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Further Results on Finite-Time Stability of Neutral Nonlinear Multi-Term Fractional Order Time-Varying Delay Systems

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Abstract. In this paper, the finite-time stability for nonlinear neutral multi-term fractional order systems with time-varying input and state delays is investigated. By use of the generalized Gronwall inequality and extended form of the generalized Gronwall inequality, new sufficient conditions for finite-time stability of such systems are obtained. Finally, numerical examples are given to illustrate the effectiveness and applicability of the proposed theoretical results.

1. Introduction

In this contribution, we consider system stability in the non-Lyapunov sense-*finite-time stability* (FTS) because the boundedness properties of system responses are very important from the engineering point of view, [1]. In the past decades, there has been a growing research interest in the field of stability and stabilization of time-delay systems which often leads to poor performance or even instability, [2-4]. Also, in the past four decades, applications of fractional (non-integer) calculus have attracted increasing attention of experts worldwide since they provide an excellent tool in modeling the complex dynamics, (for the description of memory and hereditary properties of various materials and processes), [5,6] and a lot of significant contributions have been made in non-integer (fractional) order control theory, [7,8]. In recent decades, stability problems of the non-integer time-delay systems (NITDS) have extensively been studied by using methods of the (generalized) Gronwall inequality, linear matrix inequalities, the Lyapunov method, the Holder inequality, the comparison principles, [9-13].

Here, we are interested in FTS where FTS analysis of NITDS is initially investigated and presented in [14,15] using (generalized) Gronwall inequality. In this context, several researchers have investigated FTS of NITDS, and presented their results, see [9,10,16-20].

Particularly, some authors have devoted attention to stability and control issues of the neutral TDS (NTDS) integer and fractional order [21-30]. Integer order NTDS in mechanical problems were presented in [21,22]; the stability chart of an elastic beam was obtained in [21] and the problem of ship rolling with control based on values of delayed acceleration was considered in [22]. Also, in [23] the delayed acceleration feedback control has been applied for chatter suppression in turning machines. The human self-balancing models have been modeled as integer order neutral TDS due to stabilizing time-delayed feedback control which depends on position, velocity and acceleration [24-26]. Moreover, the generalized Scot-Blair model

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has been studied [27], which can be modeled as neutral NITDS where a viscoelastic material is used as damping in vibration systems, on the assumption that the damping is proportional to a fractional rder derivative of the displacement variable with the non-integer order of derivative $0 < \alpha < 1$.

Recently, a few results have been obtained for the neutral NITDS with different fractional orders, [31]. In [32] authors studied FTS of linear neutral NITDS by using the method of steps. Also, in [33] FTS analysis of homogeneous NITDS with nonlinear perturbation based on generalized Bellman-Gronwall inequality have been investigated. In [34] the authors obtained sufficient conditions for FTS of the neutral NITDS two-term fractional order $0 < \mu \leq \lambda < 1$ with Lipschitz nonlinearities using the method of steps and the more generalized Gronwall inequality. Additionally, in [35] authors considered FTS of generalized neutral NITDS with fractional orders $0 < \beta \leq \alpha < 1$. Also, we have obtained a new criterion which is related to robust FTS of uncertain neutral nonhomogeneous NITDS $0 < \beta \leq \alpha < 1$ with time-varying input and state delay, [36]. In the meantime, the FTS of a class multiterm nonlinear fractional system with multistate time delay and $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2$ have been studied in [37,38].

Based on the above motivations and discussions, first time in this paper we shall address the FTS problem of nonlinear neutral NITDS with time-varying input and state delay and multi-term fractional order $0 < \gamma \leq 1 < \beta < \alpha \leq 2$ using the generalized Gronwall inequality and extended form of the generalized Gronwall inequality. The main contributions and features of this paper can be stated as follows.

(1) There are a few works of FTS for multi-term fractional order nonlinear systems. It is more essential to study the FTS of NITDS with damping behavior and time delay effects. Thus, a novel generalized neutral NITDS with three different fractional orders $0 < \gamma \leq 1 < \beta < \alpha \leq 2$ with time-varying input and state delays is studied, where for the first time we consider a case of multi-term neutral NITDS which includes delay terms ${}^cD_t^{\beta}\mathbf{x}(t-\tau_{xN1}(t))$, ${}^cD_t^{\gamma}\mathbf{x}(t-\tau_{xN2}(t))$ at the same time, $0 < \gamma \leq 1 < \beta < \alpha \leq 2$.
(2) Three novel criteria of FTS of neutral NITDS with multi-term fractional order $0 < \gamma \leq 1 < \beta < \alpha \leq 2$ with time-varying input and state delays are obtained by use of the generalized Gronwall inequality and extended form of the generalized Gronwall inequality.
(3) There are delay terms in the obtained novel FTS criteria of neutral NITDS with multi-term different fractional orders. Therefore, the proposed criteria in this paper are more general.
(4) Two numerical examples are presented to illustrate the correctness of the obtained results.

The rest of the paper is arranged as follows. Some basic concepts with properties of fractional calculus and problem descriptions are presented in Section 2. In Section 3, sufficient conditions ensuring the FTS of neutral NITDS are obtained and new criteria for FTS of NITDS with multi-term fractional order $0 < \gamma \leq 1 < \beta < \alpha \leq 2$ are given. In Section 4, two examples are provided to illustrate the validity of the obtained results. In Section 5, some conclusions are drawn.

2. Preliminaries and problem description

2.1. Preliminaries

In this subsection, some basic notations and definitions including the definition of Caputo fractional derivative are given. In this paper, the norm $\|(\cdot)\|$ will denote any vector norm, i.e. $\|(\cdot)\|_1$, $\|(\cdot)\|_2$, or $\|(\cdot)\|_\infty$ or the corresponding matrix norm induced by the equivalent vector norm, i.e. $1-$, -2, or $\infty-$ norm, respectively. Throughtout this paper, ${}_a^CD_t^{\alpha}f(t)$ or ${}_aD_t^{\alpha}f(t)$ denote Caputo's derivative of fractional order α with the lower limit a for function $f(\cdot)$, ${}_a^{RL}D_t^{-\alpha}f(t)$ or ${}_aI_t^{\alpha}f(t)$ denote an integral of order α with the lower limit a for function $f(\cdot)$.

