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θ-Monotone Operators and Bifunctions

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Abstract. The purpose of this paper is to introduce and investigate θ-monotone operators and θ-monotone bifunctions in the context of Banach space. Local boundedness of θ-monotone bifunctions in the interior of their domains is proved. Also, the difference of two θ-monotone operators is studied. Moreover, some relations between θ-monotonicity and θ-convexity are investigated.

1. Introduction

The literature on monotone operator theory is quiet rich. During the last three decades, generalized monotone operators and their applications in many branches of mathematics, have received a lot of attention (see [13] and the references cited therein). In [14] the concept of pre-monotone operators is introduced and studied in $\mathbb{R}^n$ and then in [6] this notion is generalized and studied in Banach spaces. Recently, S. László in [18] presented the notions of θ-monotone operators and θ-convex functions and then studied the relations between these notions. Also, he generalized some basic results of monotone operators to θ-monotone operators. The class of θ-monotone operators consists of various classes of generalized monotonicity such as the class of ε-monotone, m-relaxed monotone, γ-paramonotone, and pre-monotone (see the next section).

Furthermore, it is known that the peruse of monotone bifunctions is closely connected to the investigation of monotone operators [1, 4–7, 12].

In this paper, we will introduce and study the notion of θ-monotone bifunctions and then we will relate it to θ-monotone operators. We will show that under some mild assumptions each θ-monotone bifunction is locally bounded in the interior of its domain. Then immediately we conclude that any θ-monotone operator T is locally bounded on $\operatorname{int} D(T)$. Besides, we will study the sum and difference of θ-monotone operators.

The paper is organized as follows: In the next section, after fixing some notations we remarked that the definition of θ-monotonicity does not permit positive values for θ in many natural cases. Also, we show

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some auxiliary results and useful features of θ-monotone operators. In Section 3, we study some results on the sum of θ-monotone operators. In Section 4, after introducing the notion of θ-monotone bifunction, we will show that under very mild assumptions, any θ-monotone bifunction is locally bounded in the interior of its domain. In this way, one can obtain an easy proof of the corresponding property of θ-monotone operators. Also, we prove that in a reflexive Banach space if the graph of a θ-monotone operator with the full domain is closed, then the operator is upper semicontinuous. In Section 5, we show that under some assumptions, the difference of two maximal θ-monotone operators is a maximal θ-monotone operator. Finally, in Section 6 we study the relations between θ-monotonicity and θ-convexity.

2. Preliminaries

Throughout this paper, X is a Banach space with norm $\|.\|$ and X^* is its dual space. By $\langle x^*, x\rangle$ we denote the value of linear continuous functional $x^* \in X^*$ at $x \in X$. We denote by $\to$, $\xrightarrow{w}$ and $\xrightarrow{w^*}$ the strong convergence, weak convergence and weak*convergence of nets, respectively, and $\mathbb{R}_+ := [0, +\infty)$. Given $x, y \in X$, (x, y) will be the open line segment $(x, y) := \{(1-t)x + ty : t \in (0, 1)\}$. The line segments $[x, y]$, $(x, y]$ and $[x, y)$ are defined analogously. Let $T : X \multimap X^*$ be a multivalued operator. The *domain* of T is the set $\{x \in X : T(x) \neq \emptyset\}$ and is denoted by $D(T)$, the *range* of T is defined by $R(T) := \bigcup\{T(x) : x \in X\}$. The *graph* of T, denoted by $\mathrm{gr}(T)$, is $\{(x, y) : x \in D(T), y \in T(x)\}$. T is *monotone* if $\langle x^* - y^*, x - y\rangle \geq 0$ for every $(x, x^*), (y, y^*) \in \mathrm{gr}(T)$ and *maximally monotone*, if T is monotone and it has no proper monotone extension (in the sense of graph inclusion). In this note, $\mathrm{cl}(C)$, $\mathrm{int}(C)$ and $\mathrm{co}(C)$ are the *closure*, the *interior* and the *convex hull* of a set C, respectively. $\overline{B}(x, r) := \{y \in X : \|x - y\| \leq r\}$ and $B(x, r) := \{y \in X : \|x - y\| < r\}$ are the closed ball and the open ball centered at x with radius r, respectively.

A subset A of X is called *absorbing* if $\bigcup_{\lambda>0} \lambda A = X$. Note that any neighborhood of 0 is absorbing. If A is convex, then A is absorbing if and only if $0 \in \mathrm{core}(A)$, where $\mathrm{core}(A)$ is the *algebraic interior* (or the *core*) of A defined by

$$\mathrm{core}(A) := \{a \in X : \forall\, x \in X\, \exists\, \lambda > 0 \text{ such that } a + tx \in A \text{ for all } t \in [0, \lambda]\}.$$

For more details, see [28].

Definition 2.1. *[6, Definition 2.1] Given an operator $T : X \multimap X^*$ and a map $\sigma : D(T) \to \mathbb{R}_+$. T is called*

(i) σ-monotone, if

$$\langle x^* - y^*, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|, \ \forall (x, x^*), (y, y^*) \in \mathrm{gr}(T).$$

(ii) pre-monotone, if it is σ-monotone for some $\sigma : D(T) \to \mathbb{R}_+$.

Note that T is σ-monotone if and only if [6, Remark 2.2.(i)]

$$\langle x^* - y^*,\ x - y\rangle \geq -\sigma(y)\|x - y\|, \ \forall (x, x^*), (y, y^*) \in \mathrm{gr}(T).$$

Now, we present some basic notions and results of θ-monotone operators which was studied by László in [19]. For a given operator $T : X \multimap X^*$, let $\theta : C \times C \to \mathbb{R}$ be a bifunction fulfilling $\theta(x, y) = \theta(y, x)$, for each $x, y \in C$, where $D(T) \subseteq C$. In the following definition, we made a slight modification on the definition of θ-monotonicity (see [19, Definition 2.1.1 and Definition 2.1.2]).

Definition 2.2. *Let $T : X \multimap X^*$ be a multivalued operator and let $\theta : C \times C \to \mathbb{R}$ be a bifunction fulfilling $\theta(x, y) = \theta(y, x)$, for each $x, y \in C$, where $D(T) \subseteq C$. We say that T is*

(i) θ-monotone, if

$$\langle x^* - y^*, x - y\rangle \geq \theta(x, y)\|x - y\|, \ \forall (x, x^*), (y, y^*) \in \mathrm{gr}(T). \tag{1}$$

Also T is called strictly θ-monotone, if in (1) *equality holds only for $x = y$.*

(ii) maximal θ-monotone, if it is θ-monotone and its graph is not properly contained in the graph of any other θ-monotone operator.

Note that, every θ-monotone operator has a maximal θ-monotone extension [19, Proposition 2.1.6]. It is worth mentioning that this concept covers various concepts of monotonicity.

Remark 2.3. The θ-monotone operator $T : X \multimap X^*$ is

(i) Minty-Browder monotone operator, if for each $x, y \in D(T)$, $\theta(x, y) = 0$ in (1) (cf. [21, 22]).

(ii) ε-monotone, if $\theta(x, y) = -2\varepsilon$ and $\varepsilon > 0$, for every $x, y \in D(T)$ [15].

(iii) m-relaxed monotone, if $\theta(x, y) = -m\|x - y\|^2$, for every $x, y \in D(T)$ and $m > 0$ [30].

(iv) γ-paramonotone, if $\theta(x, y) = -C\|x - y\|^{\gamma-1}$, for each $x, y \in D(T)$, $C > 0$ and $\gamma > 1$ [16].

(v) σ-monotone, if $\theta(x, y) = -\min\{\sigma(x), \sigma(y)\}$, for all $x, y \in D(T)$ and for some $\sigma : D(T) \to \mathbb{R}_+$ (see Definition 2.1).

Note that the converse of Remark 2.3(v) is not true in general. The following is an example of a θ-monotone operator which is not pre-monotone.

Example 2.4. Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) := 1/x$ for each $x \in (0, +\infty)$ and $T(x) = 0$ otherwise. We prove that T is not pre-monotone. Let there exists $\sigma : \mathbb{R} \to \mathbb{R}_+$ such that T be σ-monotone. According to [6, Remark 2.2(iii)], the definition of σ-monotonicity does not allow negative values for σ. By using this fact and choosing $y = 1$ and $x = 1/(2 + \sigma(1))$, some easy calculations leads to a contradiction. Hence T is not pre-monotone. On the other hand, by taking $\theta(x, y) := -|T(x) - T(y)|$, relation (1) is always true. Therefore T is θ-monotone.

Remark 2.5. (i) *Similar to [6, Remark 2.2(iii)], the definition of θ-monotonicity dose not allow positive values for θ in many natural situations. For example, suppose that every two members of line segment $[x_0, y_0] \subseteq D(T)$ satisfy the inequality* (1) *and $\theta(x, y) \geq \varepsilon > 0$, for each $x, y \in [x_0, y_0]$. Select $x_0^* \in T(x_0)$, $y_0^* \in T(y_0)$ and for each $n \in \mathbb{N}$ and each $k \in \{0, 1, \dots n\}$, set $x_k := x_0 + \frac{k}{n}(y_0 - x_0)$ such that $x_k^* \in T(x_k)$. From* (1) *we have*

$$\langle x_{k+1}^* - x_k^*, x_{k+1} - x_k \rangle \geq \varepsilon \|x_{k+1} - x_k\|, \quad \forall k \in \{0, 1, \dots, n-1\}.$$

Take $x_n := y_0$ and $x_n^* := y_0^*$. Then, $\langle x_{k+1}^* - x_k^*, y_0 - x_0 \rangle \geq \varepsilon \|y_0 - x_0\|$. Summing the previous inequality for $k = 0, 1, \dots, n-1$, we obtain that $\langle y_0^* - x_0^*, y_0 - x_0 \rangle \geq n\varepsilon \|y_0 - x_0\|$. The latter inequality is satisfied for each $n \in \mathbb{N}$ and this is impossible.

