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Some Remarks on Star-Menger Spaces Using Box Products

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Abstract. This article is a continuation of study of star-Menger selection properties in line of (Kočinac, 2009, 2015), where we mainly use covers consisting of G_δ sets with certain additional condition. It is observed that star-Mengerness is equivalent to the fact that every such type of cover of a space has a countable subcover. We improve this result by considering 'subcovers of cardinality less than $\mathfrak{b}$' instead of 'countable subcovers', which is our primary observation. We also show that it is possible to produce non normal spaces using box products and dense star-Menger subspaces.

1. Introduction

The study of star selection principles was initiated in 1999 by Kočinac (see [8]). However, the study of selection principles was initiated by Borel [2], Menger [14], Hurewicz [5], Rothberger [15], and others. The systematic study of selection principles in topology was started by Scheepers [17] (see also [6]) and in the last twenty five years it has become one of the most active research areas of set theoretic topology. Various topological properties have been defined or characterized in terms of selection principles. The selection principles also have various applications in several branches of Mathematics. Nowadays, many authors have made investigations involving selection principles and star selection principles and interesting results have been obtained, see for instance [1, 7, 9–11, 11, 12, 16, 18–27], among other works. The star version of the Menger property [17], called the star-Menger property [8], plays a central role in this article.

In this paper we concentrate on a certain kind of covering of a space by G_δ sets to study star-Menger spaces. We first observe that if every such type of cover of a space X has a countable subcover, then this is equivalent to the star-Menger property of X (Theorem 3.4). Our primary concern is to investigate whether the above result holds if the cardinality of the subcover is replaced by larger cardinals. We give a partial answer to it using box products (Theorem 3.12). It is also observed that star operations can be used to produce non normal spaces using box products as well.

2. Preliminaries

Throughout the paper (X, τ) stands for a Hausdorff topological space. Let ω denote the first infinite ordinal, ω_1 denote the first uncountable ordinal and $\omega + 1$ $(= [0, \omega])$ denote the one point compactification

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of ω when ω ($= [0,\omega)$) viewed as a topological space with the order topology. For undefined notions and terminologies see [4].

If $\{X_\alpha : \alpha \in \Lambda\}$ is a family of spaces, then the box product $\times_{\alpha\in\Lambda} X_\alpha$ of these spaces is the set $\prod_{\alpha\in\Lambda} X_\alpha$ with basis consisting of all sets of the form $\prod_{\alpha\in\Lambda} U_\alpha$, where for each $\alpha \in \Lambda$, U_α is open in X_α. If for each $\alpha \in \Lambda$, $X_\alpha = X$, then we will use $\times_\Lambda X$ for $\times_{\alpha\in\Lambda} X_\alpha$. For results concerning box products in the context of classical selection principles, see [13, 28].

A space X is said to be Lindelöf if every open cover has a countable subcover. A space X is said to be para-Lindelöf if every open cover has a locally countable open refinement. We shall use the symbol $\mathcal{O}$ to denote the collection of all open covers of X. A space X is said to be Menger [17] if for each sequence $(\mathcal{U}_n)$ of open covers of X there is a sequence $(\mathcal{V}_n)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup_{n\in\omega}\mathcal{V}_n$ is an open cover of X.

For a subset A of a space X and a collection $\mathcal{P}$ of subsets of X, $St(A,\mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is the set $\cup\{B \in \mathcal{P} : A \cap B \neq \varnothing\}$. For $A = \{x\}$, $x \in X$, we write $St(x,\mathcal{P})$ instead of $St(\{x\},\mathcal{P})$ [4].

A space X is said to be star-Menger [8] if for each sequence $(\mathcal{U}_n)$ of open covers of X there is a sequence $(\mathcal{V}_n)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup_{n\in\omega}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an open cover of X.

The eventual dominance relation $\leq^*$ on the Baire space ω^ω is defined by $f \leq^* g$ if and only if $f(n) \leq g(n)$, for all but finitely many n. A subset A of ω^ω is said to be bounded if there is a $g \in \omega^\omega$ such that $f \leq^* g$, for all $f \in A$. The minimum cardinality of a unbounded subset of ω^ω is denoted by $\mathfrak{b}$.

