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Exterior Square Graph of a Finite Group

M. Zameni[a], P. Niroomand[b], M. Alizadeh Sanati[a], M. Parvizi[c]

[a]*Department of Mathematics, Faculty of Sciences, Golestan University,Gorgan, Iran*
[b]*School of Mathematics and Computer Science, Damghan University, Damghan, Iran*
[c]*Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran*

Abstract. In this paper, we define the exterior square graph $\Gamma_G^{E^\wedge}$ which is a graph associated to a non-cyclic finite group with the vertex set $G \setminus Z^\wedge(G)$, where $Z^\wedge(G)$ denotes the exterior center of G, and two vertices x and y are joined whenever $x \wedge y = 1$, where $\wedge$ denotes the operator of non-abelian exterior square. We investigate how the group structure can be affected by completeness, regularity and bipartition of this graph.

1. Introduction

Niroomand et al. in [4] assigned the non-exterior square graph of finite group $\Gamma_G^\wedge$ to an arbitrary non-cyclic group G by the vertex set $G \setminus Z^\wedge(G)$ and two vertices x and y join whenever $x \wedge y \neq 1$. We are going to consider the complement of this graph in this paper, which is called the exterior graph of the group G. So the vertex set of this graph which is denoted by $\Gamma_G^{E^\wedge}$ is $G \setminus Z^\wedge(G)$ and two distinct vertices x and y are adjacent whenever $x \wedge y = 1$. At first, we need to recall the concept of non-abelian exterior square of G.

The non-abelian exterior square $G \wedge G$ of a group G is the group generated by the symbols $a \wedge b$ subject to the relations

$$ab \wedge c = ({}^a b \wedge {}^a c)(a \wedge c), \quad a \wedge bc = (a \wedge b)({}^b a \wedge {}^b c), \quad a \wedge a = 1_{G \wedge G}$$

for all $a, b, c \in G$, where ${}^a b = aba^{-1}$. This construction was introduced by Brown and Loday in [4]. It is known that there exists a group homomorphism $\overline{\kappa} : G \wedge G \to G'$ sending $a \wedge b$ to $[a, b]$ such that the $\ker \overline{\kappa}$ is isomorphic to $\mathcal{M}(G)$, the Schur multiplier of the group G. The reader can find more details on the Schur multiplier in [4]. Recall that a group G is called capable if $G \cong \frac{E}{Z(E)}$, for some group E. It was proved by Ellis in [4] that G is capable if and only if the exterior center subgroup, namely

$$Z^\wedge(G) = \{a \in G | \ a \wedge x = 1 \ , \text{for all} \ \ x \in G\}$$

is trivial. It is clear that $Z^\wedge(G) = \bigcap_{x \in G} C_G^\wedge(x)$, in which $C_G^\wedge(x) = \{a \in G | a \wedge x = 1\}$ is the exterior centralizer of an element x.

On the base of [4], we consider simple graphs which are undirected, with no loops or multiple edges. For any graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. We use the following notations and terminology in the rest, which can be found in [4].

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Email addresses: m.zameni@gu.ac.ir, m.zameni66@yahoo.com (M. Zameni), niroomand@du.ac.ir, p_niroomand@yahoo.com (P. Niroomand), m.alizadeh@gu.ac.ir (M. Alizadeh Sanati), parvizi@math.um.ac.ir (M. Parvizi)