(ii) *Accordance to (i), throughout this paper, for a given operator $T : X \multimap X^*$, we assume that $D(T) \subseteq C \subseteq X$ and $\theta : C \times C \to \mathbb{R}_-$ is a bifunction fulfilling $\theta(x, y) = \theta(y, x)$, for each $x, y \in C$.*

Definition 2.6. *Let $A \subseteq X$ and $\theta : A \times A \to \mathbb{R}_-$ be a bifunction. Two pairs $(x, x^*), (y, y^*) \in A \times X^*$ are θ-monotonically related, if*

$$\langle x^* - y^*, x - y \rangle \geq \theta(x, y)\|x - y\|.$$

Proposition 2.7. *The following conditions for a θ-monotone operator $T : X \multimap X^*$ are equivalent:*

(i) T is maximal θ-monotone.

(ii) If a pair $(x, x^*) \in X \times X^*$ is θ-monotonically related to all pairs $(y, y^*) \in \mathrm{gr}(T)$, then $x^* \in T(x)$.

(iii) For every θ'-monotone operator T', with $\mathrm{gr}(T) \subseteq \mathrm{gr}(T')$ and $\theta(x, y) \leq \theta'(x, y)$ for all $x, y \in D(T')$, one has $T = T'$.

Proof. It is similar to the proofs of Proposition 2.1.7 and Proposition 2.1.8 in [19]. □

Theorem 2.8. *Suppose that the operator $T : X \multimap X^*$ is maximal θ-monotone. Then $T(x)$ is convex and weak*closed for all $x \in D(T)$.*

Proof. The proof of convexity can be found in [19, Theorem 2.1.2]. We show the weak*closedness. Assume that x^* is in the weak*closure of $T(x)$, for arbitrary $x \in D(T)$. Then there exists a net $\{x^*_\alpha\}$ in $T(x)$ such that $x^*_\alpha \xrightarrow{w^*} x^*$. It is enough to show that $x^* \in T(x)$. Using θ-monotonicity of T, for each $(y, y^*) \in \mathrm{gr}(T)$ we have

$$\langle x^*_\alpha - y^*, x - y\rangle \geq \theta(x, y)\|x - y\|.$$

Taking the limit, we deduce that

$$\langle x^* - y^*, x - y\rangle \geq \theta(x, y)\|x - y\|.$$

This implies that (x, x^*) is θ-monotonically related with all $(y, y^*) \in \mathrm{gr}(T)$. By Proposition 2.7(ii) we infer that $(x, x^*) \in \mathrm{gr}(T)$. This completes the proof. □

Proposition 2.9. *Given a maximal θ-monotone operator $T : X \multimap X^*$. If the mapping $\mathrm{cl}(D(T)) \ni x \mapsto \theta(x, y)$ is lower semicontinuous on $\mathrm{cl}(D(T))$, for every $y \in D(T)$, then $\mathrm{gr}(T)$ is sequentially norm×weak*closed.*

Proof. Let (x_n, x^*_n) be a sequence in $\mathrm{gr}(T)$, where $x_n \to x$ and $x^*_n \xrightarrow{w^*} x^*$. Then

$$\langle x^*_n - y^*, x_n - y\rangle \geq \theta(x_n, y)\|x_n - y\|,$$

for every $(y, y^*) \in \mathrm{gr}(T)$. By lower semicontinuity of $\theta(\cdot, y)$ we get

$$\langle x^* - y^*, x - y\rangle \geq \theta(x, y)\|x - y\|.$$

Then maximal θ-monotonicity of T implies that $(x, x^*) \in \mathrm{gr}(T)$. □

A worthy conclusion from Proposition 2.9 is the fact that lower semicontinuity of θ is a necessary condition, which we can observe this in [6, Example 2.8] by setting $\theta(x, y) := -\min\{\sigma(x), \sigma(y)\}$ for every $x, y \in \mathbb{R}$. For the sake of completeness we present it below.

Remark 2.10. *The graph of a θ-monotone operator (even monotone operator) in general is only sequentially norm×weak*closed but it is not necessarily norm×weak*closed (see [10]). However, we will prove that maximal θ-monotone operators are upper semicontinuous at each interior point of their domain.*

Consider an operator $T : X \multimap X^*$. Define the bifunction $\hat{\theta}_T : D(T) \times D(T) \to \mathbb{R}_-$ by

$$\hat{\theta}_T(x, y) := \sup\{a \in \mathbb{R} : \langle x^* - y^*, x - y\rangle \geq a\|x - y\|,\ \forall (x, x^*), (y, y^*) \in \mathrm{gr}(T)\}.$$

It is easy to see that $\hat{\theta}_T(x, y) = \hat{\theta}_T(y, x)$ for all $x, y \in D(T)$, also we have:

$$\hat{\theta}_T := \sup\{\theta : T \text{ is a } \theta\text{-monotone operator}\}. \tag{2}$$

$\hat{\theta}_T$ is finite on $D(T) \times D(T)$ and T is $\hat{\theta}_T$-monotone. For every $x, y \in D(T)$ we have:

$$\hat{\theta}_T(y, x) = \min\Big\{\inf\Big\{\frac{\langle x^* - y^*, x - y\rangle}{\|x - y\|} : x \neq y,\ \forall (x, x^*), (y, y^*) \in \mathrm{gr}(T)\Big\}, 0\Big\}. \tag{3}$$

Indeed, fix $x_0, y_0 \in D(T)$ such that $x_0 \neq y_0$. Assume that T is θ-monotone for some θ. Then for every $x^* \in T(x_0)$ and every $y^* \in T(y_0)$ we have:

$$\langle y^* - x^*, y_0 - x_0\rangle \geq \theta(x_0, y_0)\left\|x_0 - y_0\right\|.$$

Multiply both sides by $\left(\left\|x_0 - y_0\right\|\right)^{-1}$. We get

$$\frac{\langle y^* - x^*, y_0 - x_0\rangle}{\left\|x_0 - y_0\right\|} \geq \theta\left(x_0, y_0\right).$$

Now by taking the infimum over $x^* \in T\left(x_0\right)$, and $y^* \in T\left(y_0\right)$ on the left hand side of the above inequality, and the fact that $\theta\left(x_0, y_0\right) \leq 0$ we obtain:

$$\min\left\{\inf\left\{\frac{\langle y^* - x^*, y_0 - x_0\rangle}{\left\|x_0 - y_0\right\|}, x_0 \neq y_0, \left(x_0, x^*\right), \left(y_0, y^*\right) \in \operatorname{gr} T\right\}, 0\right\} \geq \theta\left(x_0, y_0\right) = \theta\left(y_0, x_0\right). \tag{4}$$

By (2) we conclude (3).

Proposition 2.11. Given an operator $T : X \multimap X^*$.

(i) $\hat{\theta}_T$ is finite on $D(T) \times D(T)$ and T is $\hat{\theta}_T$-monotone, if and only if T is a θ-monotone operator, for some θ.

(ii) $\hat{\theta}_T$ is finite on $D(T) \times D(T)$ and T is maximal $\hat{\theta}_T$-monotone, if and only if T is a maximal θ-monotone operator, for some θ.

Proof. (i): Note that from (4) we infer that, $\hat{\theta}_T(x, y) > -\infty$ for all $x, y \in D(T)$, i.e. $\hat{\theta}_T$ is finite on $D(T) \times D(T)$. The second part is direct consequence of the definitions of $\hat{\theta}_T$ and θ-monotonicity.

(ii): It is enough to show that if T is a maximal θ-monotone operator for some θ, then T is a maximal $\hat{\theta}_T$-monotone operator. Suppose that S is a θ'-monotone operator, where θ' is an extension of $\hat{\theta}_T$ and $\mathrm{gr}(T) \subseteq \mathrm{gr}(S)$. Using the fact that $\theta' = \hat{\theta}_T \geq \theta$ on $D(T) \times D(T)$ and Proposition 2.7, we obtain $T = S$ and so T is maximal $\hat{\theta}_T$-monotone. □

Definition 2.12. *A set $A \subseteq X^*$ is bounded weak*closed, if every bounded and weak*convergent net in A has its limit in A.*

Theorem 2.13. (Krein-Šmulian) [20, Theorem 2.7.11] A convex set in X^* is weak*closed, if and only if its intersection with $B(0, \varepsilon)$ is weak*closed for every $\varepsilon > 0$.

The Krein-Šmulian theorem obviously implies the following.

Corollary 2.14. [24, Theorem 1.11] A convex set in X^* is weak*closed if and only if it is bounded weak*closed.

3. Results of θ-monotone operator

In this section, one can follow a few conclusions about sum of two maximal θ-monotone operators.

Having a function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, we denote its *domain* by $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$ and its *epigraph* by $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. The function f is called *proper*, if $\operatorname{dom} f \neq \emptyset$. For a proper function f, if $f(x) \in \mathbb{R}$, then the *subdifferential* of f, $\partial f : X \multimap X^*$ is defined by $\partial f(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x)\ \forall y \in X\}$. When $f(x) \notin \mathbb{R}$ we define $\partial f(x) = \emptyset$.