Definition 2.1. *The gamma function $\Gamma(\cdot)$ known as Euler's gamma function is defined as*

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\,\mathrm{d}t, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha \in C \tag{1}$$

where C is the set of complex numbers. The Caputo fractional derivative is defined for a function $f(\cdot) : [a,b] \to C$ *that belongs to the space of absolutely continuous functions:*

$$f(t) \in AC^n[a,b] = \left\{ f(t) : \mathrm{d}^{n-1} f(t)/\mathrm{d}t^{n-1} \right\}, \quad n \in N$$

Definition 2.2. *The Caputo fractional derivative of order* α, $\alpha \in C$, $\mathrm{Re}(\alpha) \geq 0$, *for any function* $f(t) \in AC^n[a,b]$ *is defined as [39]:*

$$ {}_a^C D_t^{-\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int\limits_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, & \alpha \neq N_0, \quad n = [\mathrm{Re}(\alpha)] + 1, \ n \in N, \\ f^{(n)}(t) = \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \alpha = n \in N_0. \end{cases} \tag{2}$$

Definition 2.3. *Let* $f(t)$ *be a continuous function on* $[a,b]$ *The Riemann–Liouville (RL) fractional integral of order* α *is [39]:*

$$ {}_a^{RL} D_t^{-\alpha} f(t) \equiv {}_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \in [a,b], \quad \alpha \in C, \quad t > 0, \mathrm{Re}(\alpha) > 0. \tag{3}$$

Definition 2.4. *The Mittag-Leffler function with one parameter is given as [39]:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} z^k / \Gamma(k\alpha + 1) \quad \textit{where} \quad \alpha > 0, z \in C \tag{4}$$

and the two-parameter Mittag-Leffler function is presented as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} z^k / \Gamma(k\alpha + \beta) \quad \textit{where} \quad \alpha > 0, \beta > 0, z \in C. \tag{5}$$

Remark 2.1. *When* $\beta = 1$ *we have* $E_{\alpha,1}(z) = E_\alpha(z)$, *especially* $E_1(z) = e^z$.

The following lemmas are introduced and help prove our main stability criterion.

Lemma 2.1. *[35] Assume* $x(t) \in C^1([0,+\infty), R)$, $\dot{x}(t) \geq 0$ *and* $\alpha > 0$. *Then,* $\int\limits_0^t ((t-s)^{\alpha-1}/\Gamma(\alpha))\, x(s)\,\mathrm{d}s$ *is monotonically increasing with respect to t.*

Lemma 2.2. *[39]*

$$ {}_0 I_t^\alpha \left({}^c D_0^\alpha x(t) \right) = x(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} x^{(k)}(0), \quad n-1 < \alpha < n, \ t > 0. \tag{6}$$

Here, when $1 < \alpha < 2$, *it follows*

$$ {}_0 I_t^\alpha \left({}^c D_0^\alpha x(t) \right) = x(t) - x(0) - t x'(0), \quad t > 0. \tag{7}$$

Lemma 2.3. *Let* $\alpha > \beta > 0$, $n - 1 < \beta < n$ *and* $x(t) \in AC^n[a,b]$. *Then*

$$ {}_0 I_t^\alpha \left({}^c D_0^\beta x(t) \right) = {}_0 I_t^{\alpha-\beta} x(t) - \sum_{k=0}^{n-1} \frac{t^{k+\alpha-\beta}}{\Gamma(\alpha-\beta+k+1)} x^{(k)}(0). \tag{8}$$

Proof: From the definition 2.2, we have

$$^{c}D_0^{\beta}x(t) = {}_0I_t^{n-\beta}\, {}^{c}D_0^{n}x(t). \tag{9}$$

Applying fractional operator ${}_0I_t^{\alpha}$ and taking into account the commutative property of RL integral, we obtain

$${}_0I_t^{\alpha}\, {}^{c}D_0^{\beta}x(t) = {}_0I_t^{\alpha}\, {}_0I_t^{n-\beta}\, {}^{c}D_0^{n}x(t) = {}_0I_t^{\alpha-\beta}\left({}_0I_t^{n}\, {}^{c}D_0^{n}x(t)\right). \tag{10}$$

Actually from Lemma 2.2 we note that

$${}_0I_t^{n}\, {}^{c}D_0^{n}x(t) = x(t) - \sum_{k=0}^{n-1} \frac{t^k x^{(k)}(0)}{\Gamma(k+1)}. \tag{11}$$

Then, we have

$$\begin{aligned} {}_0I_t^{\alpha}\, {}^{c}D_0^{\beta}x(t) &= {}_0I_t^{\alpha-\beta}\left(x(t) - \sum_{k=0}^{n-1} \frac{t^k x^{(k)}(0)}{\Gamma(k+1)}\right) = \\ &= {}_0I_t^{\alpha-\beta}x(t) - {}_0I_t^{\alpha-\beta}\left(\sum_{k=0}^{n-1} \frac{t^k x^{(k)}(0)}{\Gamma(k+1)}\right) = \\ &= {}_0I_t^{\alpha-\beta}x(t) - \sum_{k=0}^{n-1} \frac{t^{k+\alpha-\beta}}{\Gamma(\alpha-\beta+k+1)} x^{(k)}(0). \end{aligned} \tag{12}$$

The proof is complete.