- The **null graph** is the graph which has no vertices.
- The vertex that has no edges is called the **single vertex**.
- A **complete** graph is a graph in which each pair of distinct vertices is connected by an edge. The complete graph with n vertices is denoted by K_n.
- For each natural number n, the **edgeless graph** (or empty graph) $\overline{K}_n$ of order n is the graph with n vertices and zero edges.
- The **degree** $d_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $d_\Gamma(v)$ simply by $d(v)$. The order of Γ is $|V(\Gamma)|$ and its maximum and minimum degrees will be denoted by $\Delta(\Gamma)$ and $\delta(\Gamma)$, respectively.
- A graph Γ is **regular** if the degrees of all vertices of Γ are the same. A regular graph with vertices of degree k is called a $k-$regular graph.
- A **planar** graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident.
- Let X be a subset of $V(\Gamma)$. Then the **induced subgraph** $\Gamma[X]$ is the graph whose vertex set is X and whose edge set consists of all of the edges in $E(\Gamma)$ that have both endpoint in X.
- A subest X of vertices of Γ is called a **clique**, if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $w(\Gamma)$.
- A subest X of vertices of Γ is called an **independent set**, if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and denoted by $\alpha(\Gamma)$.
- A **vertex cover** of a graph is a subset X of $V(\Gamma)$ such that every edge of the graph is incident to at least one vertex in X. The covering number $\beta(\Gamma)$ is the number of vertices in a smallest vertex cover for Γ.
- The length of a cycle is defined the number of its edges. The length of the shortest cycle in a graph Γ is called **girth** of Γ and denoted by $girth(\Gamma)$. If Γ has no cycle we define the girth of Γ to be infinite. A **Hamailton** cycle of Γ is a cycle that contains every vertex of Γ.
- If v and u are vertices in Γ, then $d(u,v)$ denotes the length of the shortest path between v and u. If there is no path connecting u and v we define $d(u,v)$ to be infinite. The largest distance between all pairs of the vertices of Γ is called the **diameter** of Γ, and is denoted by $diam(\Gamma)$.
- A **dominating set** for a graph is a subset D of $V(\Gamma)$ such that every vertex which does not belong to D joins to at least one number of D by some edges. The domination number $\gamma(\Gamma)$ is the number of vertices in the smallest dominating set for Γ.
- The **chormatic number** a graph Γ is the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color and denoted by $\chi(\Gamma)$.
- The **edge chormatic number** of a graph Γ is the smallest number of colors necessary to color each edge of such that no two edges incident on the same vertex have the same color and denoted by $\chi'(\Gamma)$.

2. Exterior square graph

Throughout this section, G is a finite group. We state some of basic graph theoretical properties of $\Gamma_G^{E^\wedge}$, such as independence number, regularity and domination number. Moreover, we give its effect on the group theoretical properties of G.

According to the definition, it is obvious that $deg(v) = |C_G^\wedge(v)| - |Z^\wedge(G)| - 1$ for every $v \in V(\Gamma_G^{E^\wedge})$. Clearly $\Gamma_G^{E^\wedge}$ is precisely the null graph if and only if G is cyclic. In the following lemma, we list some elementary properties of this graph.

Lemma 2.1. *Let G be a finite group then:*
(i) $\Gamma_G^{E^\wedge}$ *is the empty graph if and only if G is an elementary abelian 2-group.*
(ii) *If v is a single vertex of* $\Gamma_G^{E^\wedge}$*, then the order of v is* 2 *and G is capable group.*
(iii) *The exterior square graph of G is not complete.*
(iv) $\gamma(\Gamma_G^{E^\wedge}) \geqslant 2$.

Proof. (i) According to [4, *Theorem* 2.5] and the fact $\Gamma_G^{E^\wedge}$ is the complement of $\Gamma_G^\wedge$, the result follows.
(ii) Since $deg(v) = 0$, we have $|C_G^\wedge(v)| = |Z^\wedge(G)| + 1$. Hence $Z^\wedge(G) = 1$ and $o(v) = 2$.
(iii) Let $\Gamma_G^{E^\wedge}$ be a complete graph. We have $deg(v) = |G| - |Z^\wedge(G)| - 1$ for every $v \in G \setminus Z^\wedge(G)$. Hence $C_G^\wedge(v) = Z^\wedge(G)$, which implies that $v \in Z^\wedge(G)$, a contradiction.
(iv) If $\{x\}$ is a dominating set for $\Gamma_G^{E^\wedge}$, then $deg(x) = |G| - |Z^\wedge(G)| - 1$. Hence $|C_G^\wedge(x)| = |G|$, which implies that $x \in Z^\wedge(G)$, a contradiction. □

According to [4, *Example* 3.3], we give some results on $\Gamma_G^{E^\wedge}$ when G is an elementary abelian p-group, for odd prime p.
The subgroup generated by an element x of G is denoted by $\langle x\rangle$.

Lemma 2.2. *Let G be an elementary abelian p-group of rank n. Then* $|Z^\wedge(G)| = 1$ *and* $|C^\wedge(x)| = |\langle x\rangle| = p$ *for every* $x \in G \setminus \{1\}$.

By using Lemma 2.2, the exterior square graph associated to an elementary abelian p-group G is partitioned into $p^{n-1} + p^{n-2} + p^{n-3} + \cdots + p + 1$ of complete graphs each is K_{p-1} of order $p-1$, that is $\Gamma_G^{E^\wedge} = \underbrace{K_{p-1} \cup K_{p-1} \cup \ldots \cup K_{p-1}}_{(p^{n-1}+p^{n-2}+\cdots+p+1)-times}$.

Corollary 2.3. *Let G be an elementary abelian p-group.* $\Gamma_G^{E^\wedge}$ *is planer if and only if* $p = 2, 3$ *or* 5.