Let X be a space, $A \subseteq X$ and κ be a regular cardinal. Then X is said to be κ-contcentrated on A, if for every open set U containing A, $|X \setminus U| < \kappa$. If $\kappa = \omega_1$, then we say that X is concentrated on A.

For a collection $\{X_n : n \in \omega\}$ of sets, let ρ be a relation on $\prod_{n\in\omega} X_n$ defined as follows. For any $x = (x_n)$, $y = (y_n) \in \prod_{n\in\omega} X_n$, we say that $_x\rho_y$ if and only if $\{n \in \omega : x_n \neq y_n\}$ is finite. Clearly ρ is an equivalence relation on $\prod_{n\in\omega} X_n$ and the set of all equivalence classes is denoted by $\prod_{n\in\omega} X_n/\rho$. For $x \in \prod_{n\in\omega} X_n$, the equivalence class containing x is denoted by $\rho(x)$. If $Y \subseteq \prod_{n\in\omega} X_n$, we define $\rho(Y) = \cup_{y\in Y}\rho(y)$. For a collection of topological spaces $\{X_n : n \in \omega\}$, let $\times_{n\in\omega} X_n/\rho$ denote the quotient space of $\times_{n\in\omega} X_n$ determined by ρ. If $Y_n \subseteq X_n$ for each n, then we call the product $\prod_{n\in\omega} Y_n \subseteq \prod_{n\in\omega} X_n$ a cylinder in $\times_{n\in\omega} X_n$. Moreover if for each n, Y_n is open in X_n, then we call $\prod_{n\in\omega} Y_n$ an open cylinder in $\times_{n\in\omega} X_n$.

3. Main Results

We start with a basic reformulation of the star-Menger property which will be used subsequently.

Lemma 3.1. *A space X is star-Menger if and only if for every sequence $(\mathcal{U}_n)$ of open covers of X there is a sequence $(\mathcal{V}_n)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m,\mathcal{U}_m)$.*

Proof. Let $(\mathcal{U}_n)$ be a sequence of open covers of X. Then for each n, $(\mathcal{U}_m)_{m\geq n}$ is a sequence of open covers of X. Since X is star-Menger, for each n, there is a sequence $(\mathcal{V}_m^{(n)})_{m\geq n}$ such that for each $m \geq n$, $\mathcal{V}_m^{(n)}$ is a finite subset of $\mathcal{U}_m$ and $X = \cup_{m\geq n}St(\cup\mathcal{V}_m^{(n)},\mathcal{U}_m)$. For each m, choose $\mathcal{V}_m = \cup_{n\leq m}\mathcal{V}_m^{(n)}$. Thus we get for each m, $\mathcal{V}_m$ is a finite subset of $\mathcal{U}_m$. Clearly for any n and $m \geq n$, $\cup\mathcal{V}_m^{(n)} \subseteq \cup\mathcal{V}_m$. It follows that $X = \cup_{m\geq n}St(\cup\mathcal{V}_m,\mathcal{U}_m)$, for all n and hence $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m,\mathcal{U}_m)$.

Conversely let $(\mathcal{U}_n)$ be a sequence of open covers of X. By the given hypothesis we have a sequence $(\mathcal{V}_n)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \cap_{n\in\omega}\cup_{m\geq n}St(\cup\mathcal{V}_m,\mathcal{U}_m)$. This gives $X = \cup_{m\geq n}St(\cup\mathcal{V}_m,\mathcal{U}_m)$, for all n and then $X = \cup_{n\in\omega}St(\cup\mathcal{V}_n,\mathcal{U}_n)$. This completes the proof. □

In the following result we show that using the star operation on a sequence of open covers, one can obtain a dominating subset of the Baire space.