Property 2.2. *Assume that* $0 < \gamma < 1 < \alpha < 2$. *Then*

$${}_0I_t^{\alpha}\left({}^{c}D_0^{\gamma}x(t)\right) = {}_0I_t^{\alpha-\gamma}x(t) - \frac{x(0)\cdot t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}, \quad t \geq 0. \tag{13}$$

Property 2.3. *Assume that* $0 < \beta < \alpha < 2$. *Then*

$${}_0I_t^{\alpha}\left({}^{c}D_0^{\beta}x(t)\right) = {}_0I_t^{\alpha-\beta}x(t) - \frac{x(0)\cdot t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{x(1)\cdot t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}, \quad t \geq 0. \tag{14}$$

Lemma 2.4. *[40] (generalized Gronwall inequality) Suppose* $x(t)$, $a(t)$ *are nonnegative and local integrable on* $0 \leq t < T$, $T \leq +\infty$ *and* $g(t)$ *is a nonnegative, nondecreasing continuous function defined on* $0 \leq t < T$, $g(t) \leq M = \text{const}$, $\alpha > 0$ *with*

$$x(t) \leq a(t) + g(t)\int_0^t (t-s)^{\alpha-1} x(s)\,\mathrm{d}s \tag{15}$$

on this interval. Then

$$x(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s)\right] \mathrm{d}s, \quad 0 \leq t < T. \tag{16}$$

Corollary 2.4. *Under the hypothesis of Lemma 2.4, let* $a(t)$ *be a nondecreasing function on* $[0, T]$. *Then it holds:*

$$x(t) \leq a(t) E_{\alpha}(g(t)\Gamma(\alpha)t^{\alpha}) \tag{17}$$

where $E_{\alpha}(z)$ *is the one-parameter Mittag-Leffler function (4).*

Lemma 2.5. *(extended form of the generalized Gronwall inequality), [41] Suppose non-integer orders* $\alpha > 0$, $\beta > 0$, $a(t)$ *is a nonnegative function locally integrable on* $[0, T]$, $g_1(t)$ *and* $g_2(t)$ *are nonnegative, nondecreasing, continuous functions defined on* $[0, T)$; $g_1(t) \le N_1$, $g_2(t) \le N_2$, $(N_1, N_2 = \text{const})$. *Suppose* $x(t)$ *is nonnegative and locally integrable on* $[0, T)$ *with*

$$x(t) \le a(t) + g_1(t) \int_0^t (t-s)^{\alpha-1} x(s)\,\mathrm{d}s + g_2(t) \int_0^t (t-s)^{\beta-1} x(s)\,\mathrm{d}s, \quad t \in [0, T). \tag{18}$$

Then,

$$x(t) \le a(t) + \int_0^t \sum_{n=1}^{\infty} [g(t)]^n \cdot \sum_{k=0}^{n} \frac{C_n^k [\Gamma(\alpha)]^{n-k} [\Gamma(\beta)]^k}{\Gamma((n-k)\alpha + k\beta)} (t-s)^{(n-k)\alpha + k\beta - 1} a(s)\,\mathrm{d}s, \quad t \in [0, T) \tag{19}$$

where $g(t) = g_1(t) + g_2(t)$ *and* $C_n^k = n(n-1)(n-2)\cdots(n-k+1)/k!$.

Corollary 2.5. *Under the hypothesis of Lemma 2.5, let* $a(t)$ *be a nondecreasing function on* $[0, T)$. *Then*

$$x(t) \le a(t) E_\kappa \left[g(t) \left(\Gamma(\alpha) t^\alpha + \Gamma(\beta) t^\beta \right) \right], \quad \kappa = \min(\alpha, \beta) \tag{20}$$

2.2. Problem description

Consider the following neutral multi-term fractional order system with time-varying input and state delay with nonlinear perturbation and disturbance, presented by the following equation:

$$\begin{aligned} {}^cD_t^\alpha x(t) &= A_0 x(t) + A_1 \mathbf{x}(t - \tau_x(t)) + A_{N_1} {}^cD_t^\beta \mathbf{x}(t - \tau_{xN_1}(t)) + A_{N_2} {}^cD_t^\gamma \mathbf{x}(t - \tau_{xN_2}(t)) + \\ &+ B_0 \mathbf{u}(t) + B_1 \mathbf{u}(t - \tau_u(t)) + f(t, x(t)) + Cw(t.) \end{aligned} \tag{21}$$

with associated continuous functions of initial state and input (control):

$$\mathbf{x}(t) = \mathbf{\Psi}_x(t),\ t \in [-\tau_{xm}, 0], \quad \mathbf{x}'(t) = \varphi_x(t),\ t \in [-\tau_{xm}, 0], \quad \mathbf{u}(t) = \mathbf{\Psi}_u(t),\ t \in [-\tau_{uM}, 0] \tag{22}$$

where $\tau_x(t)$ is the time varying state delay, $\tau_{xN(\cdot)}(t)$ is the neutral time varying delay, $\tau_u(t)$ is the time varying input delay and they are continuous functions satisfying (13):

$$\begin{aligned} &0 \le \tau_x(t) \le \tau_{xM}, \quad 0 \le \tau_{xM}(t) \le \tau_{xM}, \quad \forall t \in J = [t_0, t_0 + T], \quad t_0 \in \mathbb{R},\ T > 0, \\ &0 \le \tau_u(t) \le \tau_{uM}. \end{aligned} \tag{23}$$

For the sake of simplicity $\tau_{xN}(t) = \tau_{xN_1}(t) = \tau_{xN_2}(t)$, of system (21) is assumed in this contribution, where τ_{xM}, τ_{xN} and τ_{uM} are constants; τ_{xm} is defined to be $\max(\tau_{xM}, \tau_{xN})$ and t_0 is the initial time of observation of the system. ${}^CD_t^\alpha$, ${}^CD_t^\beta$, ${}^CD_t^\gamma$ denote Caputo fractional derivatives of order α, β, γ, $0 < \gamma \le 1 < \beta < \alpha \le 2$; $x(t) \in R^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input; $A_0, A_1, A_{N_1}, B_0, B_1$ and C are constant matrices with appropriate dimensions; $w(t) \in R^n$ is the disturbance vector, which has the upper bound as follows: $\|w(t)\| < \eta_w$, $\eta_w = \text{const} > 0$ $\forall t \in J$. $\mathbf{\Psi}_x(t) \in C([-\tau_{xm}, 0], R^n)$ is the initial function of $x(t)$ with the norm $\|\Psi_x\|_C = \sup_{-\tau \le \theta \le 0} \|\Psi_x(\theta)\|$, and $\varphi_x(t) \in C([-\tau_{xm}, 0], R^n)$ is the initial function of $x'(t) = \mathrm{d}x(t)/\mathrm{d}t$ with the norm $\|\varphi_x\|_C = \sup_{-\tau \le \theta \le 0} \|\varphi_x(\theta)\|$.