For the regularity of this graph we have the following lemma.

Lemma 2.4. *The following conditions are equivalent.*
(i) $\Gamma_G^{E^\wedge}$ *is regular.*
(ii) $|C_G^\wedge(x)| = |C_G^\wedge(y)|$*, for every* $x, y \in G \setminus Z^\wedge(G)$.
(iii) $\Gamma_G^\wedge$ *is regular.*

Proof. Straightforward. □

According to [4, *Theorem* 2.6, 2.7] and Lemma 2.4, we have the following lemma.

Lemma 2.5. (i) *Let G be an abelian p-group. Then* $\Gamma_G^{E^\wedge}$ *is a regular graph if and only if* $G = C_{p^k} \oplus C_p^{(n)}$*, in which* $k \geqslant 1, n \geqslant 0$.
(ii) *Let* $G = \prod_{i=1}^{k} G_i$ *in which* G_i *have coprime orders. Then* $\Gamma_G^{E^\wedge}$ *is regular if and only if* $\Gamma_{G_i}^{E^\wedge}$ *is regular for each i,* $1 \leq i \leq k$.

The proof of the following lemma is similar to the proof of [4, Theorem 2.14] and [4, Corollary 2.15].

Lemma 2.6. (*i*) *Let G and H be two non-cyclic groups with $\Gamma_G^{E^\wedge} \cong \Gamma_H^{E^\wedge}$ and $|V(\Gamma_G^{E^\wedge})|$ is a prime number. Then $|G| = |H|$.*
(*ii*) *Let G be a non-cyclic group and $\Gamma_G^{E^\wedge} \cong \Gamma_{S_3}^{E^\wedge}$. Then $G \cong S_3$.*
(*iii*) *Let G be a non-cyclic group. If $\Gamma_G^{E^\wedge} \cong \Gamma_H^{E^\wedge}$, then H is also non-cyclic and $|Z^\wedge(H)|$ divides*

$$(|G| - |Z^\wedge(G)|, |C_G^\wedge(x)| - |Z^\wedge(G)|, |G| - |C_G^\wedge(x)|),$$

for every $x \in G \setminus Z^\wedge(G)$.
(*iv*) *Let G be a dihedral group of order $2m$. If $\Gamma_G^{E^\wedge} \cong \Gamma_H^{E^\wedge}$ for some group H, then $|G| = |H|$.*

Now we are going to state some relations between $\Gamma_G^{E^\wedge}$ and $d^\wedge(G)$. We recall that the concept of commutativity degree $d(G)$ and the exterior degree $d^\wedge(G)$ were defined by the following ratios, respectively.

$$d(G) = \frac{|\{(x,y)\in G\times G:[x,y]=1\}|}{|G|^2},\ d^\wedge(G) = \frac{|\{(x,y)\in G\times G:x\wedge y=1\}|}{|G|^2}.$$

It is clear that $d^\wedge(G) \leqslant d(G)$. Let $C = \{(x, y) \in G \times G : x \wedge y = 1\}$, then the number of edges of the exterior square graph of G is

$$E(\Gamma_G^{E^\wedge}) = |C| - 2|Z^\wedge(G)|(|G| - 1) - |G|.$$

Since $\Gamma_G^{E^\wedge}$ is not complete, we give an upper bound for $d^\wedge(G)$ in the following lemma.

Lemma 2.7. *Let G be a finite group. Then we have*

$$d^\wedge(G) < \frac{1}{2} + \frac{|Z^\wedge(G)|}{|G|} + \frac{|Z^\wedge(G)|^2}{2|G|^2} - \frac{3|Z^\wedge(G)|}{2|G|^2} + \frac{1}{|G|}.$$

Lemma 2.8. *There is no group with $\Gamma_G^{E^\wedge}$ a star graph.*

Proof. Let $\Gamma_G^{E^\wedge}$ be a star graph. Then there exists a vertex $v \in G \setminus Z^\wedge(G)$ such that $deg(v) = |G| - |Z^\wedge(G)| - 1$. Then $|C_G^\wedge(v)| = |G|$, which implies that $v \in Z^\wedge(G)$, a contradiction. □

3. On exterior graph of groups with trivial Schur multiplier

In this secion, we give some graph theoretical properties such as girth and planarity on $\Gamma_G^{E^\wedge}$ when $\mathcal{M}(G) = 0$. We have the following lemma.