The following lemma is a well-known result which is applicable in subsequent theorem.

Lemma 3.1. [29, Corollary 4] *Let X be a Banach space, $f_1, f_2 : X \to \mathbb{R} \cup \{\infty\}$ be proper, convex, lower semicontinuous functions and* $\operatorname{dom} f_1 - \operatorname{dom} f_2$ *be absorbing. Then there exists $n \geq 1$ such that*

$$\{x \in X : f_1(x) \leq n, \|x\| \leq n\} - \{x \in X : f_2(x) \leq n, \|x\| \leq n\}$$

is a neighborhood of 0.

The idea of the proof of the following theorem was inspired by [9, Theorem 2.11] and [31, Proposition 2.2].

Theorem 3.2. Suppose that $S, T : X \multimap X^*$ are θ-monotone operators and the bifunction $\theta(x, \cdot)$ is bounded below on $D(T)$ and $D(S)$, for every $x \in D(T) \cup D(S)$. If $0 \in \mathrm{core}[\mathrm{co}(D(T)) - \mathrm{co}(D(S))]$, then there exist $r > 0$ and $c > 0$ such that

$$\max(\|t^*\|, \|s^*\|) \leq c(r + \|x\|)(2r + \|t^* + s^*\|), \quad \forall x \in D(T) \cap D(S),\ t^* \in T(x),\ s^* \in S(x).$$

Proof. Define $\psi_T : X \longrightarrow \overline{\mathbb{R}}$ by

$$\psi_T(x) := \sup\Big\{\frac{\langle y^*, x - y\rangle}{1 + \|y\|} : (y, y^*) \in \mathrm{gr}(T)\Big\}.$$

This function is lower semicontinuous and convex because it is supremum of affine functions. If $(x, x^*) \in \mathrm{gr}(T)$, then for all $(y, y^*) \in \mathrm{gr}(T)$ we get

$$\begin{aligned}\frac{\langle y^*, x - y\rangle}{1 + \|y\|} &= \frac{\langle y^* - x^*, x - y\rangle}{1 + \|y\|} + \frac{\langle x^*, x - y\rangle}{1 + \|y\|}\\ &\leq \frac{-\theta(x, y)}{1 + \|y\|}\|x - y\| + \|x^*\|\frac{\|x - y\|}{1 + \|y\|}\\ &\leq (\|x^*\| - L_T)(\|x\| + 1),\end{aligned}$$

where L_T is a lower bound of $\theta(x, \cdot)$ on $D(T)$. From this it follows that $\psi_T(x) < +\infty$, so $D(T) \subset \mathrm{dom}(\psi_T)$. Convexity of $\mathrm{dom}(\psi_T)$ and $\mathrm{dom}(\psi_s)$ imply that $\mathrm{co}(D(T)) \subset \mathrm{dom}(\psi_T)$ and $\mathrm{co}(D(S)) \subset \mathrm{dom}(\psi_S)$, respectively. Hence $\mathrm{co}(D(T)) - \mathrm{co}(D(S)) \subset \mathrm{dom}(\psi_T) - \mathrm{dom}(\psi_S)$. From assumption and the previous inclusions, we conclude that $0 \in \mathrm{core}(\mathrm{dom}(\psi_T) - \mathrm{dom}(\psi_S))$. Applying Lemma 3.1 there exist $\varepsilon > 0$ and $r \geq 1$ so that

$$B(0, \varepsilon) \subset (\{x : \psi_T(x) \leq r, \|x\| \leq r\} - \{x : \psi_S(x) \leq r, \|x\| \leq r\}).$$

Select $z \in B(0, \varepsilon)$, $x \in D(T) \cap D(S)$, $t^* \in T(x)$ and $s^* \in S(x)$. Therefore $z = a - b$ such that $\psi_T(a) \leq r$, $\|a\| \leq r$, $\psi_S(b) \leq r$ and $\|b\| \leq r$. We have

$$\begin{aligned}\langle t^*, z\rangle &= \langle t^*, a - x\rangle + \langle s^*, b - x\rangle + \langle t^* + s^*, x - b\rangle\\ &\leq \psi_T(a)(1 + \|x\|) + \psi_S(b)(1 + \|x\|) + \|t^* + s^*\|(\|x\| + r)\\ &\leq (r + \|x\|)(2r + \|t^* + s^*\|).\end{aligned}$$

This gives us

$$\|t^*\| \leq \frac{(r + \|x\|)(2r + \|t^* + s^*\|)}{\varepsilon}. \tag{5}$$

Take $c = \frac{1}{\varepsilon}$ in (5). Arguing similarly, we can obtain relation (5) for $\|s^*\|$. □

Our proof of next theorem is very close to the proof of A. Verona and M. E. Verona in [31].

Theorem 3.3. Let $S, T : X \multimap X^*$ be maximal θ-monotone and for every $x \in D(T) \cap D(S)$, the function $D(T) \cup D(S) \ni y \mapsto \theta(x, y)$ be bounded from below. If $0 \in \mathrm{core}[\mathrm{co}(D(T)) - \mathrm{co}(D(S))]$, then $T(x) + S(x)$ is a weak*closed subset of X^*.

Proof. By Theorem 2.8, $T(x)$ and $S(x)$ are convex, hence $T(x) + S(x)$ is also convex. Using Corollary 2.14, we show that $T(x) + S(x)$ is bounded weak*closed. Select two nets $\{t^*_\alpha\} \subseteq T(x)$ and $\{s^*_\alpha\} \subseteq S(x)$ such that $\{t^*_\alpha + s^*_\alpha\}$ is bounded and weak*convergent to x^*. Theorem 3.2 implies that the nets $\{t^*_\alpha\}$ and $\{s^*_\alpha\}$ are bounded. Hence, by [20, Corollary 2.6.19] $\{t^*_\alpha\}$ and $\{s^*_\alpha\}$ are relatively weak*compact. Without loss of generality, replace them with subnets and we suppose $t^*_\alpha \xrightarrow{w^*} t^*$ and $s^*_\alpha \xrightarrow{w^*} s^*$. Applying Theorem 2.8, we have $t^* \in T(x)$ and $s^* \in S(x)$ and hence $x^* = t^* + s^* \in T(x) + S(x)$. □

4. θ-monotone bifunction and local boundedness

In this section, first we present the concept of θ-monotone bifunctions. Further, we study some properties of θ-monotone bifunctions and their correspondences with θ-monotone operators. In the sequel, we prove that under some conditions, θ-monotone bifunctions are locally bounded at interior points of their domain.

Throughout this section, we assume that C is a nonempty subset of a Banach space X and $\theta : C\times C \to \mathbb{R}_-$ is a bifunction with the property that $\theta(x, y) = \theta(y, x)$, for all $x, y \in C$.

Definition 4.1. *[6] Given a map $\sigma : C \to \mathbb{R}_+$, a bifunction $F : C \times C \to \mathbb{R}$ is σ-monotone, if*

$$F(x, y) + F(y, x) \le \min\{\sigma(x), \sigma(y)\}\|x - y\|, \quad \forall x, y \in C.$$

Equivalently, F is σ-monotone if

$$F(x, y) + F(y, x) \le \sigma(y)\|x - y\|, \quad \forall x, y \in C.$$

Definition 4.2. Let $\theta : C \times C \to \mathbb{R}_-$ be a bifunction with the property that $\theta(x, y) = \theta(y, x)$, for all $x, y \in C$. The bifunction $F : C \times C \to \mathbb{R}$ is called *θ-monotone*, if

$$F(x, y) + F(y, x) \le -\theta(x, y)\|x - y\|, \quad \forall x, y \in C.$$

It is quickly checked that, if $\theta(x, y) = 0$, for any $x, y \in C$, the above definition coincides with the definition of bifunctions [8], and if $\theta(x, y) = -\min\{\sigma(x), \sigma(y)\}$ with $\sigma : C \to \mathbb{R}_+$. Then the concept of θ-monotone bifunction reduces to σ-monotone bifunction, which is introduced and studied in [6].

According to [1], for each bifunction $F : C \times C \to \mathbb{R}$ one can attach the *diagonal subdifferential operator* $A^F : X \multimap X^*$ defined by

$$A^F(x) := \begin{cases} \{x^* \in X^* : F(x, y) \ge \langle x^*, y - x\rangle,\ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Note that in case $F(x, x) = 0$ for all $x \in C$, one has $A^F(x) = \partial F(x, \cdot)(x)$ (i.e., the subdifferential of the function $F(x, \cdot)$ at x) [12].

Proposition 4.3. Let $F : C \times C \to \mathbb{R}$ be a θ-monotone bifunction. Then the operator A^F is θ-monotone.

Proof. Let $(x, x^*), (y, y^*) \in \mathrm{gr}(A^F)$. Then $F(x, y) \ge \langle x^*, y - x\rangle$ and $F(y, x) \ge \langle y^*, x - y\rangle$ for all $x, y \in C$. Therefore,

$$\langle x^* - y^*, x - y\rangle \ge -F(x, y) - F(y, x) \ge \theta(x, y)\|x - y\|.$$

Consequently, A^F is θ-monotone. □

Remark 4.4. Suppose that $F, G : C \times C \to \mathbb{R}$ are two θ-monotone bifunctions and $\alpha > 0$. The bifunctions

$$F + G : C \times C \to \mathbb{R}$$
$$(x, y) \mapsto F(x, y) + G(x, y)$$

and

$$\alpha F : C \times C \to \mathbb{R},$$
$$(x, y) \mapsto \alpha.(F(x, y)),$$

are 2θ-monotone and $\alpha\theta$-monotone, respectively. Moreover, $A^F(x) + A^G(x) \subseteq A^{F+G}(x)$, for each $x \in X$.