Lemma 3.2. *Let X be a Lindelöf space which is not star-Menger. If $(\mathcal{U}_n)$ is a sequence of open covers of X which witnesses that X is not star-Menger, then using $\mathcal{U}_n$'s it is possible to construct a dominating subset of ω^ω.*

Proof. Since the sequence $(\mathcal{U}_n)$ of open covers of X witnesses that X is not star-Menger, by Lemma 3.1, we can say that for any sequence $(\mathcal{V}_n)$ with for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $X \neq \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m,\mathcal{U}_m)$.

Without loss of generality, we assume that $\mathcal{U}_n = \{U_m^{(n)} : m \in \omega\}$. For each $x \in X$, define $f_x : \omega \to \omega$ by $f_x(n) = \min\{m : x \in U_m^{(n)}\}$. Let $D = \{f_x : x \in X\}$. We claim that D is a dominating subset of ω^ω. Suppose that D is not dominating. Then there exists a $f \in \omega^\omega$ such that for every $f_x \in D$, $f_x(n) \leq f(n)$ for infinitely many n. For each n, choose $\mathcal{V}_n = \{U_m^{(n)} : m \leq f(n)\}$ and thus each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. We now show that $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$. Fix $n \in \omega$ and let $x \in X$. Since $f_x(m) \leq f(m)$ for infinitely many m, there exists m_0 such that $m_0 \geq n$ and $f_x(m_0) \leq f(m_0)$. It follows that $U_{f_x(m_0)}^{(m_0)} \in \mathcal{V}_{m_0}$. Thus $x \in St(\cup\mathcal{V}_{m_0}, \mathcal{U}_{m_0})$ and consequently $x \in \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$, for all n. This gives $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$, which is a contradiction. Hence D is dominating. □

Lemma 3.3. *Let $\{X_n : n \in \omega\}$ be a family of spaces, let X be a space and let $y = (y_n) \in \times_{n\in\omega} X_n$. If O is an open subset of $X \times \times_{n\in\omega} X_n$ containing $X \times \rho(y)$, then there exists an open cylinder V in $\times_{n\in\omega} X_n$ such that $X \times \rho(y) \subseteq X \times \rho(V) \subseteq O$.*

Proof. For each $n \in \omega$, let $U^{(n)} = X_0 \times X_1 \times \cdots \times X_{n-1} \times U_{y_n,n}^{(n)} \times U_{y_{n+1},n+1}^{(n)} \times \cdots$, where for each $n \in \omega$, $U_{y_n,n}^{(n)}$ is an open subset of X_n such that $y_n \in U_{y_n,n}^{(n)}$ and $X \times U^{(n)} \subseteq O$. Now, for each $n \in \omega$, let $V_n = \cap_{m\leq n} U_{y_n,n}^{(m)}$ and we put $V = \prod_{n\in\omega} V_n$. It is easy to verify that $X \times \rho(V) \subseteq O$. □

For a space X, we denote the collection of all star-Menger subspaces of X by $\mathcal{M}(X)$. Another reformulation of the star-Menger property, which of our primary concern, is the following.

Theorem 3.4. *For a space X the following assertions are equivalent.*

(1) *X is star-Menger.*

(2) *Every cover $\{G_M : M \in \mathcal{M}(X)\}$ of X, where G_M is a G_δ subset of X with $M \subseteq G_M$, has a countable subcover.*

Proof. We only give proof of (2) ⇒ (1). Let $(\mathcal{U}_n)$ be a sequence of open covers of X. Then for each $M \in \mathcal{M}(X)$, $(\mathcal{U}_n)$ is a sequence of covers of M by open sets in X. Apply the star-Menger property of M to $(\mathcal{U}_n)$ to obtain a sequence $(\mathcal{V}_n^{(M)})$ such that for each n, $\mathcal{V}_n^{(M)}$ is a finite subset of $\mathcal{U}_n$ and $M \subseteq \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m^{(M)}, \mathcal{U}_m)$. For each M, let $G_M = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m^{(M)}, \mathcal{U}_m)$ and consequently $\{G_M : M \in \mathcal{M}(X)\}$ is a cover of X with each G_M is a G_δ subset of X containing M. By the given hypothesis we obtain a countable subfamily $\{G_{M_i} : i \in \omega\}$ of $\{G_M : M \in \mathcal{M}(X)\}$ that covers X. For each n, $\mathcal{V}_n = \cup_{i\leq n} \mathcal{V}_n^{(M_i)}$ is a finite subset of $\mathcal{U}_n$. We now show that $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$. Let $n \in \omega$ be fixed. Next we pick a $M_i \in \mathcal{M}(X)$ such that $G_{M_i} \in \{G_{M_i} : i \in \omega\}$. For $m \geq n + i$, $\cup\mathcal{V}_m^{(M_i)} \subseteq \cup\mathcal{V}_m$. It follows that $\cup_{m\geq n+i} St(\cup\mathcal{V}_m^{(M_i)}, \mathcal{U}_m) \subseteq \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$ and subsequently $G_{M_i} \subseteq \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$. This gives us $\cup_{i\in\omega} G_{M_i} \subseteq \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$ and hence $X = \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$. Since n was chosen arbitrarily, $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$. This completes the result. □