Here, it is assumed that the nonlinear perturbation $f : [0, T] \times R^n \times R^n \to R^n$ is Lipschitz continuous on $[0, T]$ and there exists a continuous function $l(t)$ such that

$$\|f(t, x(t))\| \le l(t) \|x(t)\| \tag{24}$$

for any $\forall t \in [0, T]$ and $f(t, 0) = (0, 0)^T$. Before proceeding further, the definitions of FTS will be given for nonhomogeneous system (21) with associated initial functions (22).

Definition 2.5. *[9,42]: The fractional neutral time-delay system given by nonhomogeneous state equation (21) satisfying initial conditions (22) is finite-time stable w.r.t.* $\{\delta, \varepsilon, \eta_u, \eta_0, t_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$ *if and only if:*

$$\left.\begin{matrix} \rho < \delta, \quad \|\boldsymbol{\Psi}_u\|_C < \eta_0, \\ \|\mathbf{u}(t)\|_C < \eta_u, \quad \forall t \in J, \end{matrix}\right\} \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \in J, \tag{25}$$

where $\rho = \max\{\|\Psi\|_C, \|\varphi\|_C\}$ *and* $\delta, \varepsilon, \eta_0, \eta_u$ *are positive constants.*

Definition 2.6. *[9,42]: The fractional neutral time-delay system given by nonhomogeneous state equation (21),* $(\mathbf{u}(t - \tau_u(t)) \equiv 0)$ *satisfying initial conditions (22) is finite-time stable w.r.t.* $\{\delta, \varepsilon, \eta_u, t_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$ *if and only if:*

$$\rho < \delta, \quad \|\mathbf{u}(t)\| < \eta_u, \quad \forall t \in J \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \in J, \tag{26}$$

where $\rho = \max\{\|\Psi\|_C, \|\varphi\|_C\}$ *and* $\delta, \varepsilon, \eta_u, \eta_0$ *are positive constants.*

Definition 2.7. *[9,40]: The fractional neutral time-delay system given by homogeneous state equation (21),* $(\mathbf{u}(t - \tau_u(t)) \equiv 0, \mathbf{u}(t) \equiv 0)$ *satisfying initial conditions (22) is finite-time stable w.r.t.* $\{\delta, \varepsilon, t_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$ *if and only if:*

$$\rho < \delta, \quad \forall t \in J \Rightarrow \|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \in J, \tag{27}$$

where $\rho = \max\{\|\Psi\|_C, \|\varphi\|_C\}$ *and* δ, ε *are positive constants.*

3. Main Results

In this section, using generalized Gronwall inequality including an extended form, new criteria for FTS of NITDS are derived.

Theorem 3.1. *The nonhomogeneous nonlinear neutral multi-term fractional order time varying delay system (21) satisfying initial conditions (22) is finite-time stable w.r.t.* $\{\delta, \varepsilon, \eta_u, \eta_0, t_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$ *if the following condition holds:*

$$\begin{aligned} &\left[1 + |t| + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \right] \cdot E_\kappa[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta} + \Gamma(\alpha-\gamma)t^{\alpha-\gamma})] E_\alpha(\mu_\Sigma t^\alpha) + \\ &+ \frac{\eta_{0u}|t^\alpha|}{\Gamma(\alpha+1)} + \frac{\eta_{01}\tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{\eta_{1u}|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)} + \frac{\eta_{0w}|t|^\alpha}{\Gamma(\alpha+1)} < \frac{\varepsilon}{\delta}, \quad \forall t \in J_0, \end{aligned} \tag{28}$$

where: $\|A_0\| = a_0$, $\|A_1\| = a_1$, $\|A_{N_1}\| = a_{n_1}$, $\|A_{N_2}\| = a_{n_2}$, $\|B_0\| = b_0$, $\|B_1\| = b_1$, $\|C\| = c$,

$$\begin{aligned} &\sup_{t\in[0,T]} (a_0 + l(t)) = \mu_0, \quad a_1 = \mu_1, \quad \mu_\Sigma = \mu_0 + \mu_1, \\ &\eta_{0u} = b_0\eta_u/\delta, \quad \eta_{0w} = c\eta_w/\delta, \quad \eta_{01} = b_1\eta_0/\delta, \quad \eta_{1u} = b_1\eta_u/\delta. \end{aligned} \tag{29}$$

Proof: Following the property of the non-integer order $0 < \gamma \le 1 < \beta < \alpha \le 2$ and applying the fractional integral ${}_0I_t^\alpha$ on the system (21), we have

$$\begin{aligned} &{}_0I_t^\alpha\left({}^cD_t^\alpha \mathbf{x}(t) - A_{N_1}{}^cD_t^\beta \mathbf{x}(t-\tau_{xN_1}(t)) - A_{N_2}{}^cD_t^\gamma \mathbf{x}(t-\tau_{xN_2}(t))\right) = \\ &= {}_0I_t^\alpha\left(A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_x(t)) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau_u(t)) + f(t, x(t)) + Cw(t)\right) \end{aligned} \tag{30}$$

Using Lemma 2.2 and Lemma 2.3, we can obtain solution for (30) in the form of the equivalent Volterra integral equation, where $t_0 = 0$ as:

$$\begin{aligned}\mathbf{x}(t) &= \psi_x(0) + t\varphi_x(0) - A_{N_2}\cdot \psi_x(-\tau_{xm})\frac{t^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t (t-s)^{\alpha-\gamma-1} A_{N_2}x(s-\tau_{xN}(s))\,\mathrm{d}s- \\ &- A_{N_1}\frac{\psi_x(-\tau_{xm})\cdot t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - A_{N_1}\frac{\varphi_x(-\tau_{xm})\cdot t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} A_{N_1}x(s-\tau_{xN}(s))\,\mathrm{d}s+ \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[A_0\mathbf{x}(s) + A_1\mathbf{x}(t-\tau_x(s)) + B_0\mathbf{u}(s) + B_1\mathbf{u}(s-\tau_u(s)) + f(t,x(s)) + Cw(s)\right]\mathrm{d}s\end{aligned} \tag{31}$$

