for some } a \in A\\
&\Rightarrow \ \text{for every } j \in J, u_j = \varphi_j(\{a\}) \quad \text{for some } a \in A\\
&\Rightarrow (u_j)_{j\in J} \in \varphi\Big(\overline{apr}_s(\{a\})\Big) \quad \text{for some } a \in A\\
&\Rightarrow (u_j)_{j\in J} \in \varphi\Big(\overline{apr}_s(A)\Big).
\end{aligned}$$

Thus, φ is a left upper natural transformation. The rest is evident.

(2). First, we show that π_j is an upper natural transformation from (U, t) into (U_j, t_j) for every $j \in J$. Suppose that $(a_j)_{j\in J} \in U$. It is easy to see that

$$\pi_j\Big(\overline{apr}_t(\{(a_j)_{j\in J}\})\Big) = \pi_j(\{(a_j)_{j\in J}\}) = \{a_j\} = \overline{apr}_{t_j}(\{a_j\}) = \overline{apr}_{t_j}\Big(\pi_j((a_j)_{j\in J})\Big).$$

Hence, by Proposition 2.5, π_j is an upper natural transformation for every $j \in J$. Now let $\varphi_j\colon V \to U_j$, for any $j \in J$, be an upper natural transformation from (V, s) into (U_j, t_j) for $j \in J$. Then $\varphi_j\Big(\overline{apr}_s(\{x\})\Big) = \overline{apr}_{t_j}\Big(\varphi_j(\{x\})\Big) = \{\varphi_j(x)\}$ for every $x \in V$. Define $\varphi\colon V \to U$ by $\varphi(x) = (\varphi_j(x))_{j\in J}$. We show that φ is an upper natural transformation from (V, s) into (U, t). To see this, let $x \in V$. Then

$$\begin{aligned}
\varphi\Big(\overline{apr}_s(\{x\})\Big) &= \{\varphi(y)\colon y \in \overline{apr}_s(\{x\})\}\\
&= \{(\varphi_j(y))_{j\in J}\colon y \in \overline{apr}_s(\{x\})\}\\
&= \{(\varphi_j(x))_{j\in J}\}\\
&= \{\varphi(x)\}\\
&= \overline{apr}_t\Big(\varphi(x)\Big).
\end{aligned}$$

Thus, by Proposition 2.5, φ is an upper natural transformation. It is clear that $\pi_j\varphi = \varphi_j$ for every $j \in J$. Now, we prove that φ with this property is unique. Let ψ be an upper natural transformation from (V, s) into (U, t) such that $\pi_j\psi = \varphi_j$. Then, by the universal property of product in the sets category, $\psi = \varphi$.
□

5. Equalizer and coequalizer of a pair of morphisms

Let $\mathbb{C}$ be a category and let $f, g : A \to B$ be a pair of morphisms in $\mathbb{C}$. We recall from [1] that an object E, also denoted by $eq(f, g)$, together with a morphism $e : E \to A$ is called an ***equalizer*** of f and g if $f \circ e = g \circ e$ and for every morphism $h : C \to A$ with $f \circ h = g \circ h$, there exists a unique morphism $\bar{h} : C \to E$ such that $e \circ \bar{h} = h$.

$$\begin{tikzcd}
C \arrow[d, dotted, "\bar{h}"'] \arrow[dr, "h"] & & \\
E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

Example 5.1. Consider $U := \{1, 2, 3\}$ and the map $t\colon U \to \mathcal{P}^*(U)$ given by $t = \begin{pmatrix} 1 & 2 & 3 \\ \{1\} & \{2\} & \{3\} \end{pmatrix}$. If $\varphi, \phi\colon U \to U$ are given by

$$\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } \phi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

then φ and ϕ are upper natural transformations from (U, t) to (U, t). Let (E, r) together with an upper natural transformation $\psi : E \to U$ be an equalizer of φ and ϕ in $\overline{\mathbf{NT}}\mathbf{AprS}$. Then $\varphi\psi = \phi\psi$, which implies that

$$\psi(E) \subseteq \{\, x \in U\colon \varphi(x) = \phi(x) \,\} = \emptyset,$$

but this contradicts with $\psi(E) \neq \emptyset$. Therefore, φ and ϕ do not have equalizer.