A space X is called a P-space if every G_δ set is open.

Corollary 3.5. *Let $f : X \to Y$ be a closed continuous mapping from X onto Y such that $f^{-1}(y)$ is star-Menger for each $y \in Y$. If Y is a Lindelöf P-space, then X is star-Menger.*

Proof. Let $\mathcal{A} = \{f^{-1}(y) : y \in Y\}$. So $\mathcal{A} \subseteq \mathcal{M}(X)$. For each $y \in Y$, let $G_{f^{-1}(y)}$ be a G_δ subset of X containing $f^{-1}(y)$. It is easy to see that $\{G_{f^{-1}(y)} : y \in Y\}$ is a cover of X. Since f is closed and Y is a P-space, $f(X \setminus G_{f^{-1}(y)})$ is closed in Y. Choose $V_{f^{-1}(y)} = Y \setminus f(X \setminus G_{f^{-1}(y)})$. It is immediate that for each $y \in Y$, $f^{-1}(y) \subseteq f^{-1}(V_{f^{-1}(y)}) \subseteq G_{f^{-1}(y)}$. Since $\{V_{f^{-1}(y)} : y \in Y\}$ is an open cover of Y, there exists a countable subfamily $\{V_{f^{-1}(y_n)} : n \in \omega\}$ of $\{V_{f^{-1}(y)} : y \in Y\}$ that covers Y. Consequently $\{f^{-1}(V_{f^{-1}(y_n)}) : n \in \omega\}$ is an open cover of X and it follows that $\{G_{f^{-1}(y_n)} : n \in \omega\}$ is a countable subfamily of $\{G_{f^{-1}(y)} : y \in Y\}$ that covers X. From this one can easily observe that every cover $\{G_M : M \in \mathcal{M}(X)\}$ of X, where G_M is a G_δ subset of X with $M \subseteq G_M$, has a countable subcover. By Theorem 3.4, X is star-Menger. □

Corollary 3.6. *If $X = \cup_{n\in\omega} X_n$, where each X_n is star-Menger, then X is star-Menger.*

It is interesting to ask the following question.

Problem 3.7. *What is the largest possible cardinal κ for which the following assertions are equivalent?*

(1) *X is star-Menger.*

(2) *Every cover $\{G_M : M \in \mathcal{M}(X)\}$ of X, where G_M is a G_δ subset of X with $M \subseteq G_M$, has a subcover of cardinality less than κ.*

In Theorem 3.12 we will provide a partial answer to this question. Before, we provide some results concerning about box products.

Theorem 3.8. *If $\{X_n : n \in \omega\}$ is a family of compact spaces and X is a star-Menger space, then the quotient mapping $q : X \times \times_{n\in\omega} X_n \to (X \times \times_{n\in\omega} X_n)/\rho$ is closed.*

Proof. The proof follows directly from Lemma 3.3. □

Corollary 3.9. ([13]) *If $\{X_n : n \in \omega\}$ is a family of compact spaces, then the quotient mapping $q : \times_{n\in\omega} X_n \to \times_{n\in\omega} X_n/\rho$ is closed.*