Now, using the norm $\|(\cdot)\|$ on equation (30), we can obtain an estimate of the solution

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq \|\psi_x(0)\| + |t|\,\|\varphi_x(0)\| + \|A_{N_2}\|\,\|\psi_x(-\tau_{xm})\|\frac{|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}+ \\ &+ \frac{1}{\Gamma(\alpha-\gamma)}\int_0^t |(t-s)|^{\alpha-\gamma-1}\|A_{N_2}\|\,\|x(s-\tau_{xN}(s))\|\,\mathrm{d}s + \|A_{N_1}\|\,\|\psi_x(-\tau_{xm})\|\frac{|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+ \\ &+ \|A_{N_1}\|\,\|\varphi_x(-\tau_{xm})\|\frac{|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t |(t-s)|^{\alpha-\beta-1}\|A_{N_1}\|\,\|x(s-\tau_{xN}(s))\|\,\mathrm{d}s+ \\ &+ \int_0^t |(t-s)|^{\alpha-1}\left[A_0\mathbf{x}(s) + A_1\mathbf{x}(t-\tau_x(s)) + B_0\mathbf{u}(s) + B_1\mathbf{u}(s-\tau_u(s)) + f(s,x(s)) + Cw(s)\right]\mathrm{d}s.\end{aligned} \tag{32}$$

Also, we can obtain:

$$\begin{aligned}&\left\|A_0\mathbf{x}(s) + A_1\mathbf{x}(t-\tau_x(s)) + B_0\mathbf{u}(s) + B_1\mathbf{u}(s-\tau_u(s)) + f(s,x(s)) + Cw(s)\right\| \leq \\ &\leq a_0\|\mathbf{x}(t)\| + a_1\|\mathbf{x}(t-\tau_x(t))\| + b_0\|\mathbf{u}(t)\| + b_1\|\mathbf{u}(t-\tau_u(t))\| + c\|\mathbf{w}(t)\| + \|f(t,x(t))\| \leq \\ &\leq (a_0 + l(t))\|x(t)\| + a_1\|\mathbf{x}(t-\tau_x(t))\| + b_0\|u(t)\| + b_1\|\mathbf{u}(t-\tau_u(t))\| + c\|\mathbf{w}(t)\| = \\ &= \mu_0\|x(t)\| + \mu_1\|\mathbf{x}(t-\tau_x(t))\| + b_0\|\mathbf{u}(t)\| + b_1\|\mathbf{u}(t-\tau_u(t))\| + c\|\mathbf{w}(t)\|.\end{aligned} \tag{33}$$

Let $y(t) = \sup\limits_{\theta\in[-\tau_{xm},t]}\|x(\theta)\|$, [18]. For $\forall t^\bullet \in [0,t]$ the following conditions satisfy

$$\left\|x\left(t^\bullet - \tau_{x^*(t^\bullet)}\right)\right\| \leq y(t^\bullet),\quad \|x(t^\bullet)\| \leq \sup_{t^\bullet\in[t-\tau_{xm},t]}\left\{\ \left\|\ \mathbf{x}(t^\bullet)\right\|\right\} \leq y(t^\bullet) \tag{34}$$

Applying this inequality, expression (32) can be rewritten as follows:

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq \|\psi_x\|_C\left[1 + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C\left[|t| + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right]+ \\ &+ \frac{a_{n_1}}{\Gamma(\alpha-\beta)}\int_0^t |(t-s)|^{\alpha-\beta-1}y(s)\,\mathrm{d}s + \frac{a_{n_2}}{\Gamma(\alpha-\gamma)}\int_0^t |(t-s)|^{\alpha-\gamma-1}y(s)\,\mathrm{d}s+ \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)|^{\alpha-1}\left[(\mu_0+\mu_1)y(s) + b_0\|\mathbf{u}(s)\| + b_1\|\mathbf{u}(s-\tau_u(s))\| + c\|w(s)\|\right]\mathrm{d}s.\end{aligned} \tag{35}$$

In view of the $\|\mathbf{u}(s)\| < \eta_u$ conditions for $\|\mathbf{w}(s)\| < \eta_w$, one may rewrite the above inequality as

$$\begin{aligned}\|\mathbf{x}(t)\| \le \|\psi_x\|_C \left[1 + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C \left[|t| + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] + \\ + \frac{a_{n_1}}{\Gamma(\alpha-\beta)} \int_0^t |(t-s)|^{\alpha-\beta-1}\, y(s)\, \mathrm{d}s + \frac{a_{n_2}}{\Gamma(\alpha-\gamma)} \int_0^t |(t-s)|^{\alpha-\gamma-1}\, y(s)\, \mathrm{d}s + \\ + \frac{\mu_\Sigma}{\Gamma(\alpha)} \int_0^t |(t-s)|^{\alpha-1}\, y(s)\, \mathrm{d}s + \frac{b_0\, \eta_u\, |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\, \eta_0\, \tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{c\, \eta_w\, |t|^\alpha}{\Gamma(\alpha+1)} + \\ + \frac{b_1}{\Gamma(\alpha)} \int_{\tau_{uM}}^t \left|(t-s)^{\alpha-1}\right| \|\mathbf{u}(s-\tau_{uM})\|\, \mathrm{d}s,\end{aligned} \tag{36}$$

or

$$\begin{aligned}\|\mathbf{x}(t)\| \le \|\psi_x\|_C \left[1 + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C \left[|t| + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] + \\ + \frac{a_{n_1}}{\Gamma(\alpha-\beta)} \int_0^t |(t-s)|^{\alpha-\beta-1}\, y(s)\, \mathrm{d}s + \frac{a_{n_2}}{\Gamma(\alpha-\gamma)} \int_0^t |(t-s)|^{\alpha-\gamma-1}\, y(s)\, \mathrm{d}s + \\ + \frac{\mu_\Sigma}{\Gamma(\alpha)} \int_0^t |(t-s)|^{\alpha-1}\, y(s)\, \mathrm{d}s + \frac{b_0\, \eta_u\, |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\, \eta_0\, \tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{c\, \eta_w\, |t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \eta_u\, |t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}.\end{aligned} \tag{37}$$