Proposition 5.2. *Let (U, t) and (V, s) be approximation spaces and let $\varphi, \phi\colon U \to V$ be two functions. We set*

$$E := \{\, x \in U\colon \varphi(x) = \phi(x) \,\},$$

and define $r\colon E \to \mathcal{P}^*(E)$ by $r(x) = t(x) \cap E$. Suppose that the following statements hold:

(1) *$E \neq \emptyset$,*
(2) *$t(x) \cap E \neq \emptyset$ for every $x \in E$, and*
(3) *$t(x) \cap E \neq \emptyset$ if and only if $t(x) \subseteq E$ for every $x \in U$.*

Then the following statements hold:

(1) *If $\varphi, \phi\colon U \to V$ are two right upper natural transformations from (U, t) to (V, s), then $\psi\colon E \to U$ given by $\psi(x) = x$ is a right upper natural transformation from (E, r) to (U, t), and (E, r) together with ψ is an equalizer of φ and ϕ in $\overline{\boldsymbol{RNT}}\boldsymbol{AprS}$.*

(2) *If $\varphi, \phi\colon U \to V$ are two left upper natural transformations from (U, t) to (V, s) and suppose that $x \in t(x)$ for every $x \in U$, then $\psi\colon E \to U$ given by $\psi(x) = x$ is a left upper natural transformation from (E, r) to (U, t), and (E, r) together with ψ is an equalizer of φ and ϕ in $\overline{\boldsymbol{LNT}}\boldsymbol{AprS}$.*

(3) *If $\varphi, \phi\colon U \to V$ are two upper natural transformations from (U, t) to (V, s) and $x \in t(x)$ for every $x \in U$, then $\psi\colon E \to U$ given by $\psi(x) = x$ is an upper natural transformation from (E, r) to (U, t), and (E, r) together with ψ is an equalizer of φ and ϕ in $\overline{\boldsymbol{NT}}\boldsymbol{AprS}$.*

Proof. (1). At first, we prove that ψ is a right upper natural transformation. In order to approach this goal, let us assume $x \in E$. In view of Proposition 2.6 and

$$\psi\Big(\overline{apr}_r(\{x\})\Big) = \overline{apr}_r(\{x\}) \subseteq \overline{apr}_t(\{x\}) = \overline{apr}_t\Big(\psi(x)\Big),$$

we infer that ψ is a right upper natural transformation from (E, r) to (U, t). Let $\rho\colon W \to U$ be a right upper natural transformation from (W, q) to (U, t) such that $\varphi\rho = \phi\rho$, which implies that $\rho(W) \subseteq E$. We define $\bar{\rho}\colon W \to E$ by $\bar{\rho}(x) = \rho(x)$. It is clear that $\bar{\rho}$ is a unique right upper natural transformation from (W, q) to (E, r) such that the following diagram is commutative.

$$\begin{tikzcd}
W \arrow[d, dotted, "\bar{\rho}"'] \arrow[dr, "\rho"] & & \\
E \arrow[r, "\psi"'] & U \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\varphi"'] & V
\end{tikzcd}$$

(2). Let $A \subseteq E$ be given. Then

$$\overline{apr}_t\Big(\psi(\{A\})\Big) = \overline{apr}_t\Big(\{A\}\Big) = \overline{apr}_r\Big(\{A\}\Big) = \psi\Big(\overline{apr}_r(\{A\})\Big).$$

We infer that ψ is a left upper natural transformation from (E, r) to (U, t). Let $\rho\colon W \to U$ be a left upper natural transformation from (W, q) to (U, t) such that $\varphi\rho = \phi\rho$. We define $\bar{\rho}\colon W \to E$ by $\bar{\rho}(x) = \rho(x)$. It is clear that $\bar{\rho}$ is a unique left upper natural transformation from (W, q) to (E, r) such that the following diagram is commutative:

$$\begin{tikzcd} W \arrow[d, dotted, "\bar{\rho}"'] \arrow[dr, "\rho"] & & \\ E \arrow[r, "\psi"'] & U \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\varphi"'] & V \end{tikzcd}$$