Lemma 3.10. ([3]) *Let $f : X \to Y$ be a closed continuous map from X onto Y. If Y is para-Lindelöf and $f^{-1}(y)$ is Lindelöf for each $y \in Y$, then X is para-Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. For each $y \in Y$, choose a countable set $\mathcal{V}_y \subseteq \mathcal{U}$ such that $f^{-1}(y) \subseteq \cup\mathcal{V}_y$. Since f is closed, for each $y \in Y$, there exists an open set U_y in Y containing y such that $f^{-1}(y) \subseteq f^{-1}(U_y) \subseteq \cup\mathcal{V}_y$. The open cover $\{U_y : y \in Y\}$ of Y has a locally countable open refinement $\mathcal{V}$. Clearly $\{f^{-1}(V) : V \in \mathcal{V}\}$ is a locally countable open cover of X and for each $V \in \mathcal{V}$, we get a $y(V) \in Y$ with $f^{-1}(V) \subseteq f^{-1}(U_{y(V)}) \subseteq \cup\mathcal{V}_{y(V)}$. It can be easily observed that $\{f^{-1}(V) \cap U : V \in \mathcal{V} \text{ and } U \in \mathcal{V}_{y(V)}\}$ is a locally countable open refinement of $\mathcal{U}$. Hence X is para-Lindelöf. □

Theorem 3.11. *If X is a star-Menger Lindelöf space and $\{X_n : n \in \omega\}$ is a family of compact spaces such that $\times_{n\in\omega} X_n$ is paracompact, then $X \times \times_{n\in\omega} X_n$ is para-Lindelöf.*

Proof. By Theorem 3.8, the quotient mappings $q_1 : \times_{n\in\omega} X_n \to \times_{n\in\omega} X_n/\rho$ and $q_2 : X\times\times_{n\in\omega} X_n \to (X\times\times_{n\in\omega} X_n)/\rho$ are closed. Clearly $\times_{n\in\omega} X_n/\rho$ is paracompact as paracompactness is preserved under closed continuous mappings. It is also easy to see that $\times_{n\in\omega} X_n/\rho$ is homeomorphic to $(X \times \times_{n\in\omega} X_n)/\rho$. Consequently $(X\times\times_{n\in\omega} X_n)/\rho$ is paracompact. For any $\rho(y) \in \times_{n\in\omega} X_n/\rho$, $q_1^{-1}(\rho(y))$ is a σ-compact subspace of $\times_{n\in\omega} X_n$, the reason is as follows. Let $y = (y_n)$ and choose $Y = \times_{n\in\omega} Z_n$, where $Z_n = \{y_n\}$, for all but finitely many n and for these finitely many n, $Z_n = X_n$. Then obviously Y is compact. It is immediate that we can find at most countably many Y for y, say $\{Y_n : n \in \omega\}$, and clearly $q_1^{-1}(\rho(y)) = \cup_{n\in\omega} Y_n$. For any $X\times\rho(y) \in (X\times\times_{n\in\omega} X_n)/\rho$, $q_2^{-1}(X \times \rho(y)) = X \times \cup_{n\in\omega} Y_n$ is a Lindelöf subspace of $X \times \times_{n\in\omega} X_n$. Thus $X \times \times_{n\in\omega} X_n$ is para-Lindelöf by Lemma 3.10. □

We now attempt to give a partial answer to the Problem 3.7.

Theorem 3.12. *For a Lindelöf space X the following assertions are equivalent.*

(1) *X is star-Menger.*

(2) *Every cover $\{G_M : M \in \mathcal{M}(X)\}$ of X, where G_M is a G_δ subset of X with $M \subseteq G_M$, has a subcover of cardinality less than $\mathfrak{b}$.*