For $\forall \theta \in [0, t]$ we have

$$\begin{aligned}\|\mathbf{x}(\theta)\| \le \|\psi_x\|_C \left[1 + \frac{a_{n_2}|\theta|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|\theta|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C \left[|\theta| + \frac{a_{n_1}|\theta|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] + \\ + \frac{a_{n_1}}{\Gamma(\alpha-\beta)} \int_0^\theta |s|^{\alpha-\beta-1}\, y(\theta-s)\, \mathrm{d}s + \frac{a_{n_2}}{\Gamma(\alpha-\gamma)} \int_0^\theta |s|^{\alpha-\gamma-1}\, y(\theta-s)\, \mathrm{d}s + \\ + \frac{\mu_\Sigma}{\Gamma(\alpha)} \int_0^\theta |s|^{\alpha-1}\, y(\theta-s)\, \mathrm{d}s + \frac{b_0\, \eta_u\, |\theta|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\, \eta_0\, \tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{c\, \eta_w\, |\theta|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 \eta_u\, |\theta-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}.\end{aligned} \tag{38}$$

Taking into account that the nonnegative function $y(t)$ is increasing, then functions $\int_0^t |s|^{\alpha-\gamma-1} y(t-s)\, \mathrm{d}s$. $\int_0^t |s|^{\alpha-\beta-1} y(t-s)\, \mathrm{d}s$, $\int_0^t |s|^{\alpha-1} y(t-s)\, \mathrm{d}s$, are increasing with respect to $t \ge 0$,

Lemma 3.1. *Therefore,* $\alpha > 0$, $\alpha-\beta > 0$, $\alpha-\gamma > 0$, $\theta^\alpha \le t^\alpha$, $\theta^{\alpha-\beta} \le t^{\alpha-\beta}$, $\theta^{\alpha-\gamma} \le t^{\alpha-\gamma}$, *it follows*

$$\int_0^\theta |s|^\omega\, y(\theta-s)\, \mathrm{d}s \le \int_0^t |s|^\omega\, y(t-s)\, \mathrm{d}s, \quad \omega = (\alpha-1, \alpha-\beta-1, \alpha-\gamma-1) \tag{39}$$

i.e.

$$\begin{aligned}\|\mathbf{x}(\theta)\| \leq \|\psi_x\|_C \left[1 + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C \left[|t| + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] + \\ + \frac{a_{n_1}}{\Gamma(\alpha-\beta)} \int_0^t |s|^{\alpha-\beta-1}\, y(t-s)\,\mathrm{d}s + \frac{a_{n_2}}{\Gamma(\alpha-\gamma)} \int_0^t |s|^{\alpha-\gamma-1}\, y(t-s)\,\mathrm{d}s + \\ + \frac{\mu_\Sigma}{\Gamma(\alpha)} \int_0^t |s|^{\alpha-1}\, y(t-s)\,\mathrm{d}s + \frac{b_0\eta_u\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\eta_0\,\tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{c\eta_w\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\eta_u\,|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}.\end{aligned} \tag{40}$$

Also, we get

$$\begin{aligned}y(t) = \sup_{\theta\in[-\tau_{xm},t]} \|x(\theta)\| \leq \max\left\{\sup_{\theta\in[-\tau_{xm},0]} \|x(\theta)\|, \sup_{\theta\in[0,t]} \|x(\theta)\|\right\} \leq \\ \leq \max\Bigg\{\|\psi_x\|_C, \|\psi_x\|_C \left[1 + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C \left[|t| + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] + \\ + \frac{a_{n_1}}{\Gamma(\alpha-\beta)} \int_0^t |(t-s)|^{\alpha-\beta-1}\, y(s)\,\mathrm{d}s + \frac{a_{n_2}}{\Gamma(\alpha-\gamma)} \int_0^t |(t-s)|^{\alpha-\gamma-1}\, y(s)\,\mathrm{d}s + \\ + \frac{\mu_\Sigma}{\Gamma(\alpha)} \int_0^t |(t-s)|^{\alpha-1}\, y(s)\,\mathrm{d}s + \frac{b_0\eta_u\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\eta_0\,\tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{c\eta_w\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\eta_u\,|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}\Bigg\} = \\ = \|\psi_x\|_C \left[1 + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C \left[|t| + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] + \\ + \frac{a_{n_1}}{\Gamma(\alpha-\beta)} \int_0^t |(t-s)|^{\alpha-\beta-1}\, y(s)\,\mathrm{d}s + \frac{a_{n_2}}{\Gamma(\alpha-\gamma)} \int_0^t |(t-s)|^{\alpha-\gamma-1}\, y(s)\,\mathrm{d}s + \\ + \frac{\mu_\Sigma}{\Gamma(\alpha)} \int_0^t |(t-s)|^{\alpha-1}\, y(s)\,\mathrm{d}s + \frac{b_0\,\eta_u\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\,\eta_0\,\tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{c\,\eta_w\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\,\eta_u\,|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}.\end{aligned} \tag{41}$$

Now, we introduce $e(t)$ *which is a nondecreasing function on* $J_0 = [0, T]$

$$e(t) = \|\psi_x\|_C \left[1 + \frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} + \frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] + \|\varphi_x\|_C \left[|t| + \frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] \tag{42}$$