(3). Let us assume $x \in E$. In view of Proposition 2.5, at first, we prove ψ is an upper natural transformation. In order to approach this goal, let us assume $x \in E$. In view of Proposition 2.5 and

$$\psi\Big(\overline{apr}_r(\{x\})\Big) = \overline{apr}_r(\{x\}) = \overline{apr}_t(\{x\}) = \overline{apr}_t\Big(\psi(x)\Big),$$

we infer that the map ψ is an upper natural transformation from (E, r) to (U, t). Let $\rho\colon W \to U$ be an upper natural transformation from (W, q) to (U, t) such that $\varphi\rho = \phi\rho$. We define $\bar{\rho}\colon W \to E$ by $\bar{\rho}(x) = \rho(x)$. It is clear that $\bar{\rho}$ is a unique upper natural transformation from (W, q) to (E, r) such that the following diagram is commutative:

$$\begin{tikzcd} W \arrow[d, dotted, "\bar{\rho}"'] \arrow[dr, "\rho"] & & \\ E \arrow[r, "\psi"'] & U \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\varphi"'] & V \end{tikzcd}$$

□

Proposition 5.3. *Let $\varphi, \phi\colon U \to V$ be two upper natural transformations from (U, t) to (V, s). We set*

$$E := \{\, x \in U\colon \varphi(x) = \phi(x) \,\}.$$

If an approximation space of (W, r) together with an upper natural transformation $\psi\colon W \to U$ is an equalizer of φ and ϕ in $\overline{\textbf{NTAprS}}$, then the following statements hold:

(1) *$E \neq \emptyset$.*

(2) *$\varphi\Big(\overline{apr}_t(\{x\})\Big) = \phi\Big(\overline{apr}_t(\{x\})\Big)$ for every $x \in E$.*

(3) *If $r(x) = \{x\}$ and $t\Big(\psi(x)\Big) \cap \psi(W) \neq \emptyset$ for every $x \in W$, then ψ is an injection function.*

Proof. Let the approximation space of (W, r) together with an upper natural transformation $\psi\colon W \to U$ be an equalizer of φ and ϕ.

(1). Since $\psi(W) \subseteq E$, we infer that $E \neq \emptyset$.

(2). Let $x \in E$ be given. Then,

$$\varphi\Big(\overline{apr}_t(\{x\})\Big) = \overline{apr}_s\Big(\{\varphi(x)\}\Big) = \overline{apr}_s\Big(\{\phi(x)\}\Big) = \phi\Big(\overline{apr}_t(\{x\})\Big).$$

(3). By Proposition 3.3, ψ is a injection function, since ϕ in $\overline{\textbf{NTAprS}}$ is monomorphism. □

Given two morphisms $f, g : B \to C$, their ***coequalizer*** is an ordered pair (Z, e) with $ef = eg$ that is universal with the property that if $p : C \to X$ satisfies $pf = pg$, then there exists a unique $\bar{p} : Z \to X$ with $p'e = p$.

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \arrow[r, "e"] \arrow[dr, "p"'] & Z \arrow[d, dotted, "\bar{p}"] \\ & & X \end{tikzcd}$$

Proposition 5.4. *Let $\varphi, \phi \colon U \to V$ be two upper natural transformations from (U, t) to (V, s). If an approximation space of (W, r) together with an upper natural transformation $\psi \colon V \to W$ is a coequalizer of φ and ϕ in $\overline{\mathbf{NT}}\mathbf{AprS}$, then the following statements hold:*

(1) $\psi\Big(\overline{apr}_s(\{\varphi(x)\})\Big) = \psi\Big(\overline{apr}_s(\{\phi(x)\})\Big)$ *for every $x \in U$.*

(2) *If $\overline{apr}_r(A) \neq \emptyset$ and*

$$\overline{apr}_r(A) \subseteq \psi(U) \Rightarrow A \subseteq \psi(U)$$

for every $\emptyset \neq A \subseteq V$, then ψ is a surjective function.