Proof. It is enough to give proof of (2) ⇒ (1). Let $(\mathcal{U}_n)$ be a sequence of open covers of X. Without loss of generality, we assume that each $\mathcal{U}_n = \{U_m^{(n)} : m \in \omega\}$. Let $M \in \mathcal{M}(X)$ be fixed. For each $x \in M$, we define $f_x : \omega \to \omega$ by $f_x(n) = \min\{m : x \in U_m^{(n)}\}$. Let $D_M = \{f_x : x \in M\}$ and $P = \{(x, f_x) : x \in M\}$. We now show that D_M is not dominating. Suppose that D_M is dominating. Let $g : \omega \to \omega + 1$ be such that $g(n) = \omega$, for all n. We claim that $\overline{P} \cap (M \times \rho(g)) = \emptyset$, where $\overline{P}$ is the closure of P in $M \times \times_\omega(\omega + 1)$. Let $(y, f) \in M \times \rho(g)$. Since

$f \in \rho(g)$, there is a $n_0 \in \omega$ such that $f(n) = g(n) = \omega$, for all $n \geq n_0$. From $f_y(n_0) = \min\{m : y \in U_m^{(n_0)}\}$ we have $y \in U_{f_y(n_0)}^{(n_0)}$ and denote by $U_y = U_{f_y(n_0)}^{(n_0)} \cap M$. Let $V = \prod_{n\in\omega} V_n$, where $V_n = \omega+1$ if $n \neq n_0$ and $V_{n_0} = [f_y(n_0)+1, \omega]$. Then $U_y \times V$ is an open set in $M \times X_\omega(\omega+1)$ containing (y, f). We pick a $(x, f_x) \in P$ with $x \in U_y$. Subsequently $x \in U_{f_y(n_0)}^{(n_0)}$ and $f_x(n_0) \leq f_y(n_0)$. It follows that $(x, f_x) \notin U_y \times V$ and thus we have $P \cap (U_y \times V) = \emptyset$. This shows that $\overline{P} \cap (M \times \rho(g)) = \emptyset$. Then $M \times \rho(g) \subseteq (M \times X_\omega(\omega+1)) \setminus \overline{P}$ and by Lemma 3.3, we can find an open cylinder W in $X_\omega(\omega+1)$ containing g such that $M \times \rho(g) \subseteq M \times \rho(W) \subseteq (M \times X_\omega(\omega+1)) \setminus \overline{P}$. This gives us $\overline{P} \cap (M \times \rho(W)) = \emptyset$. Without loss of generality, we can assume that $W = \prod_{n\in\omega}[k_n, \omega]$ as $g \in W$. Suppose that $D_M \cap \rho(W) = \emptyset$. This gives us $D_M \subseteq (\omega+1)^\omega \setminus \rho(W)$ i.e. $D_M \subseteq \omega^\omega \setminus \rho(W)$. Let $\phi \in \omega^\omega$ be such that $\phi(n) = k_n$, for all n. Since D_M is a dominating subset of ω^ω, there exists a $\psi \in D_M$ such that $\phi(n) \leq \psi(n)$, for all but finitely many n i.e. $k_n \leq \psi(n)$, for all but finitely many n. It follows that $\psi \in \rho(W)$ i.e. $\psi \notin D_M$, which is absurd. This shows that $D_M \cap \rho(W) \neq \emptyset$. Then there is a $f_{x_0} \in D_M$ such that $f_{x_0} \in \rho(W)$ and hence $(x_0, f_{x_0}) \in P \cap (M \times \rho(W))$, which is a contradiction. Thus D_M is not dominating. Then there exists a $h_M \in \omega^\omega$ such that for every $x \in M$, $f_x(n) \leq h_M(n)$ for infinitely many n. For each n, $\mathcal{V}_n^{(M)} = \{U_m^{(n)} : m \leq h_M(n)\}$ is a finite subset of $\mathcal{U}_n$. Now proceeding similarly as in the proof of Lemma 3.2, we obtain $M \subseteq \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m^{(M)}, \mathcal{U}_m)$ with $G_M = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m^{(M)}, \mathcal{U}_m)$ is a G_δ subset of X. Then $\{G_M : M \in \mathcal{M}(X)\}$ is a cover of X and hence by the given hypothesis there is a subfamily $\{G_{M_\alpha} : \alpha < \kappa\}$, $\kappa < \mathfrak{b}$, of $\{G_M : M \in \mathcal{M}(X)\}$ that covers X. Since $\{h_{M_\alpha} : \alpha < \kappa\}$ is of cardinality less than $\mathfrak{b}$, $\{h_{M_\alpha} : \alpha < \kappa\}$ is a bounded subset of ω^ω. Consequently there exists a $h \in \omega^\omega$ such that $h_{M_\alpha} \leq^* h$, for all $\alpha < \kappa$. For each n, $\mathcal{W}_n = \{U_m^{(n)} : m \leq h(n)\}$ is a finite subset of $\mathcal{U}_n$. Next we show that $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{W}_m, \mathcal{U}_m)$. Let $n \in \omega$ be fixed and $G_{M_\beta} \in \{G_{M_\alpha} : \alpha < \kappa\}$. Since $h_{M_\beta} \leq^* h$, we can find a $n_1 \in \omega$ such that $h_{M_\beta}(m) \leq h(m)$, for all $m \geq n_1$. Without loss of generality, we assume that $n_1 \geq n$. It is clear that $\cup\mathcal{V}_m^{(M_\beta)} \subseteq \cup\mathcal{W}_m$, for all $m \geq n_1$ and hence $St(\cup\mathcal{V}_m^{(M_\beta)}, \mathcal{U}_m) \subseteq St(\cup\mathcal{W}_m, \mathcal{U}_m)$, for all $m \geq n_1$. It becomes $G_{M_\beta} \subseteq \cup_{m\geq n_1} St(\cup\mathcal{V}_m^{(M_\beta)}, \mathcal{U}_m) \subseteq \cup_{m\geq n_1} St(\cup\mathcal{W}_m, \mathcal{U}_m)$ and then $G_{M_\beta} \subseteq \cup_{m\geq n} St(\cup\mathcal{W}_m, \mathcal{U}_m)$. Since $\beta < \kappa$ is arbitrarily chosen, $\cup_{\alpha<\kappa} G_{M_\alpha} \subseteq \cup_{m\geq n} St(\cup\mathcal{W}_m, \mathcal{U}_m)$. It follows that $X = \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{W}_m, \mathcal{U}_m)$. Hence the result. □