Lemma 3.1. *Let G be a finite group with trivial Schur multiplier. Then*
(*i*) $G \wedge G \cong G'$.
(*ii*) $[x, y] = 1$ *if and only if* $x \wedge y = 1$ *for all* $x, y \in G$.
(*iii*) $Z^\wedge(G) = Z(G)$.

Proof. Straightforward. □

If $\mathcal{M}(G) = 0$, then $V(\Gamma_G^{E^\wedge}) = G \setminus Z(G)$ and two vertices x and y are joined whenever $[x, y] = 1$. In this case, $\Gamma_G^{E^\wedge}$ is just the commuting graph which is defined in [4].

Consider the generalized quarternion group $Q_{4n} = \langle a, b : a^n = b^2, a^{2n} = b^4 = 1, ba = a^{-1}b\rangle$ for some integer $n \geqslant 2$. It ius known that $\mathcal{M}(Q_{4n}) = 0$. Clearly, the center of Q_{4n} is $\{1, a^n\}$ and $C_{Q_{4n}}(x) = \langle x\rangle$ for every $x \in Q_{4n} \setminus \langle a\rangle$. On the other hand, we know $deg(x) = |C_{Q_{4n}}(x)| - |Z(Q_{4n})| - 1$ for every $x \in Q_{4n} \setminus Z(Q_{4n})$. Thus $deg(a^i) = 2n - 3$ and $deg(x) = 1$ for $1 \leqslant i \leqslant 2n, i \neq n$ and $x \in G \setminus \langle a\rangle$. Therefore $\Gamma_{Q_{4n}}^{E^\wedge}$ is partitioned into n copies of K_2 and a K_{2n-2} that is, $\Gamma_{Q_{4n}}^{E^\wedge} = \underbrace{K_2 \cup K_2 \cup \ldots \cup K_2}_{n-times} \cup K_{2n-2}$.

Lemma 3.2. *For the generalized quaternion group we have:*
(*i*) *girth*$(\Gamma^{E^\wedge}_{Q_{4n}}) = 3$, *for each* $n > 2$.
(*ii*) *diam*$(\Gamma^{E^\wedge}_{Q_{4n}}) = \infty$.
(*iii*) $\Gamma^{E^\wedge}_{Q_{4n}}$ *is regular if and only if* $n = 2$.
(*iv*) $\Gamma^{E^\wedge}_{Q_{4n}}$ *is planar if and only if* $n = 2$ *or* 3.
(*v*) $\beta(\Gamma^{E^\wedge}_{Q_{4n}}) = \gamma(\Gamma^{E^\wedge}_{Q_4}) = n + 1$.
(*vi*) $\chi(\Gamma^{E^\wedge}_{Q_{4n}}) = 2n - 2$.

Proof. Straightforward. □

Lemma 3.3. *Let G be a non-cyclic group and* $\Gamma^{E^\wedge}_G \cong \Gamma^{E^\wedge}_{Q_{4n}}$. *Then* $|G| = 4n$.

Proof. It is enough to show that $|Z^\wedge(G)| = 2$. We know that Q_{4n} contains non-central elements x, y such that $deg(x) = 1$ and $deg(y) = 2n - 3$. Since $\Gamma^{E^\wedge}_G \cong \Gamma^{E^\wedge}_{Q_{4n}}$, there exist $g, g' \in G \setminus Z^\wedge(G)$ such that $deg(g) = 1$ and $deg(g') = 2n - 3$. On the other hand, $deg(g) = |C_{Q_{4n}}(x)| - |Z(G)| - 1 = 1$ and $|Z^\wedge(G)|$ divides $|C^\wedge_G(g)|$. Thus $|Z^\wedge(G)| = 1$ or 2. If $|Z^\wedge(G)| = 1$, then $|C^\wedge_G(g')| = 2n - 1$ and $|G| = 4n - 1$, which implies that $|C^\wedge_G(g')|$ does not divide $|G|$, a contradiction. □

Kumar Das et al. in [4, *Section*5] characterize all finite non-abelian groups whose commuting graphs are planar. We state the proof of planarity on $\Gamma^{E^\wedge}_G$ when G has trivial Schur multiplier, in a simple way.

Theorem 3.4. *Let G be non-capable non-abelian group with trivial Schur multiplier. Then* $\Gamma^{E^\wedge}_G$ *is planar if and only if G is isomorphic to one of the groups* $Q_8, Q_{12}, SL(2,3)$ *or* $M_{16} = \langle a, b : a^2 = 1, aba = b^{-3}\rangle$.