Proof. (1). Let $x \in U$ be given. Then,

$$\psi\Big(\overline{apr}_s(\{\varphi(x)\})\Big) = \psi\Big(\varphi\Big(\overline{apr}_s(\{x\})\Big)\Big) = \psi\Big(\phi\Big(\overline{apr}_s(\{x\})\Big)\Big) = \psi\Big(\overline{apr}_s(\{\phi(x)\})\Big).$$

(2). By Proposition 3.2, ψ is a surjective function, since ϕ in $\overline{\mathbf{NT}}\mathbf{AprS}$ is an epimorphism.
□

Proposition 5.5. *Let $\varphi, \phi \colon U \to V$ be two upper natural transformations from (U, t) to (V, s) such that $\varphi\Big(\overline{apr}_t(\{u\})\Big) = \{\varphi(u)\}$ for every $u \in U$, and assume that φ is an onto function. We set*

$$C := \Big\{\Big(\varphi(x), \phi(x)\Big) \colon x \in U\Big\}.$$

Let θ be the equivalence relation on V generated by C (the least equivalence relation on V containing C), and suppose that $W := V/\theta$. Define $r \colon W \to \mathcal{P}^*(W)$ by $r([x]_\theta) = \{\,[x]_\theta\,\}$. If $\psi \colon V \to W$ is given by $\psi(v) = [v]_\theta$, then ψ is an upper natural transformation from (V, s) to (W, r), and (W, r) together with ψ is a coequalizer of φ and ϕ in $\overline{\mathbf{NT}}\mathbf{AprS}$.

Proof. It is clear that ψ is a function. Then

$$\psi\Big(\varphi(u)\Big) = [\varphi(u)]_\theta = [\phi(u)]_\theta = \psi\Big(\phi(u)\Big),$$

and

$$\psi\Big(\overline{apr}_s(\{\varphi(u)\})\Big) = \psi\Big(\varphi\Big(\overline{apr}_t(\{u\})\Big)\Big) = \{[\varphi(u)]_\theta\} = \overline{apr}_r\Big(\psi(\varphi(u))\Big).$$

Hence, ψ is an upper natural transformation from (V, s) to (W, r) such that $\psi\varphi = \psi\phi$. Let α be an upper natural transformation from (V, s) to (W', r') such that $\alpha\varphi = \alpha\phi$. We define $\alpha' \colon W \to W'$ by $\alpha'([v]_\theta) = \alpha(v)$. Then for every $u \in U$,

$$\begin{aligned}
\alpha'\Big(\overline{apr}_r\Big(\Big\{[\varphi(u)]_\theta\Big\}\Big)\Big) &= \{\alpha(\varphi(u))\} \\
&= \alpha(\{\varphi(u)\}) \\
&= \alpha\Big(\varphi\Big(\overline{apr}_t(\{u\})\Big)\Big) \\
&= \alpha\Big(\overline{apr}_s\Big(\varphi(\{u\})\Big)\Big) \\
&= \overline{apr}_{r'}\Big(\alpha\Big(\varphi(\{u\})\Big)\Big) \\
&= \overline{apr}_{r'}\Big(\alpha'\Big(\Big\{[\varphi(u)]_\theta\Big\}\Big)\Big).
\end{aligned}$$

Therefore, α' is an upper natural transformation from (W, r) to (W', r') such that $\alpha'\psi = \alpha$, that is, the following diagram is commutative:

$$\begin{tikzcd}
U \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\phi"'] & V \arrow[r, "\psi"] \arrow[dr, "\alpha"'] & W \arrow[d, dotted, "\alpha'"] \\
 & & W'
\end{tikzcd}$$

It is clear that α' is a unique a right upper natural transformation from (W, r) to (W', r') such that $\alpha'\psi = \alpha$. Therefore, (W, r) together with ψ is a coequalizer of φ and ϕ in $\overline{\mathbf{NTAprS}}$. $\square$

Example 5.6. Consider $U := \{a, b, c\}$, $V := \{x, y, z\}$, and $U := \{a, b\}$. Define $t\colon U \to \mathcal{P}^*(U)$, $s\colon V \to \mathcal{P}^*(V)$, and $r\colon W \to \mathcal{P}^*(W)$ by

$$t = \begin{pmatrix} a & b & c \\ \{a, b\} & \{b\} & \{c\} \end{pmatrix}, \; s = \begin{pmatrix} x & y & z \\ \{x, y\} & \{x, y\} & \{z\} \end{pmatrix} \text{ and } r = \begin{pmatrix} a & b \\ \{a\} & \{b\} \end{pmatrix}.$$