Indeed, by θ-monotonicity of F and G we have

$$(F+G)(x,y)+(F+G)(y,x)\leq -2\theta(x,y)\|x-y\|,\quad \forall x,y\in C.$$

Hence $F+G$ is 2θ-monotone. Clearly, θ-monotonicity of F implies that αF is $\alpha\theta$-monotone. If $x\in X\backslash C$, then the relation $A^F(x)+A^G(x)\subseteq A^{F+G}(x)$ is true. Now, we assume that $(x,x^*)\in \mathrm{gr}(A^F+A^G)$, then there exist $x_1^*\in A^F(x)$ and $x_2^*\in A^G(x)$ such that $x^*=x_1^*+x_2^*$. From the definition of A^F and A^G we conclude

$$(F+G)(x,y)\geq \langle x_1^*+x_2^*, y-x\rangle=\langle x^*,y-x\rangle,\quad \forall y\in C.$$

Therefore, $A^F(x)+A^G(x)\subseteq A^{F+G}(x)$ for all $x\in C$.

Definition 4.5. A θ-monotone bifunction $F: C\times C\to\mathbb{R}$ is said to be *maximal θ-monotone*, if the operator A^F is maximal θ-monotone.

As we know from [12] for any operator $T: X\multimap X^*$, there corresponds a bifunction $G_T: D(T)\times D(T)\to\mathbb{R}$ defined by

$$G_T(x,y):=\sup_{x^*\in T(x)}\langle x^*,y-x\rangle.$$

The relations between θ-monotonicity of the bifunction G_T and the operator T are given in the following proposition.

Proposition 4.6. For a θ-monotone operator $T: X\multimap X^*$, the following statements hold.

(i) G_T is θ-monotone and real-valued.

(ii) If T is maximal θ-monotone, then G_T is maximal θ-monotone and $A^{G_T}=T$.

(iii) If $T(x)$ is closed and convex for all $x\in D(T)=X$ and G_T is maximal θ-monotone, then T is also maximal θ-monotone.

Proof. (i): By hypothesis, there exists $\theta: D(T)\times D(T)\to\mathbb{R}_-$ such that

$$\langle x^*-y^*,x-y\rangle\geq\theta(x,y)\|x-y\|,$$

for every $(x,x^*),(y,y^*)\in \mathrm{gr}(T)$. Then $\langle x^*,y-x\rangle+\langle y^*,x-y\rangle\leq-\theta(x,y)\|x-y\|$ and hence

$$\sup_{x^*\in T(x)}\langle x^*,y-x\rangle+\sup_{y^*\in T(y)}\langle y^*,x-y\rangle\leq-\theta(x,y)\|x-y\|.$$

Therefore, $G_T(x,y)+G_T(y,x)\leq-\theta(x,y)\|x-y\|$, for all $x,y\in D(T)$. Hence G_T is θ-monotone and $G_T(x,y)\in\mathbb{R}$ for each $x,y\in D(T)$.

(ii): Take $(x,z^*)\in \mathrm{gr}(T)$. By definition of G_T, we have $G_T(x,y)\geq\langle z^*,y-x\rangle$, for every $y\in D(T)$. Then $z^*\in A^{G_T}(x)$, this implies that $T(x)\subseteq A^{G_T}(x)$. By Proposition 4.3 and part (i), A^{G_T} is θ-monotone. Since T is maximal θ-monotone, we get $T=A^{G_T}$.

(iii): Take $x\in X$ and $z^*\in A^{G_T}(x)$, thus $G_T(x,y)\geq\langle z^*,y-x\rangle$. Now, using separation theorem [3, Corollary 5.80], we have $z^*\in T(x)$. So that $\mathrm{gr}(A^{G_T})\subseteq \mathrm{gr}(T)$. Then $T=A^{G_T}$, since A^{G_T} is maximal θ-monotone. □

Remark 4.7. It follows from Propositions 4.3 and 4.6 that the operator A^F and the bifunction G_{A^F} are θ-monotone, for any θ-monotone bifunction F. It is easy to see that $G_{A^F}(x,y)\leq F(x,y)$. According to [12, Example 2.5], we see that the correspondence $F\to A^F$ is not one-to-one even when F is a monotone bifunction, i.e., $\theta\equiv 0$.

We recall the concept of local boundedness for bifunctions.

Definition 4.8. [6, Definition 3.5] A bifunction $F : C \times C \to \mathbb{R}$ is called

(i) *locally bounded* at $(x_0, y_0) \in X \times X$, if there exist an open neighborhood V of x_0, an open neighborhood W of y_0 and $M \in \mathbb{R}$ such that $F(x, y) \leq M$, for all $(x, y) \in (V \times W) \cap (C \times C)$.

(ii) *locally bounded* on $K \times L \subseteq X \times X$, if it is locally bounded at every point $(x, y) \in K \times L$.

(iii) *locally bounded* at $x_0 \in X$, if it is locally bounded at (x_0, x_0). In other words, if there exist an open neighborhood V of x_0 and $M \in \mathbb{R}$ such that $F(x, y) \leq M$ for all $x, y \in V \cap C$.

(iv) *locally bounded* on $K \subseteq X$, if it is locally bounded at each $x \in K$.

Remark 4.9. [7, Remark 6] If a bifunction $F : C \times C \to \mathbb{R}$ is locally bounded at $x_0 \in \text{int}(C)$, then A^F is locally bounded at x_0. Hence if G_T is locally bounded at $x_0 \in \text{int}(D(T))$, then T is locally bounded at x_0, because $T(x) \subseteq A^{G_T}(x)$, for all $x \in X$. This fact is a main tool for showing local boundedness of operators.

A function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconvex*, if

$$f((1-\lambda)x + \lambda y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in X, \ \forall \lambda \in [0, 1].$$

In the following, we prove that under some sufficient conditions, the θ-monotone bifunction is locally bounded at the interior points of its domain. In the next proposition for the finite dimensional case, we provide a constructive proof.

Proposition 4.10. Let $C \subseteq \mathbb{R}^n$ and $F : C \times C \to \mathbb{R}$ be θ-monotone such that $C \ni y \mapsto F(x, y)$ be lower semicontinuous and quasiconvex, and $\text{int}(C) \ni x \mapsto \theta(x, y)$ be lower semicontinuous. Then F is locally bounded at every point of $\text{int}(C) \times \text{int}(C)$.

Proof. Take $(x_0, y_0) \in \text{int}(C) \times \text{int}(C)$. Since the space is finite-dimensional, we can choose $U := \{z_1, z_2, \ldots, z_m\} \subseteq C$ and $V := \text{co}(U) \subseteq C$ be a neighborhood of y_0. Assume that $W \subseteq C$ is a compact neighborhood of x_0 in C and M_k and L_k are minimums of $F(z_k, \cdot)$ and $\theta(\cdot, z_k)$ on W, respectively. By hypothesis and for each $x \in W$ and $y \in V$, we have

$$\begin{aligned} F(x, y) \leq \max_{1 \leq k \leq m} F(x, z_k) &\leq \max_{1 \leq k \leq m} \{-\theta(x, z_k)\|x - z_k\| - F(z_k, x)\} \\ &\leq \max_{1 \leq k \leq m} (-L_k) \sup_{z \in W, v \in V} \|z - v\| + \max_{1 \leq k \leq m} (-M_k). \end{aligned}$$

Since W and V are bounded, $\sup_{z \in W, v \in V} \|z - v\|$ is finite and hence the proof is complete. □

Remark 4.11. Note that, in the hypothesis of Proposition 4.10, it is enough to assume that $\text{int}(D(T)) \ni x \mapsto \theta(x, y)$ is locally bounded from bellow for all $y \in \text{int}(D(T))$ (see [19, Remark 2.1.3]).

Lemma 4.12. [7, Lemma 9] Let $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous and quasiconvex function. If $x_0 \in \text{int}(\text{dom}(f))$, then f is bounded from above in a neighborhood of x_0.

Theorem 4.13. Consider the θ-monotone bifunction $F : C \times C \to \mathbb{R}$ such that $C \ni y \mapsto F(x, y)$ is lower semicontinuous and quasiconvex, for all $x \in C$. Let $x_0 \in C$ and $y_0 \in \text{int}(C)$ be such that $B(y_0, \varepsilon) \subseteq C$ for some $\varepsilon > 0$ and let $F(y, \cdot)$ and $\theta(\cdot, y)$ be bounded from below on $B(x_0, \varepsilon) \cap C$, for every $y \in B(y_0, \varepsilon)$, (note that these bounds may be dependent to y). Then F is locally bounded at (x_0, y_0).