From Lemma 2.5 we obtain:

$$\begin{aligned}y(t) \leq e(t)\, E_\kappa \left[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta} + \Gamma(\alpha-\gamma)t^{\alpha-\gamma})\right] + \\ + \frac{\mu_\Sigma}{\Gamma(\alpha)} \int_0^t |(t-s)|^{\alpha-1}\, y(s)\,\mathrm{d}s + \frac{b_0\,\eta_u\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\,\eta_0\,\tau_{uM}^\alpha}{\Gamma(\alpha+1)} + \frac{c\,\eta_w\,|t|^\alpha}{\Gamma(\alpha+1)} + \frac{b_1\,\eta_u\,|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}\end{aligned} \tag{43}$$

where $g(t) = g_1(t) + g_2(t)$, $g_1 = \dfrac{a_{n1}}{\Gamma(\alpha-\beta)}$, $g_2 = \dfrac{a_{n2}}{\Gamma(\alpha-\gamma)}$ *and* $\kappa = \min(\alpha-\gamma,\ \alpha-\beta)$*. Now, applying Lemma 2.4, we* *have:*

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq y(t) \leq e(t)E_\kappa\left[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha-\gamma)t^{\alpha-\gamma})\right]E_\alpha(\mu_\Sigma\, t^\alpha)+\\ &+\frac{b_0\,\eta_u\,|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\,\eta_0\,\tau_{uM}^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\,\eta_u\,|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}+\frac{c\,\eta_w\,|t|^\alpha}{\Gamma(\alpha+1)}\leq\\ &\leq \varrho\left[1+|t|+\frac{a_{n_1|t|^{\alpha-\beta}}}{\Gamma(\alpha-\beta+1)}+\frac{a_{n_1|t|^{\alpha-\beta+1}}}{\Gamma(\alpha-\beta+2)}+\frac{a_{n_2|t|^{\alpha-\gamma}}}{\Gamma(\alpha-\gamma+1)}+\right]\cdot\\ &E_\kappa\left[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha-\gamma)t^{\alpha-\gamma})\right]E_\alpha(\mu_\Sigma\, t^\alpha)+\\ &+\frac{b_0\,\eta_u\,|t|^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\,\eta_0\,\tau_{uM}^\alpha}{\Gamma(\alpha+1)}+\frac{b_1\,\eta_u\,|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)}+\frac{c\,\eta_w\,|t|^\alpha}{\Gamma(\alpha+1)}.\end{aligned} \tag{44}$$

Finally, using the basic condition of Theorem 3.1, we can obtain the required FTS condition:

$$\|\mathbf{x}(t)\| < \varepsilon, \quad \forall t \in J. \tag{45}$$

From Theorem 3.1, we obtain the following result.

Theorem 3.2. *The homogeneous system given by (21), when* $\mathbf{u}(t) \equiv 0$, $\mathbf{u}(t-\tau_u) \equiv 0\, \forall t \in J_0$ *without perturbations and disturbance* $f(t,x(t)) \equiv 0$, $w(t) \equiv 0$ *satisfying function of the initial state (22) is finite-time stable w.r.t.* $\{\delta, \varepsilon, J_0, \|(\cdot)\|\}$, $\delta < \varepsilon$, *if the following condition is satisfied:*

$$\begin{aligned}&\left[1+|t|+\frac{a_{n_1}|t|^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{a_{n_1}|t|^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}+\frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right]\cdot\\ &E_\kappa\left[g(t)(\Gamma(\alpha-\beta)t^{\alpha-\beta}+\Gamma(\alpha-\gamma)t^{\alpha-\gamma})\right]E_\alpha(\mu_\Sigma\, t^\alpha) < \frac{\varepsilon}{\delta}, \quad \forall t \in J_0.\end{aligned} \tag{46}$$

Proof: The proof immediately follows from the proof of previous Theorem 3.1.

Theorem 3.3. *The nonhomogeneous nonlinear neutral two-term fractional order time varying delay system without term* $A_{N_1}{}^cD_t^\beta\,\mathbf{x}(t-\tau_{xn_1}(t))$, *(i.e* $A_{N_1}=0$*) given by (47) with satisfying function of the initial state (22) is finite-time stable w.r.t.* $\{\delta, \varepsilon, \eta_u, \eta_0, t_0, J, \|(\cdot)\|\}$, $\delta < \varepsilon$, *if the following condition is satisfied, (48):*

$$\begin{aligned}{}^cD_t^\beta\,\mathbf{x}(t) &= A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau_x(t))+A_{N_2}{}^cD_t^\gamma\,\mathbf{x}(t-\tau_{t-\tau_{xN_2}}(t))+\\ &+B_0\,\mathbf{u}(t)+B_1\,\mathbf{u}(t-\tau_u(t))+f(t,x(t))+Cw(t)\end{aligned} \tag{47}$$

$$\begin{aligned}&\left[1+|t|+\frac{a_{n_2}|t|^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\right]E_\kappa\left[\left(\frac{a_{n_2}}{\Gamma(\alpha-\gamma)}+\frac{\mu_\Sigma}{\Gamma(\alpha)}\right)(\Gamma(\alpha-\gamma)t^{\alpha-\gamma}+\Gamma(\alpha)t^\alpha)\right]+\\ &+\frac{\eta_{0u}\,|t|^\alpha}{\Gamma(\alpha+1)}+\frac{\eta_{01}\,\tau_{uM}^\alpha}{\Gamma(\alpha+1)}+\frac{\eta_{0w}\,|t|^\alpha}{\Gamma(\alpha+1)}+\frac{\eta_{1u}|t-\tau_{uM}|^\alpha}{\Gamma(\alpha+1)} < \frac{\varepsilon}{\delta}, \quad \forall t \in J_0,\end{aligned} \tag{48}$$

where $\kappa = \min(\alpha-\gamma, \alpha)$.

Proof: Similar to the proof of Theorem 3.1 with applying only the extended form of generalized Gronwall inequality we get the proposed result (48) of Theorem 3.3.