If $\varphi, \phi\colon U \to V$ are given by

$$\varphi = \begin{pmatrix} a & b & c \\ y & x & z \end{pmatrix} \text{ and } \phi = \begin{pmatrix} a & b & c \\ x & x & z \end{pmatrix},$$

then φ and ϕ are right upper natural transformations from (U, t) to (V, s). If $\psi\colon W \to U$ is given by $\psi = \begin{pmatrix} a & b \\ c & b \end{pmatrix}$, then ψ is a right upper natural transformations from (W, r) to (U, t). Let $h\colon W' \to U$ be a right upper natural transformation from (W', r') to (U, t) such that $\varphi h = \phi h$. We define $\overline{h}\colon W' \to W$ by $\overline{h} = \psi^{-1}h$. It is clear that ψ is an equalizer of φ and ϕ, but $\varphi\big(\overline{apr}_t(b)\big) \neq \phi\big(\overline{apr}_t(b)\big)$.

6. Applications and advantage

The rough set methodology has found many image processing, real-life recognition, and others. The proposed method has many important advantages. Some of them are listed below.

- Provides efficient algorithms for finding hidden patterns in data.
- Finds minimal sets of data (data reduction).
- Evaluates significance of data.
- Generates minimal sets of decision rules from data.
- Easy to understand and offers a straightforward interpretation of results.

The method is particularly suited for parallel processing but in order to fully exploit this feature, a new hardware solutions are necessary.

In applied mathematics, we encounter many examples of mathematical objects that can be multiplied with each other. First of all, the real numbers themselves are such objects. Other examples are real-valued functions, the complex numbers, matrices, infinite series, vectors in n-dimensional spaces, and vector-valued functions. On the other hand, rough sets are originally proposed in the presence of an equivalence relation. An equivalence relation is sometimes difficult to be obtained in real-world problems due to the vagueness and incompleteness of human knowledge.

Example 6.1. In Table 1, there are six objects with three properties. It reflects on the experience of Q sales staff to determine the quality of the goods and the existence of the train station near the store. The class label is called the decision attribute. The information system that contains the decision attribute is called the decision system. In Table 1, the pf (profitability) is the decision attribute. In other words, their decision making characteristics are different. According to the table, we can introduce the set-valued mapping as follows: Let $U = \{1, 2, 3, 4, 5, 6\}$ and let $W = U$. Consider the set-valued mapping $t\colon U \to \mathcal{P}^*(W)$, such that t represents all objects and has similar properties, defined by $t := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \{1\} & \{2, 3\} & \{2, 3\} & \{4\} & \{5\} & \{6\} \end{pmatrix}$. Let $X = \{1, 3, 6\}$, (stores have profitability), we have the following properties: $\overline{apr_t}(X) = \{1, 2, 3, 6\}$ and $\underline{apr_t}(X) = \{1, 6\}$. This means stores 1 and 6 are profitable and stores 1,2,3,6 are likely to profit. However, stores 2,3 cannot be assigned under the mapping t.

store	E	Q	L	pf
1	High	Good	No	profit
2	medium	Good	No	loss
3	medium	Average	No	profit
4	Low	Average	No	loss
5	medium	Average	Yes	loss
6	High	Average	Yes	profit

Table 1: Numerical Results.

7. Conclusion

Sciences have a natural tendency toward diversification and specialization. In particular, contemporary rough sets consist of many different branches and are intimately related to various other fields. Each of these branches and fields is growing rapidly and is itself diversifying. Fortunately, however, there is a considerable amount of common ground similar ideas, concepts, and constructions. These provide a basis for a general theory of structures. The equivalence relation, however, seems to be a very stringent condition that may limit the application domain of the rough set model. From both theoretic and practical needs, many authors have generalized the notion of approximation operators by using nonequivalence binary relations.

The purpose of this paper was to present the fundamental concepts and results of such a theory, expressed in the language of category theory. This paper created three categories $\overline{\textbf{NT}}\textbf{AprS}$, $\overline{\textbf{RNT}}\textbf{AprS}$, and $\overline{\textbf{LNT}}\textbf{AprS}$.

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