Proof. Define $g : B(y_0, \varepsilon) \to \overline{\mathbb{R}}$ by

$$g(y) := \sup\{F(x, y) : x \in B(x_0, \varepsilon) \cap C\}. \tag{6}$$

For each $y \in B(y_0, \varepsilon)$ and $x \in B(x_0, \varepsilon) \cap C$, θ-monotonicity of F implies that

$$F(x, y) \leq -\theta(x, y)\|x - y\| - F(y, x) \leq -L_y(\varepsilon + \|y - x_0\|) - M_y,$$

where M_y and L_y are lower bounds of $F(y, \cdot)$ and $\theta(\cdot, y)$ on $B(x_0, \varepsilon) \cap C$, respectively. Then g is real-valued. On the other hand, g is lower semicontinuous, quasiconvex and $y_0 \in \operatorname{int}(\operatorname{dom}(g))$. Applying Lemma 4.12, there exist $\delta < \varepsilon$ and $M \in \mathbb{R}$ such that $g(y) \leq M$, for all $y \in B(y_0, \delta)$. According to (6), $F(x, y) \leq M$, for all $y \in B(y_0, \delta)$ and $x \in B(x_0, \delta) \cap C$, i.e., F is locally bounded at (x_0, y_0). □

If either X is a reflexive Banach space or $F(x, \cdot)$ is lower semicontinuous and convex for all $x \in C$, then we can eliminate the condition "$F(y, \cdot)$ and $\theta(\cdot, y)$ are bounded from below on $B(x_0, \varepsilon) \cap C$ for some $x_0 \in C$".

Corollary 4.14. Suppose that X is a reflexive Banach space, $C \ni y \mapsto \theta(y, x)$ and $C \ni y \mapsto F(x, y)$ are lower semicontinuous and quasiconvex for each $x \in C$. Then F is locally bounded at any point of $\operatorname{int}(C) \times \operatorname{int}(C)$. Moreover, if C is weakly closed, then F is locally bounded on $C \times \operatorname{int}(C)$.

Proof. Take $x_0 \in \operatorname{int} C$ and choose $\varepsilon > 0$ such that $\overline{B}(x_0, \varepsilon) \subseteq C$. Since $F(x, \cdot)$ and $\theta(\cdot, x)$ are lower semicontinuous and quasiconvex, they are weakly lower semicontinuous. Hence for every $y \in C$, $F(y, \cdot)$ and $\theta(\cdot, y)$ attain their minimum values throughout weakly compact set $\overline{B}(x_0, \varepsilon)$ and so we have $F(y, \cdot)$ and $\theta(\cdot, y)$ are bounded from below on $B(x_0, \varepsilon)$. Theorem 4.13 implies that F is locally bounded at any point of $\operatorname{int}(C) \times \operatorname{int}(C)$. For the second part, since C is weakly closed, $\overline{B}(x_0, \varepsilon) \cap C$ is weakly compact (see [20, Theorem 2.8.2]), for any $x_0 \in C$ and $\varepsilon > 0$. The proof of the second part is similar. □

Corollary 4.15. Let X be a Banach space, $C \subseteq X$ and θ be the same as in the Definition 4.2. Let $F : C \times C \to \mathbb{R}$ be θ-monotone, $C \ni y \mapsto \theta(y, x)$ and $C \ni y \mapsto F(x, y)$ be lower semicontinuous and convex for all $x \in C$. Then F is locally bounded at any point of $C \times \operatorname{int}(C)$.

Proof. Take $x_0 \in C$, $y_0 \in \operatorname{int}(C)$ and $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq C$. For any $y \in B(y_0, \varepsilon)$, we have $\partial F(y, \cdot)(y) \neq \emptyset$ and $\partial\theta(\cdot, y)(y) \neq \emptyset$. Hence there exist $y^* \in \partial F(y, \cdot)(y)$ and $z^* \in \partial\theta(\cdot, y)(y)$ such that for any $x \in B(x_0, \varepsilon) \cap C$, we obtain

$$F(y, x) - F(y, y) \geq \langle y^*, x - y\rangle \geq -\|y^*\|\|x - y\| \geq -\|y^*\|(\varepsilon + \|x_0 - y\|),$$

and

$$\theta(x, y) - \theta(y, y) \geq \langle z^*, y - x\rangle \geq -\|z^*\|\|x - y\| \geq -\|z^*\|(\varepsilon + \|x_0 - y\|).$$

It follows that $F(y, \cdot)$ and $\theta(\cdot, y)$ are bounded from below on $B(x_0, \varepsilon) \cap C$. Applying Theorem 4.13, the bifunction F is locally bounded at (x_0, y_0). □

An immediate consequence of this result is a generalization of [6, Proposition 3.11], [14, Proposition 3.5] and [19, Theorem 2.1.1].

Corollary 4.16. Let $T : X \multimap X^*$ be a θ-monotone operator such that for any $x \in X$, $\operatorname{int}(D(T)) \ni y \mapsto \theta(x, y)$ is locally bounded from below, then T is locally bounded at every point of $\operatorname{int}(D(T))$.

Proof. Apply Corollary 4.15 for G_T. □

Corollary 4.17. (Rockafellar) [11, Theorem 4.2.10] Every monotone operator $T : X \multimap X^*$ is locally bounded at any point of $\operatorname{int}(D(T))$.

Proposition 4.18. Let $T : X \multimap X^*$ be maximal θ-monotone and the bifunction $D(T) \ni y \mapsto \theta(y, x)$ be lower semicontinuous and convex. Then $T(x)$ is weak*compact for all $x \in \operatorname{int}(D(T))$.

Proof. It is easy to see that

$$\mathrm{gr}(T) = \bigcap_{(t,t^*)\in\mathrm{gr}(T)} \{(x,x^*) \in X \times X^* : \langle x^* - t^*, x - t\rangle \geq \theta(x,t)\|x - t\|\},$$

because T is maximal θ-monotone. Hence, we get

$$T(x) = \bigcap_{(t,t^*)\in\mathrm{gr}(T)} \{x^* \in X^* : \langle x^* - t^*, x - t\rangle \geq \theta(x,t)\|x - t\|\},$$

for every $x \in D(T)$. Since $T(x)$ is the intersection of weak*closed sets, it is weak*closed. By Corollary 4.16, T is locally bounded at any interior point of $D(T)$. Thus, there exists $K \geq 0$ such that $\|x^*\| \leq K$ for all $x^* \in T(x)$. According to the Banach-Alaoglu theorem [27, Theorem 3.15], for every $(t,t^*) \in \mathrm{gr}(T)$ and $x \in \mathrm{int}(D(T))$, the set

$$\{x^* \in X^* : \langle x^* - t^*, x - t\rangle \geq \theta(x,t)\|x - t\|, \|x^*\| \leq K\},$$

is weak*compact. It follows that

$$T(x) = \bigcap_{(t,t^*)\in\mathrm{gr}(T)} \{x^* \in X^* : \langle x^* - t^*, x - t\rangle \geq \theta(x,t)\|x - t\|, \|x^*\| \leq K\},$$

is also weak*compact. □

Consider the mapping $\theta_T : \mathbb{R} \to \mathbb{R}_-$ which is defined by $\theta_T(y) := \inf_{x\in D(T)\setminus\{y\}} \hat{\theta}_T(y,x)$, for each $y \in D(T)$. If $T : \mathbb{R} \to \mathbb{R}$ is θ-monotone, then

$$\begin{aligned}\theta_T(y) &= \inf_{x\in\mathbb{R}\setminus\{y\}} \{(T(x) - T(y))\,\mathrm{sgn}(x - y)\}\\ &= \min\Big\{\inf_{x\leq y}\{T(y) - T(x)\}, \inf_{x\geq y}\{T(x) - T(y)\}\Big\}. \qquad (7)\end{aligned}$$

The following propositions are generalized versions of some results in [5] for σ-monotone operators. For the sake of completeness we add their proofs.

Proposition 4.19. *Suppose that $T : \mathbb{R} \to \mathbb{R}$ is θ-monotone. Then T is locally bounded. Moreover, if* $\mathrm{gr}(T)$ *is closed, then T is continuous.*

Proof. For all $x, y \in \mathbb{R}$, by (7) we have

$$\theta_T(y) = \min\Big\{\inf_{x\leq y}\{T(y) - T(x)\}, \inf_{x\geq y}\{T(x) - T(y)\}\Big\}.$$

Let $a < b$. Thus $\theta_T(b) \leq \inf_{x\leq b}\{T(b) - T(x)\}$ and so $T(x) \leq T(b) - \theta_T(b)$ for all $x \leq b$. i.e., T is bounded above on $(-\infty, b]$. Likewise, $\theta_T(a) \leq \inf_{a\leq x}\{T(x) - T(a)\}$. Therefore, $T(x) \geq \theta_T(a) + T(a)$, that is T is bounded below on $[a, +\infty)$. Hence T is bounded on every interval $[a, b]$. Now, assume that $\mathrm{gr}(T)$ is closed but it is not continuous. Then there exists a sequence $\{x_n\}$ in $\mathbb{R}$ such that $x_n \to x$, while $\{T(x_n)\}$ does not converge to $T(x)$. Thus there exists $\varepsilon > 0$ such that $|T(x_n) - T(x)| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. According to local boundedness of T, there would be a subsequence (which we denote again by $\{T(x_n)\}$ for simplicity) converging to a point $a \in \mathbb{R}$ such that $|a - T(x)| \geq \varepsilon$. This means that $(x_n, T(x_n)) \to (x, a) \neq (x, T(x))$, which contradicts with the fact that $\mathrm{gr}(T)$ is closed. □

The idea of the following proposition and its proof are due to N. Hadjisavvas in the case of σ-monotone operators.