Proof. The planarity of $\Gamma^{E^\wedge}_G$ for the mentioned groups is obtained easily. Conversely Let $\Gamma^{E^\wedge}_G$ is planar. Since the complete graph of order 5 is not planar, we have $\omega(\Gamma^{E^\wedge}_G) < 5$. Now we claim that if $\Gamma^{E^\wedge}_G$ is planer, then $|Z(G)| < 5$. By contradiction, let $|Z(G)| \geqslant 5$. Put $T_x = Z(G)x$ where x is an arbitrary element of $Z(G)$. The induced subgraph $\Gamma^{E^\wedge}_{T_x}$ of $\Gamma^{E^\wedge}_G$ by T_x is a planar graph. But $\Gamma^{E^\wedge}_{T_x}$ contains K_5, which is a contradiction. Hence $|Z(G)| < 5$. We claim that $\pi(G) \subseteq \{2, 3, 5\}$, in which $\pi(X)$ is the set of all prime divisors of the order of a group X. Let $p \in \pi(G)$ and $p \geq 7$ then there exist an element $x \in G$ of order p so $\Gamma^{E^\wedge}_G$ contains K_5.

Therefore we have $2 \leqslant |Z(G)| \leqslant 4$ and $\pi(G) \subseteq \{2, 3, 5\}$. Now we consider the following cases.

Case 1. If $5 \in \pi(G)$, then there exists $x \in G$ such that $o(x) = 5$. The induced subgraph $\Gamma^{E^\wedge}_{T_x}$ of $\Gamma^{E^\wedge}_G$ by $T_x = \{x, x^2, x^3, x^4, xz\}$ for $z \in Z(G) \setminus \{1\}$ is a planar graph. But $\Gamma^{E^\wedge}_{T_x}$ contains K_5. So we must have $\pi(G) \subseteq \{2, 3\}$.
Case 2. If $\pi(G) = \{2, 3\}$, then we have three subcases.

Subcase 1. Let $|Z(G)| = 3$. If there exists element $x \in G \setminus Z(G)$ such that $o(x) > 6$ or $o(x) = 3$ or 4, then $\Gamma^{E^\wedge}_G$ is not planar. If $o(x) = 6$ such that $Z(G) \subset \langle x\rangle$, then $|C_G(x)| = 6$. By using class equation, $|G| \leqslant 18$. There is no group G such that $|G| \leq 18$, $|Z(G)| = 3$ and $\mathcal{M}(G) = 1$.

Subcase 2. Let $|Z(G)| = 2$. If there exists element $x \in G \setminus Z(G)$ such that $o(x) > 6$ or $o(x) = 3$, then $\Gamma^{E^\wedge}_G$ is not planar. If $o(x) = 6$ and $o(y) = 4$ such that $Z(G) \subset \langle x\rangle$, then $|C_G(x)| = 6$ and $|C_G(y)| = 4$ for all $x, y \in G \setminus Z(G)$. By using class equation and GAP, groups whose $|Z(G)| = 2$ and $\mathcal{M}(G) = 1$ are Q_{12} and $SL(2, 3)$. Hence $\Gamma^{E^\wedge}_{Q_{12}}$ and $\Gamma^{E^\wedge}_{SL(2,3)}$ are planar.

Subcase 3. $|Z(G)| = 4$. There exists element $x \in G \setminus Z(G)$ such that $o(x) = 3$, then the induced subgraph $\Gamma^{E^\wedge}_{T_x}$ of $\Gamma^{E^\wedge}_G$ by $T_x = \{x, x^2, xz_1, xz_2, xz_3\}$ for $z_i \in Z(G) \setminus \{1\}$, $1 \leqslant i \leqslant 3$ is a planar graph. But $\Gamma^{E^\wedge}_{T_x}$ contains K_5 and so $\Gamma^{E^\wedge}_G$ is not planar.
Case 3. If $\pi(G) = \{3\}$ or equivalently G be a 3-group, then $|Z(G)| = 3$ and $o(x) = 3^n$ for every $x \in G \setminus Z(G)$, for some $n \geqslant 1$. Put $T_x = Z(G)x \cup Z(G)x^{-1}$. The induced subgraph $\Gamma^{E^\wedge}_{T_x}$ of $\Gamma^{E^\wedge}_G$ by T_x contains K_5. Therefore $\Gamma^{E^\wedge}_G$ is not planar.
Case 4. If $\pi(G) = \{2\}$ or equivalently G be a 2-group, then $|Z(G)| = 2$ or 4. Here we have three subcases.