It is interesting to observe that Corollary 3.6 can be extended to κ-many star-Menger spaces, where $\kappa < \mathfrak{b}$.

Corollary 3.13. *Let $\kappa < \mathfrak{b}$. If $X = \cup_{\alpha<\kappa} X_\alpha$ is Lindelöf and each X_α is star-Menger, then X is also star-Menger.*

Another immediate consequence of the above result is that every Lindelöf space, which is $\mathfrak{b}$-concentrated on a star-Menger subspace, is also star-Menger.

Corollary 3.14. *Let Y be a star-Menger subspace of a Lindelöf space X. If X is $\mathfrak{b}$-concentrated on Y, then X is star-Menger.*

Proof. Let $\mathcal{A} = \{Y\} \cup \{\{x\} : x \in X\} \subseteq \mathcal{M}(X)$. For each $M \in \mathcal{M}$, let G_M be a G_δ subset of X containing M. Since X is $\mathfrak{b}$-concentrated on Y, $|X \setminus G_Y| < \mathfrak{b}$. Choose $X \setminus G_Y = \{x_\alpha : \alpha < \kappa\}$, where $\kappa < \mathfrak{b}$. Let $\mathcal{B} = \{Y\} \cup \{\{x_\alpha\} : \alpha < \kappa\}$. Since $\{G_M : M \in \mathcal{B}\}$ covers X with $|\mathcal{B}| < \mathfrak{b}$, X is star-Menger by Theorem 3.12. □

We end with another useful application of Lemma 3.1 and 3.2.