Proposition 4.20. *Suppose that X is a reflexive Banach space and $T : X \multimap X^*$ is θ-monotone such that for every $x \in X$, $X \ni y \mapsto \theta(x, y)$ is locally bounded from below. Then T is locally bounded. Moreover, if* $\mathrm{gr}(T)$ *is sequentially norm×weak*closed, then T is norm×weak*upper semicontinuous.*

Proof. Applying Corollary 4.16, the firs part is obtained. Now, suppose on the contrary, T is not upper semicontinuous at $x_0 \in X$. Then there exists a weakly open set $V \subseteq X^*$ such that $T(x_0) \subseteq V$ and $T(B(x_0, \varepsilon)) \nsubseteq V$, for every $\varepsilon > 0$. By taking $\varepsilon = 1/n$ we can construct a sequence $\{x_n\} \subseteq X$ with $\|x_n - x_0\| < \frac{1}{n}$ and $\{x_n^*\} \in T(x_n) \cap V^c$. By local boundedness of T, $\{x_n^* : n \in \mathbb{N}\}$ is bounded. According to the Banach-Alaouglu theorem [27, Theorem 3.15], the sequence $\{x_n^* : n \in \mathbb{N}\}$ is weak*compact in X^*. It follows from Eberlein–Šmulian theorem [20, Theorem 2.8.6], that there exists a subsequence $\{x_{n_k}^*\}$ such that $x_{n_k}^* \xrightarrow{w^*} x^* \in X^*$. Hence $(x_{n_k}, x_{n_k}^*) \to (x_0, x^*)$. By the closedness assumption, $x^* \in T(x_0)$, which implies that $x_{n_k}^* \in V$. We therefore arrive at a contradiction. □

Proposition 4.21. *Suppose that $T : \mathbb{R} \to \mathbb{R}$ is θ-monotone and* $\mathrm{gr}(T)$ *is closed. Then $\hat{\theta}_T$ is continuous.*

Proof. For each $y \in \mathbb{R}$, we claim that $\inf_{x \le y}\{T(y) - T(x)\}$ and $\inf_{x \ge y}\{T(x) - T(y)\}$ are continuous. By Proposition 4.19, T is continuous. Set $f(y) := \inf_{x \le y} T(x)$. The continuity of T implies that T is locally uniformly continuous, i.e., for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|T(y_0) - T(x)| < \frac{\varepsilon}{2}, \tag{8}$$

for every $x \in [y_0 - \delta, y_0 + \delta]$ and $y_0 \in \mathbb{R}$. Take $A := [y_0 - \frac{\delta}{2}, y_0 + \frac{\delta}{2}]$ and $y \in A$. It follows from (8) that

$$\Big|\inf_{x \in A, x \le y} T(x) - \inf_{x \in A, x \le y_0} T(x)\Big| < \varepsilon. \tag{9}$$

Note that

$$f(y) = \inf_{x \le y} T(x) = \min\Big\{\inf_{x < y_0 - \frac{\delta}{2}} T(x), \inf_{y_0 - \frac{\delta}{2} \le x \le y} T(x)\Big\}.$$

and

$$f(y_0) = \inf_{x \le y_0} T(x) = \min\Big\{\inf_{x < y_0 - \frac{\delta}{2}} T(x), \inf_{y_0 - \frac{\delta}{2} \le x \le y_0} T(x)\Big\}.$$

For shorthand, set $a := \inf_{x < y_0 - \frac{\delta}{2}} T(x)$, $b := \inf_{y_0 - \frac{\delta}{2} \le x \le y} T(x)$ and $c := \inf_{y_0 - \frac{\delta}{2} \le x \le y_0} T(x)$. Therefore $f(y) = \min\{a, b\}$ and $f(y_0) = \min\{a, c\}$. Using (9) we infer that $|b - c| < \varepsilon$, i.e. $-\varepsilon + c < b < \varepsilon + c$ which implies

$$-\varepsilon + \min\{a, c\} = \min\{a - \varepsilon, c - \varepsilon\} \le \min\{a, c - \varepsilon\} \tag{10}$$

and

$$\min\{a, c + \varepsilon\} \le \min\{a + \varepsilon, c + \varepsilon\} = \min\{a, c\} + \varepsilon. \tag{11}$$

Now (10) together with (11) imply that

$$-\varepsilon + \min\{a, c\} < \min\{a, b\} < \min\{a, c\} + \varepsilon,$$

so $|f(y) - f(y_0)| < \varepsilon$. This means that f is continuous. In a similar manner one can get $\inf_{x \ge y}\{T(x) - T(y)\}$ is continuous. □

Corollary 4.22. *Let $T : \mathbb{R} \to \mathbb{R}$ be a maximal θ-monotone operator such that* $\mathrm{int}(D(T)) \ni y \mapsto \theta(x, y)$ *is locally bounded from below for any $x \in D(T)$. Then $D(T) \ni x \mapsto \hat{\theta}_T(x, y)$ is continuous.*

Proof. Using Proposition 2.9 and Proposition 4.21, the proof is complete. □

Recall that, for a subset K of X, *the normal cone* $N_K : X \multimap X^*$ is defined by

$$N_K(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \quad \forall y \in K\} & x \in K, \\ \emptyset & x \notin K. \end{cases}$$

Lemma 4.23. Let $T : X \multimap X^*$ be maximal θ-monotone. Then for each $x \in D(T)$,

$$T(x) + N_{D(T)}(x) \subseteq T(x).$$

Proof. Assume that $x^* \in N_{D(T)}(z)$ and for $x \in D(T)$, the operator $T_1 : X \multimap X^*$ is defined by $T_1(z) := T(z) + \mathbb{R}_+ x^*$ and $T_1(x) := T(x)$, for $x \neq z$. Then $T(x) \subseteq T_1(x)$, for every $x \in D(T)$. If $z^* \in T(z)$, $y^* \in T(y)$ and $\lambda > 0$, we have

$$\langle z^* + \lambda x^* - y^*, z - y\rangle = \langle z^* - y^*, z - y\rangle + \lambda\langle x^*, z - y\rangle \geq \theta(z, y)\|z - y\|.$$

Hence T_1 is a θ-monotone operator. By Proposition 2.7(iii), we obtain that $T = T_1$. □

Here is a generalization of the Libor Vesely theorem which the other version of it can be found in [7, Theorem 3.14] for σ-monotone operators.

Theorem 4.24. Let $T : X \multimap X^*$ be a maximal θ-monotone operator and $\mathrm{cl}(D(T)) \ni x \mapsto \theta(x, y)$ be lower semicontinuous for any $y \in D(T)$. If T is locally bounded at $x_0 \in \mathrm{cl}(D(T))$, then $x_0 \in D(T)$. Furthermore, if $\mathrm{cl}(D(T))$ is convex, then $x_0 \in \mathrm{int}(D(T))$.

Proof. By assumption, there exists a neighborhood U of x_0 such that $T(U \cap D(T))$ is bounded. Choose $\{x_n\} \subseteq D(T) \cap U$ so that $x_n \to x_0$ and $x_n^* \in T(x_n)$. Applying the Banach-Alaoglu theorem [27, Theorem 3.15], there exist a subnet $\{(x_\alpha, x_\alpha^*)\}$ of $\{(x_n, x_n^*)\}$ and $x_0^* \in X^*$ such that $x_\alpha^* \xrightarrow{w^*} x_0^*$. Therefore for every $(y, y^*) \in \mathrm{gr}(T)$, we obtain

$$\langle x_0^* - y^*, x_0 - y\rangle = \lim_\alpha \langle x_\alpha^* - y^*, x_\alpha - y\rangle \geq \liminf_\alpha \theta(x_\alpha, y)\|x_\alpha - y\| \geq \theta(x_0, y)\|x_0 - y\|.$$

Then (x_0, x_0^*) is θ-monotonically related to all $(y, y^*) \in \mathrm{gr}(T)$. Hence, By Proposition 2.7(ii), $(x_0, x_0^*) \in gr(T)$. For the second part, it is enough to show that $U \subseteq \mathrm{int}(\mathrm{cl}(D(T)))$. In fact, if not, U contains a boundary point of $\mathrm{cl}(D(T))$. Using Bishop-Phelps theorem [32, Theorem 3.1.8], U contains a support point of $\mathrm{cl}(D(T))$, i.e., there exist $z \in U \cap \mathrm{cl}(D(T))$ and $0 \neq w^* \in X^*$ such that $\langle w^*, z\rangle = \sup\{\langle w^*, y\rangle : y \in \mathrm{cl}(D(T))\}$. Since T is locally bounded at z, by the first part of this theorem, $z \in D(T)$. On the other hand, $w^* \in N_{D(T)}(z)$ and hence $N_{D(T)}(z)$ is not equal to $\{0\}$. Lemma 4.23 implies that $T(z)$ is not bounded and this is a contradiction. Then $U \subseteq \mathrm{int}(\mathrm{cl}(D(T)))$. Since T is locally bounded on U, we have $U \subseteq D(T)$, so $x_0 \in \mathrm{int}(\mathrm{cl}(D(T)))$. □

Corollary 4.25. (Libor Vesely) [25, Theorem 1.14] Suppose that T is maximal monotone and $\mathrm{cl}(D(T))$ is convex. If $x \in \mathrm{cl}(D(T))$ and T is locally bounded at x, then $x \in \mathrm{int}(\mathrm{cl}(D(T)))$.

Here, we study some properties associated with local boundedness.