Subcase 1. Let $Z(G) \cong C_2 \times C_2$. There exists $x \in G \setminus Z(G)$ such that $o(x) \geqslant 2^n, n \geqslant 2$. Put $T_x = Z(G)x \cup Z(G)x^{-1}$. The induced subgraph $\Gamma^{E^\wedge}_{T_x}$ of $\Gamma^{E^\wedge}_G$ by T_x is a planar graph. But $\Gamma^{E^\wedge}_{T_x}$ contains K_5. So $\Gamma^{E^\wedge}_G$ is not planar.

Subcase 2. $Z(G) \cong C_4$. If there exists element $x \in G \setminus Z(G)$ such that $o(x) \geqslant 16$, then the induced subgraph $\Gamma_{T_x}^{E^\wedge}$ of $\Gamma_G^{E^\wedge}$ by $T_x = \langle x \rangle \setminus Z(G)$ is a planar graph. But $\Gamma_{T_x}^{E^\wedge}$ contains K_5. Therefore $\Gamma_G^{E^\wedge}$ is not planar. If $o(x) = 4$ or 8 for every $x \in G \setminus Z(G)$, then by using class equation, $|G| \leqslant 32$. By using GAP, the only group G whose $\mathcal{M}(G) = 1$, $Z(G) \cong C_4$ and $|G| \leqslant 32$ is M_{16}.

Subcases 3. Let $Z(G) \cong C_2$. If there exists element $x \in G \setminus Z(G)$ such that $o(x) \geqslant 8$, then the induced subgraph $\Gamma_{T_x}^{E^\wedge}$ of $\Gamma_G^{E^\wedge}$ by $T_x = \langle x \rangle \setminus Z(G)$ is a planar graph. But $\Gamma_{T_x}^{E^\wedge}$ contains K_5 and so $\Gamma_G^{E^\wedge}$ is not planar. If $o(x) = 4$ or 2, for every $x \in G \setminus Z(G)$, then by using class equation, $|G| \leqslant 8$. By using GAP, the only group G whose $\mathcal{M}(G) = 1$, $Z(G) \cong C_2$ and $|G| \leqslant 8$ is Q_8. □

The following lemma shows that all non-abelian groups except Q_8 with trivial Schur multiplier satisfy $girth(\Gamma_G^{E^\wedge}) = 3$.

Lemma 3.5. *Let G be a non-abelian group such that $\mathcal{M}(G) = 1$ and $G \ncong Q_8$. Then $girth(\Gamma_G^{E^\wedge}) = 3$.*

Proof. If $\Gamma_G^{E^\wedge}$ is not planar, then $\Gamma_G^{E^\wedge}$ contains K_5 and $girth(\Gamma_G^{E^\wedge}) = 3$. Hence, we assume that $\Gamma_G^{E^\wedge}$ is planar graph.
First, if $|Z(G)| \neq 1$, then according to Lemma 3.4, we have three cases.
Case 1. Let $G \cong Q_{12}$. Since the center of $Q_{12} = \langle a, b : a^3 = b^2, ba = a^5 b \rangle$ is $\{1, a^3\}$, we have a cycle $\{a, a^2, a^4\}$ in $\Gamma_{Q_{12}}^{E^\wedge}$.
Case 2. Let $G \cong SL(2,3)$, we know that $SL(2,3) = \langle a, b : a^3 = b^3 = (ab)^2 \rangle$ and $Z(G) = \{1, a^3\}$. Therefore $\{a, a^2, a^4\}$ is a cycle in $\Gamma_{SL(2,3)}^{E^\wedge}$.
Case 3. Let $G \cong \langle a, b : bab = a^3, b^2 = 1 \rangle$. Therefore $\{a, a^2, a^5\}$ is a cycle in $\Gamma_G^{E^\wedge}$.
Now let $Z(G) = 1$. In this case, we have two possibilities.
Case 1. If there exists element $x \in G$ such that $o(x) \geqslant 4$, then $\{x, x^2, x^3\}$ is a cycle.
Case 2. If there exists element $x \in G$ such that $o(x) = 2$ or 3, by using class equation, there is no group with this condition. Hence the result follows. □

4. Some properties on $\Gamma_G^{E^\wedge}$ when G is CE-group

In this section, we give some results on $\Gamma_G^{E^\wedge}$ when G is CE-group.

Definition 4.1. *A group G is called an exterior CE-group provided that $C_G^\wedge(x)$ is cyclic for every $x \in G$.*

We recall [4, Lemma 2.20], which is an essential tool in the next.