Remark. The system (47) can be reduced to [37], ((9) for case $n = 1$, $\tau_{N_1} = 0$, $A_{N_2} = B_1 = 0$, $C = 0$) and [38], (1) assuming that $\tau_{N_1} = 0$, $A_{N_2} = B_1 = 0$, $C = 0$, $f = 0$. It is easily check that obtained criteria (29), (48) are more general.

4. Numerical examples

In this section, to demonstrate the effectiveness of the previously obtained FTS results, a nonhomogeneous nonlinear NITDS with disturbance (49) is considered. Here all notation $\|(\cdot)\|$ means the $\infty-$ norm of a matrix or a vector.

Example 4.1. *Consider the nonlinear neutral NITDS (21) with time-varying input and state delay and multi-term non-integer order* $0 < \gamma \le 1 < \beta < \alpha \le 2$

$$
\begin{aligned}
{}^cD_t^{\alpha}\,\mathbf{x}(t) &= A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_x(t)) + A_{N_1}{}^cD_t^{\beta}\mathbf{x}(t-\tau_{xN_1}(t)) + A_{N_2}{}^cD_t^{\gamma}\mathbf{x}(t-\tau_{t-\tau_{xN_2}}(t)) + \\
&+ B_0\,\mathbf{u}(t) + B_1\,\mathbf{u}(t-\tau_u(t)) + f(t,x(t)) + Cw(t)
\end{aligned}
\tag{49}
$$

where

$$
A_0 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.3 \end{bmatrix},\quad A_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix},\quad A_{N_1} = \begin{bmatrix} 0.3 & 0 \\ -0.05 & 0.2 \end{bmatrix},\quad A_{N_2} = \begin{bmatrix} 0.3 & -0.2 \\ 0.4 & 0.1 \end{bmatrix},
$$
where (50)
$$
B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},\quad B_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},\quad C = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},\quad f(t,x(t)) = \begin{bmatrix} 0.01\sin x_1(t) \\ 0.01\sin x_2(t) \end{bmatrix},\quad w(t)\sin(t)
$$

and $t_0 = 0$, $\alpha = 1.5$, $\beta = 1.1$, $\gamma = 0.5$, $\tau_x = \tau_u = 0.1$, $\tau_{xN_1} = \tau_{xN_2} = \tau_{xN} = 0.1$, $\tau_{xm} = 0.1$, *with associated functions:* $\psi_x = [0.05\,0.05]^T$*;* $t \in [-0.1, 0]$*;* $\varphi(t) = [0.07\,0.07]^T$, $\psi_u = 0.05$. *The task is to analyze the FTS with respect to* $\{\delta = 0.08,\ \varepsilon = 50,\ \eta_u = 1\}$. *From the initial functions and given state equation, we have:* $\|\psi_x\|_C = \max\limits_{t\in[-0.1,0]} \|\psi_x(t)\|_\infty = 0.05$, $\|\varphi_x\|_C = 0.07$, $\rho = \max\{\|\psi\|_C, \|\varphi\|_C\} = 0.07 < \delta = 0.08$, $\|A_0\| = 0.4$, $\|A_1\| = 0.3$, $\|A_{N_1}\| = 0.3$, $\|A_{N_2}\| = 0.5$, $\|B_0\| = 0.5$, $\|C\| = 0.5$, $l(t) = 0.01$, $\eta_w = 1.01$, $\eta_u = 1$, $\eta_0 = 0.06$. *Applying the condition of Theorem 3.1, we can obtain the estimated time of the FTS* $T_e \approx 0.8\,s$.

Example 4.2. *Consider the following homogeneous neutral NITDS with time-varying state delays multi-term non-integer order* $0 < \gamma \le 1 < \beta < \alpha \le 2$

$$
{}^cD_t^{\alpha}\,\mathbf{x}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_x(t)) + A_{N_1}{}^cD_t^{\beta}\mathbf{x}(t-\tau_{xN_1}(t)) + A_{N_2}{}^cD_t^{\gamma}\mathbf{x}(t-\tau_{t-\tau_{xN_2}}(t)) \tag{51}
$$

where

$$
A_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\quad A_1 = \begin{bmatrix} 0.1 & -0.3 \\ 0.2 & 0.1 \end{bmatrix},\quad A_{N_1} = \begin{bmatrix} -0.3 & 0 \\ 0.3 & 0.2 \end{bmatrix},\quad A_{N_2} = \begin{bmatrix} 0 & 0.1 \\ -0.2 & 0.2 \end{bmatrix} \tag{52}
$$

and $\alpha = 1.75$, $\beta = 1.5$, $\gamma = 0.75$, $\tau_x = 0.15$, $\tau_{xm} = 0.15$ $\tau_{xN_1} = \tau_{xN_2} = \tau_{xN} = 0.15$, *with associated functions:* $\psi_x = [0.03\,0.03]^T$, $t \in [-0.15,\ 0]$, $\varphi_x(t) = [0.05\,0.05]^T$. *Also, from the initial functions and given state equation, we calculate:* $\|\psi_C\|_C = 0.03$, $\|\varphi_C\|_C = 0.05$, $\rho = \max\{\|\psi_C\|_C, \|\varphi_C\|_C\} = 0.05 < \delta = 0.06$. *It is obviosly that* $\|A_0\| = 0.2$, $\|A_1\| = 0.4$, $\|A_{N_1}\| = 0.5$, $\|A_{N_2}\| = 0.4$. *If we take* $\delta = 0.06$ *and* $\varepsilon = 100$ *then condition (48) of Theorem 3.2 holds for* $T_e \approx 0.328\,s$ *so we can get the estimated time of the FTS.*

5. Conclusions

In this paper, FTS analysis for a class of (non)homogeneous nonlinear neutral multi-term fractional order systems $0 < \gamma \le 1 < \beta < \alpha \le 2$ with time-varying input and state delays has been investigated. By use of the extended form of generalized Gronwall inequality, new criteria for the FTS have been developed. Sufficient conditions for FTS for this class of neutral NITDS have been proposed. Finally, two numerical examples have been provided to illustrate the effectiveness and the benefit of the proposed novel stability criterion of FTS.

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