Proposition 4.26. Let $T : X \multimap X^*$ be a maximal θ-monotone operator and for each $y \in D(T)$, $\mathrm{cl}(D(T)) \ni x \mapsto \theta(x, y)$ be lower semicontinuous. Then T is norm×weak*upper semicontinuous in $\mathrm{int}(D(T))$.

Proof. Choose $y \in \mathrm{int}(D(T))$. It is enough to prove for every net $\{(y_\alpha, y_\alpha^*)\}$ in $\mathrm{gr}(T)$ provided with $y_\alpha \to y$ in X, there exists a weak*cluster point of $\{y_\alpha^*\}$ in $T(y)$ by [11, Theorem 2.1.8]. According to the Corollary 4.16, T is locally bounded at y. Hence we may assume that $y_\alpha^* \xrightarrow{w^*} y^*$ (choose a subnet if it is necessary). It follows from local boundedness of $\{y_\alpha^*\}$ that $\langle y_\alpha^*, y_\alpha\rangle \to \langle y^*, y\rangle$. By θ-monotonicity of T, for every $(x, x^*) \in \mathrm{gr}(T)$, we deduce that

$$\langle y_\alpha^* - x^*, y_\alpha - x\rangle \geq \theta(y_\alpha, x)\|y_\alpha - x\|.$$

Passing to the limit in the above inequality, we get $\langle y^* - x^*, y - x\rangle \geq \theta(y, x)\|y - x\|$. It follows that (y, y^*) is θ-monotonically related to all $(x, x^*) \in \mathrm{gr}(T)$. According to the Proposition 2.7(ii), $y^* \in T(y)$. □

Remark 4.27. In Proposition 4.26, when the space is reflexive and $D(T) = X$, by using $\theta(x, y) = \theta(y, x)$, one can present a shorter proof: Since T is maximal θ-monotone, by Theorem 2.9, $\mathrm{gr}(T)$ is sequentially norm×weak* closed. Hence, according to the second part of Proposition 4.20, T is norm×weak* upper semicontinuous.

Corollary 4.28. Suppose that $T : X \multimap X^*$ is maximal θ-monotone and for each $y \in D(T)$, $\mathrm{cl}(D(T)) \ni x \mapsto \theta(x, y)$ is lower semicontinuous. If X is finite dimensional, then the relation (3) can be written as

$$\hat{\theta}_T(y, x) = \inf\Big\{\frac{\langle x^* - y^*, x - y\rangle}{\|x - y\|} : x \neq y, \forall x^* \in T(x), y^* \in T(y)\Big\}.$$

Proof. Assume the sequence $\{(x_n, x_n^*)\} \subseteq \mathrm{gr}(T)$ such that $x_n \to y$ and $x_n \neq y$. By Proposition 4.20, $\{x_n^*\}$ is bounded. By selecting a subsequence (if necessary), let $x_n^* \to z^* \in T(y)$. Since

$$\inf\Big\{\frac{\langle x^* - y^*, x - y\rangle}{\|x - y\|} : x \neq y, \forall x^* \in T(x), y^* \in T(y)\Big\} \leq \frac{\langle x_n^* - z^*, x_n - y\rangle}{\|x_n - y\|} \leq \|x_n^* - z^*\| \to 0.$$

The proof is complete. □

Similar to [6, Proposition 3.16] and [7, Proposition 14], in the following result, we prove not only θ-monotone bifunctions are locally bounded, but also they are bounded by a small bound in a neighborhood of any interior point.

Proposition 4.29. Consider a θ-monotone bifunction $F : C \times C \to \mathbb{R}$ such that $F(x, x) = 0$, for each $x \in C$. Let $C \ni y \mapsto F(x, y)$ be lower semicontinuous and convex and $C \ni y \mapsto \theta(y, x)$ be lower semicontinuous, for all $x \in C$. If $x_0 \in \mathrm{int}(C)$, then there exist an open neighborhood V of x_0 and $K \in \mathbb{R}$ such that $F(y, x) \leq K\|x - y\|$, for every $x \in V$ and $y \in C$.

Proof. By hypothesis, $A^F(x) = \partial F(x, \cdot)(x)$, for all $x \in C$ and $\partial F(x, \cdot) \neq \emptyset$, for each $x \in \mathrm{int}(C)$. Therefore $\mathrm{int}(C) \subseteq D(A^F)$. According to Corollary 4.15 and Remark 4.9, A^F is locally bounded at x_0, i.e., there exist an open neighborhood $V_1 \subseteq C$ of x_0 and $K_1 \in \mathbb{R}$ such that $\|x^*\| \leq K_1$, for every $(x, x^*) \in (V_1 \times A^F)$. Since $\theta(\cdot, x)$ is lower semicontinuous at x_0, so it is bounded below on a neighborhood V_2 with lower bound K_2. Hence, for every $x \in V := V_1 \cap V_2$, $y \in C$ and $x^* \in A^F(x)$,

$$F(y, x) \leq -F(x, y) - \theta(y, x)\|y - x\| \leq -\langle x^*, y - x\rangle - K_2\|y - x\| \leq (K_1 - K_2)\|y - x\|,$$

where K_2 is a lower bound of $\theta(\cdot, x)$ and hence the proof is completed. □

5. Difference of two θ-monotone operators

Here, we are going to survey an important discussion of theory of monotone operators. Since difference of two θ-monotone operators is not necessarily θ-monotone, investigation of maximality of it is difficult. We study conditions under which difference of two θ-monotone operators is maximal θ-monotone operator.

Theorem 5.1. Let $S : X \multimap X^*$ be maximal θ-monotone and $T : X \multimap X^*$ be monotone. If $D(T) = X$ and $S - T$ is θ-monotone, then $S - T$ is maximal θ-monotone.

Proof. Let $(y, y^*) \in X \times X^*$ be θ-monotonically related to $\mathrm{gr}(S - T)$. For any $(x, x^*) \in \mathrm{gr}(S)$ and $(x, z^*) \in \mathrm{gr}(T)$, we get $(x, x^* - z^*) \in \mathrm{gr}(S - T)$. Then $\langle x^* - z^* - y^*, x - y\rangle \geq \theta(x, y)\|x - y\|$. By monotonicity of T and condition $D(T) = X$, there exists $t^* \in T(y)$ such that

$$\langle x^* - t^* - y^*, x - y\rangle = \langle x^* - z^* - y^*, x - y\rangle + \langle z^* - t^*, x - y\rangle \geq \theta(x, y)\|x - y\|.$$

It follows that $(y, y^* + t^*)$ is θ-monotonically related to $\mathrm{gr}(S)$. Maximality of S implies that $(y, y^* + t^*) \in \mathrm{gr}(S)$. Hence, by Proposition 2.7(ii), $(y, y^*) \in \mathrm{gr}(S - T)$, i.e., $S - T$ is maximal θ-monotone. □

The following example shows that the condition $D(T) = X$ in Theorem 5.1 is necessary.

Example 5.2. Define $T, S : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_-$ via

$$T(x) := \begin{cases} \{0\}, & \text{if } x = 0, \\ \emptyset, & \text{if } x \neq 0, \end{cases} \qquad S(x) := \begin{cases} \{0\}, & \text{if } x < 0, \\ [0, +\infty), & \text{if } x = 0, \\ \emptyset, & \text{if } x > 0, \end{cases}$$

and $\theta(x, y) := -|S(x) - S(y)|$ for every $x, y \in \mathbb{R}$, respectively. Then S is maximal θ-monotone, T is monotone and $S - T$ is θ-monotone but not maximal, since $\mathrm{gr}(S - T) = \{0\} \times \mathbb{R}$. Therefore, in the above theorem, the condition of $D(T) = X$ cannot be dropped.

In the following example, we observe that Theorem 5.1 and positive linearity is not necessary.

Example 5.3. Let $S : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_-$ be such that $S(x) := 2x$ for all $x \in \mathbb{R}$ and $\theta(x, y) := -|S(x) - S(y)|$ for any $x, y \in \mathbb{R}$. Suppose that the mapping $T : \mathbb{R} \to \mathbb{R}$ is defined by $T(x) := \frac{x}{2} + 1$, if $x \in (-\infty, 0)$ and $T(x) := x + 1$, otherwise. It is easy to see, S is maximal θ-monotone, T is monotone but it is not positive and not linear whiles $S - T$ is maximal θ-monotone.

The linear relation $T : X \multimap X^*$ is called a *skew linear relation* if $\langle x^*, x\rangle = 0$ for each $(x, x^*) \in \mathrm{gr}(T)$ [2].

Corollary 5.4. Let $S : X \multimap X^*$ be maximal θ-monotone, $T : X \multimap X^*$ be skew and linear and $D(T) = X$. Then $S \pm T$ is maximal θ-monotone.

Proof. Because T is skew linear relation, hence $-T$ is skew linear too. Then $\pm T$ is monotone and $S - (\pm T)$ is θ-monotone. Therefore $S \pm T$ is maximal θ-monotone by Theorem 5.1. □

According to the above corollary, the following result is clear.

Corollary 5.5. Let the operator $S : X \multimap X^*$ be maximal θ-monotone and $T : X \to X^*$ be skew linear. Then $S \pm T$ is maximal θ-monotone.

6. θ-convexity and θ-monotonicity

We start this section by recalling some important notions of subdifferential and introduce some preliminary notions and results. Then we investigate the notion of θ-convexity which covers concepts of ε-convexity [15] and σ-convexity [4].