Lemma 4.2. *The following conditions are equivalent for any group G*
(*i*) *G is CE-group.*
(*ii*) *If $x \wedge y = 1$, then $C_G^\wedge(x) = C_G^\wedge(y)$, for all $x, y \in G \setminus Z^\wedge(G)$.*
(*iii*) *If $x \wedge y = x \wedge z = 1$, then $y \wedge z = 1$, for all $x, y, z \in G \setminus Z^\wedge(G)$.*

We give some elementary properties of $\Gamma_G^{E^\wedge}$ when G is an exterior CE-group.

Lemma 4.3. *Let G be non-cyclic CE-group. Then we have:*
(*i*) *$\Gamma_G^{E^\wedge}$ is partitioned into at least two complete graphs.*
(*ii*) *$\Gamma_G^{E^\wedge}$ is disconnected.*
(*iii*) *$diam(\Gamma_G^{E^\wedge}) = \infty$.*
(*iv*) *$\Gamma_G^{E^\wedge}$ is not Hamiltonian graph.*
(*v*) *If $\Delta(\Gamma_G^{E^\wedge}) = 2n$, then $\chi'(\Gamma_G^{E^\wedge}) = 2n + 1$.*
(*vi*) *If $\Delta(\Gamma_G^{E^\wedge}) = 2n - 1$, then $\chi'(\Gamma_G^{E^\wedge}) = 2n - 1$.*
(*vi*) *$\Delta(\Gamma_G^{E^\wedge}) + 1 = \chi(\Gamma_G^{E^\wedge})$.*

Proof. It follows from Lemma 4.2 and [4, *Exercise* 6.2.1] directly. □

In the following lemmas, we give some properties on a group G when $\Gamma_G^{E^\wedge}$ is a $(p-1)$-regular graph or planar.

Lemma 4.4. *Let G be a non-cyclic CE-group. Then we have*

(i) $\Gamma_G^{E^\wedge}$ *is 1-regular if and only if one of the following cases holds: Case 1. G is an elementary abelian 3-group. Case 2. $|Z^\wedge(G)| = 2$ and $o(x) = 4$ for every $x \in G \setminus Z^\wedge(G)$.*

(ii) $\Gamma_G^{E^\wedge}$ *is $(p-1)$-regular if and only if one of the following cases hold. Case 1. $|Z^\wedge(G)| = 1$ and $\Gamma_G^{E^\wedge}$ is partitioned into the induced subgraph $\Gamma_G^{E^\wedge}[\langle x_i\rangle]$ such that $o(x_i) = p+1$ for every $x_i \in G \setminus \{1\}$. Case 2. $|Z^\wedge(G)| = p$ and $o(x) = 2$ or $2p$ for every $x \in G \setminus Z^\wedge(G)$.*

Proof. (*i*) Let $\Gamma_G^{E^\wedge}$ be 1-regular. We know $deg(x) = |C_G^\wedge(x)| - |Z^\wedge(G)| - 1 = 1$, for every $x \in G \setminus Z^\wedge(G)$ and $|Z^\wedge(G)|$ divides $|C_G^\wedge(x)|$. Then $|Z^\wedge(G)| = 1$ or 2.
If $|Z^\wedge(G)| = 1$, then $|C_G^\wedge(x)| = 3$ for every $x \in G \setminus Z^\wedge(G)$. So, G is an elementary abelian 3−group.
If $|Z^\wedge(G)| = 2$, then $C_G^\wedge(x) \cong C_4$ for every $x \in G \setminus Z^\wedge(G)$ and $o(x) = 4$.
Conversely, if G is an elementary abelian 3-group, then $Z^\wedge(G) = 1$, $o(x) = 3$ for every $x \in G \setminus Z^\wedge(G)$. So $|C_G^\wedge(x)| = 3$ and $\Gamma_G^{E^\wedge}$ is 1-regular.
If $|Z^\wedge(G)| = 2$ and $o(x) = 4$ for every $x \in G \setminus Z^\wedge(G)$, then $C_G^\wedge(x) \cong C_4$, since G is non-cyclic CE-group. Then $deg(x) = 1$ for every $x \in G \setminus Z^\wedge(G)$.
(*ii*) Let $\Gamma_G^{E^\wedge}$ be $(p-1)$-regular. Then $Z^\wedge(G) = 1$ or p. We consider two following cases.
Case 1. $|Z^\wedge(G)| = 1$, then $|C_G^\wedge(x)| = p+1$ and $C_G^\wedge(x) \cong C_{p+1}$, for every $x \in G \setminus Z^\wedge(G)$. By using Definition 4.1, the order of x divides $p+1$.
Case 2. $|Z^\wedge(G)| = p$, then $|C_G^\wedge(x)| = 2p$ and $C_G^\wedge(x) \cong C_{2p}$, for every $x \in G \setminus Z^\wedge(G)$. By using Definition 4.1, $o(x) = 2$ or $2p$.
Conversely, if $|Z^\wedge(G)| = 1$ and $\Gamma_G^{E^\wedge}$ is partitioned into $\Gamma_G^{E^\wedge}[\langle x_i\rangle]$ such that $o(x_i) = p+1$ for every $x_i \in G \setminus \{1\}$. We have $C_G^\wedge(x) \cong C_{|C_G^\wedge(x)|}$, then $|C_G^\wedge(x)| = p+1$. Otherwise if $|C_G^\wedge(x)| \neq p+1$, then there exists $x \in G \setminus \{1\}$ such that $o(x) > p+1$. It is a contradiction.
If $|Z^\wedge(G)| = p$ and $o(x) = 2$ *or* $2p$ for every $x \in G \setminus Z^\wedge(G)$, then $|C_G^\wedge(x)| = 2p$. If $|C_G^\wedge(x)| > 2p$, then there exists element $y \in G \setminus Z^\wedge(G)$ such that $o(y) > 2p$. It is a contradiction. □