Definition 6.1. [5, Definition 3.1] Given $\sigma : \mathrm{dom}\, f \to \mathbb{R}_+$, we say that function $f : X \to \overline{\mathbb{R}}$ is *σ-convex* if, for all $x, y \in \mathrm{dom}\, f$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) + \lambda(1-\lambda) \min\{\sigma(x), \sigma(y)\} \|x - y\|. \tag{12}$$

Definition 6.2. Let H be a real Hilbert space, D be an open and convex subset of H. A function $f : D \to \mathbb{R}$ is called

(i) [19, Definition 2.2.1] *θ-convex* if, for all $x, y \in D$ and and all $z \in (x, y)$ we have

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} + \theta(x, z) + \theta(z, y) \leq 0. \tag{13}$$

If in (13) we replace $\theta(x, z) + \theta(z, y)$ with $\theta(x, y)$, in this case a new notion of convexity defined by means of the function θ, the so called weak θ-convexity is obtained.

(ii) [19, Definition 2.2.2] *weak θ-convex* if, for all $x, y \in D$ and all $z \in (x, y)$ we have

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} + \theta(x, y) \leq 0. \tag{14}$$

It can easily be observed that (14) is equivalent to

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) - \lambda(1 - \lambda)\theta(x, y)\|x - y\|$$

for all $x, y \in D$ and $\lambda \in [0, 1]$.

Definition 6.3. *For a proper function $f : X \to \overline{\mathbb{R}}$ and $x, z \in X$, we define*

(i) the Clark-Rockafellar generalized directional derivative at $x \in \operatorname{dom} f$ in the direction $z \in X$ via

$$f^{\uparrow}(x, z) := \sup_{\delta > 0} \left(\limsup_{(y,\alpha) \xrightarrow{f} x, \lambda \searrow 0} \inf_{u \in B(z,\delta)} \frac{f(y + \lambda u) - \alpha}{\lambda} \right),$$

where $(y, \alpha) \xrightarrow{f} x$ means that $y \to x$, $\alpha \to f(x)$ and $\alpha \geq f(y)$.

(ii) the Clark-Rockafellar subdifferential of f at $x \in \operatorname{dom}(f)$ via

$$\partial^{CR} f(x) := \{x^* \in X^* : \langle x^*, z\rangle \leq f^{\uparrow}(x, z) \ \ \forall z \in X\}.$$

(iii) the Clark directional derivative at $x \in \operatorname{dom} f$ in the direction $z \in X$ by

$$f^{o}(x, z) := \limsup_{y \to x, \lambda \searrow 0} \frac{f(y + \lambda z) - f(y)}{\lambda}.$$

(iv) the Clark's subdifferential of f at $x \in \operatorname{dom}(f)$ by

$$\partial^{C} f(x) = \{x^* \in X^* : \langle x^*, z\rangle \leq f^{o}(x, z) \ \ \forall\, z \in X\}.$$

Remark 6.4. *If f is lower semicontinuous at $x \in \operatorname{dom} f$, then the Clark-Rockafellar generalized directional derivative at x in the direction $z \in X$ reduces to*

$$f^{\uparrow}(x, z) = \sup_{\delta > 0} \left(\limsup_{y \xrightarrow{f} x, \lambda \searrow 0} \inf_{u \in B(z,\delta)} \frac{f(y + \lambda u) - f(y)}{\lambda} \right),$$

where $y \xrightarrow{f} x$ means that $y \to x$ and $f(y) \to f(x)$. Moreover, if f is locally Lipschitz, then $f^{\uparrow}(x, z) = f^{o}(x, z)$.

Theorem 6.5. (Lebourg's Mean Value Theorem) [13, Theorem 9.5(i)] *Let C be a nonempty, closed and convex subset of X and $f : C \to \mathbb{R}$ be a locally Lipschitz function. Then for any $x, y \in C$ there exist $z \in (x, y)$ and $x^* \in \partial^C f(z)$ such that $f(x) - f(y) = \langle x^*, x - y\rangle$.*

Through this section, $f : X \to \overline{\mathbb{R}}$ is a function and $\theta : \operatorname{dom} f \times \operatorname{dom} f \to \mathbb{R}_-$ is a bifunction satisfying $\theta(x, y) = \theta(y, x)$ for all $x, y \in \operatorname{dom} f$. We say that f is *θ-convex* if

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) - \lambda(1 - \lambda)\theta(x, y)\|x - y\| \tag{15}$$

for all $x, y \in \operatorname{dom} f$ and $\lambda \in [0, 1]$.

The following results are generalizations of σ-convexity which is discussed in [4].

Lemma 6.6. Suppose that $f : X \to \overline{\mathbb{R}}$ is lower semicontinuous and θ-convex. If $X \ni y \mapsto \theta(x, y)$ is lower semicontinuous, for each $x \in X$, then

$$\partial^{CR} f(x) \subseteq \Big\{x^* \in X^* : \langle x^*, z\rangle \leq f(x+z) - f(x) - \theta(x, z+x)\|z\| \ \ \forall\, z \in X\Big\}.$$

Proof. By θ-convexity of f, for each $y, u \in X$ and $\lambda \in (0, 1)$ we get

$$f(y + \lambda u) \leq \lambda f(y + u) + (1 - \lambda) f(y) - \lambda(1 - \lambda)\theta(y + u, y)\|u\|.$$

Fix z and x in X. Set $u = z + x - y$. Then for each $\delta > 0$ and each $y \in B(x, \delta)$ we obtain

$$\begin{aligned}
\limsup_{y \xrightarrow{f} x, \lambda \searrow 0} \left(\inf_{u \in B(z,\delta)} \frac{f(y + \lambda u) - f(y)}{\lambda} \right) &\leq \limsup_{y \xrightarrow{f} x, \lambda \searrow 0} \frac{f(y + \lambda(z + x - y)) - f(y)}{\lambda} \\
&\leq \limsup_{y \xrightarrow{f} x, \lambda \searrow 0} [f(x + z) - f(y) - (1 - \lambda)\theta(x + z, y)\|x + z - y\|] \\
&\leq f(x + z) - f(x) - \theta(z + x, x)\|z\|.
\end{aligned}$$

Since f is lower semicontinuous and $\delta > 0$ is arbitrary, the above relation shows that

$$f^{\uparrow}(x, z) \leq f(x + z) - f(x) - \theta(x + z, x)\|z\|.$$

By the definition of the Clark-Rockafellar's subdifferential, the proof is complete. □

Proposition 6.7. Let $f : X \to \overline{\mathbb{R}}$ be lower semicontinuous and θ-convex. If $X \ni y \mapsto \theta(x, y)$ is lower semicontinuous, for every $x \in X$, then $\partial^{CR} f$ is 2θ-monotone.

Proof. Select $x, y \in X$, $x^* \in \partial^{CR} f(x)$ and $y^* \in \partial^{CR} f(y)$. Using Lemma 6.6, we get

$$\langle x^*, y - x\rangle \leq f(y) - f(x) - \theta(x, y)\|y - x\|,$$

and

$$\langle y^*, x - y\rangle \leq f(x) - f(y) - \theta(y, x)\|y - x\|.$$

Adding two above inequalities and applying the property that $\theta(x, y) = \theta(y, x)$, gives us $\partial^{CR} f$ is 2θ-monotone. □

Proposition 6.8. *Let C be a nonempty, closed and convex subset of X and $f : C \to \mathbb{R}$ be locally Lipschitz. If $X \ni y \mapsto \theta(x, y)$ is lower semicontinuous, for each $x \in C$ and $\partial^C f$ is θ-monotone then f is a θ-convex function.*

Proof. Assume that $\partial^C f$ is θ-monotone. Let $x_\lambda = \lambda x + (1 - \lambda) y$ with $\lambda \in (0, 1)$ and $x, y \in C$ where $x \neq y$. By Theorem 6.5, there exist $z_1 \in [x, x_\lambda)$ and $z_1^* \in \partial^C f(z_1)$ such that

$$\langle z_1^*, x_\lambda - x\rangle = f(x_\lambda) - f(x). \tag{16}$$

Similarly there exist $z_2 \in (x_\lambda, y]$ and $z_2^* \in \partial^C f(z_2)$ such that

$$\langle z_2^*, x_\lambda - y\rangle = f(x_\lambda) - f(y). \tag{17}$$

Since $x_\lambda - x = (1 - \lambda)(y - x)$ and $x_\lambda - y = \lambda(x - y)$, multiplying (16) and (17) in λ and $1 - \lambda$, respectively and adding the new equalities, we obtain

$$\lambda f(x) + (1 - \lambda) f(y) - f(x_\lambda) = \lambda(1 - \lambda)\langle z_1^* - z_2^*, x - y\rangle.$$

Now θ-monotonicity of $\partial^C f$ implies that (15) is satisfied, i.e., f is a θ-convex function. □

Example 6.9. Let $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_-$ and $T, \sigma : \mathbb{R} \to \mathbb{R}$ be defined by $\theta(x, y) := -\min\{\sigma(x), \sigma(y)\}$,

$$T(x) := \begin{cases} x \sin^2 x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \quad \text{and} \quad \sigma(x) := \max\{T(x), \max_{z \leq x}(T(z) - T(x))\},$$

respectively. It follows from [6, Example 2.8] that T is θ-monotone. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \int_0^x t \sin^2 t dt, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

According to Proposition 6.8, f is θ-convex. In [5, Example 3.7], it is shown that f is σ-convex and also it is not ε-convex (see [23, Theorem 4.4]).

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