Lemma 4.5. *Let G be non-cyclic CE-group. $\Gamma_G^{E^\wedge}$ is planar if and only if one of the following cases holds.*
(*i*) *G is a 2-group such that $Z^\wedge(G) \cong C_4$ and $o(x) = 8$ for every $x \in G \setminus Z^\wedge(G)$.*
(*ii*) *$\Gamma_G^{E^\wedge}$ is partitioned into the induced subgraph $\Gamma_G^{E^\wedge}[\langle x_i\rangle]$ such that $o(x_i) = 6$ and $Z^\wedge(G) \cong C_3$ for some $x_i \in G \setminus Z^\wedge(G)$.*
(*iii*) *$|G| = 2^n \times 3^m$, where $m, n \geqslant 0$, $Z^\wedge(G) \cong C_2$ and $o(x) \leqslant 6$ for every $x \in G \setminus Z^\wedge(G)$.*
(*iv*) *$|G| = 2^n \times 3^m \times 5^r$, where $m, n, r \geqslant 0$, $|Z^\wedge(G)| = 1$, $o(x) \leqslant 5$ and if $o(x) \neq o(y)$, $x \notin \langle y\rangle$ and $y \notin \langle x\rangle$ for every $x, y \in G \setminus Z^\wedge(G)$, then $x \wedge y \neq 1$.*

Proof. Suppose that $\Gamma_G^{E^\wedge}$ is planar. We prove that $|Z^\wedge(G)| < 5$. By contrary, let $|Z^\wedge(G)| \geqslant 5$ and consider $T_x = xZ^\wedge(G)$ for some $x \in G \setminus Z^\wedge(G)$. The induced subgraph $\Gamma_{T_x}^{E^\wedge}$ of $\Gamma_G^{E^\wedge}$ by T_x is a planar graph. On the other hand, $\Gamma_G^{E^\wedge}$ contains K_5. It is a contradiction. Therefore $deg(x) \leqslant 3$ and $|Z^\wedge(G)| \leqslant 4$. We consider four cases.
(*i*) $|Z^\wedge(G)| = 4$, then $Z^\wedge(G) \cong C_4$ and $|C^\wedge(x)| \leqslant 8$ for every $x \in G \setminus Z^\wedge(G)$. We know that $|Z^\wedge(G)|$ divides $|C^\wedge(x)|$, $C^\wedge(x) \cong C_8$.
(*ii*) $|Z^\wedge(G)| = 3$, then $Z^\wedge(G) \cong C_3$ and $|C^\wedge(x)| \leqslant 7$ for every $x \in G \setminus Z^\wedge(G)$. Hence $C^\wedge(x) \cong C_6$.
(*iii*) $|Z^\wedge(G)| = 2$, then $Z^\wedge(G) \cong C_2$ and $|C^\wedge(x)| \leqslant 6$ for every $x \in G \setminus Z^\wedge(G)$. Therefore $C^\wedge(x) \cong C_6$ or C_4.
(*iv*) $|Z^\wedge(G)| = 1$, then $C^\wedge(x) \cong C_n$, where $n = 2, 3, 4, 5$, as required.
The converse is trivial. □

Corollary 4.6. *Let G be non-cyclic CE-group such that $Z^\wedge(G) = 1$ and $|C^\wedge(x)| = 2$ or 3 or 5. Then $\Gamma_G^{E^\wedge}$ is planar if and only if G is an elementary abelian p-group.*

Proof. Straightforward. □

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