Theorem 3.15. *Let X be a Lindelöf space which is not star-Menger. If X contains a dense star-Menger subspace, then $X \times X_\omega(\omega+1)$ is not normal.*

Proof. By Lemma 3.1, there is a sequence $(\mathcal{U}_n)$ of open covers of X such that for all sequences $(\mathcal{V}_n)$ with for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $X \neq \cap_{n\in\omega} \cup_{m\geq n} St(\cup\mathcal{V}_m, \mathcal{U}_m)$. Without loss of generality, for each n, choose $\mathcal{U}_n = \{U_m^{(n)} : m \in \omega\}$. For each $x \in X$, we define $f_x : \omega \to \omega$ by $f_x(n) = \min\{m : x \in U_m^{(n)}\}$. Let $D = \{f_x : x \in X\}$ and $P = \{(x, f_x) : x \in X\}$. Then D is dominating by Lemma 3.2. Later we pick a $g : \omega \to \omega+1$ such that $g(n) = \omega$, for all n. We now show that $\overline{P} \cap (X \times \rho(g)) = \emptyset$. Let $(y, f) \in X \times \rho(g)$. Since $f \in \rho(g)$, there exists a $n_0 \in \omega$ such that $f(n) = g(n) = \omega$, for all $n \geq n_0$. Now $f_y(n_0) = \min\{m : y \in U_m^{(n_0)}\}$ implies that $y \in U_{f_y(n_0)}^{(n_0)}$. Let $V = \prod_{n\in\omega} V_n$, where $V_n = \omega+1$ if $n \neq n_0$ and $V_{n_0} = [f_y(n_0)+1, \omega]$. It is clear that $U_{f_y(n_0)}^{(n_0)} \times V$ is an open set in

$X \times \square_\omega(\omega+1)$ containing (y, f). Choose $(x, f_x) \in P$ with $x \in U^{(n_0)}_{f_y(n_0)}$. Then $f_x(n_0) \leq f_y(n_0)$ and so $f_x(n_0) \notin V_{n_0}$. It follows that $(x, f_x) \notin U^{(n_0)}_{f_y(n_0)} \times V$ and hence $P \cap (U^{(n_0)}_{f_y(n_0)} \times V) = \emptyset$. Thus $\overline{P} \cap (X \times \rho(g)) = \emptyset$.

Next we show that the disjoint closed sets $\overline{P}$ and $X \times \rho(g)$ can not be separated by open sets in $X \times \square_\omega(\omega+1)$. Let U and V be two open sets in $X \times \square_\omega(\omega+1)$ with $\overline{P} \subseteq U$ and $X \times \rho(g) \subseteq V$. By the given hypothesis we can find a dense star-Menger subspace M of X. For each $(x, f_x) \in P$, we consider an open set $U_x \times V_x$ in $X \times \square_\omega(\omega+1)$ such that $(x, f_x) \in U_x \times V_x \subseteq U$. Since M is a dense subset of X, for each $U_x \times V_x$, there exists $z_x \in M$ such that $(z_x, f_x) \in U_x \times V_x$. Choose $Q = \{(z_x, f_x) : x \in X\} \subseteq M \times \square_\omega(\omega+1)$. If we can prove that $\overline{Q} \cap (M \times \rho(g)) \neq \emptyset$, where $\overline{Q}$ is the closure of Q in $M \times \square_\omega(\omega+1)$, then we are done. Suppose that $\overline{Q} \cap (M \times \rho(g)) = \emptyset$. Then $M \times \rho(g) \subseteq (M \times \square_\omega(\omega+1)) \setminus \overline{Q}$. By Lemma 3.3, we can obtain an open cylinder W in $\square_\omega(\omega+1)$ containing g such that $M \times \rho(g) \subseteq M \times \rho(W) \subseteq (M \times \square_\omega(\omega+1)) \setminus \overline{Q}$. Consequently $\overline{Q} \cap (M \times \rho(W)) = \emptyset$. Since D is a dominating subset of ω^ω, by using similar technique of the proof of Theorem 3.12, we get $D \cap \rho(W) \neq \emptyset$. Then there is a $x_0 \in X$ such that $f_{x_0} \in \rho(W)$ and hence $(z_{x_0}, f_{x_0}) \in Q \cap (M \times \rho(W))$, which is a contradiction. Thus we have $\overline{Q} \cap (M \times \rho(g)) \neq \emptyset$. This implies that $U \cap V \neq \emptyset$ as $Q \subseteq U$ and $M \times \rho(g) \subseteq V$. Hence $X \times \square_\omega(\omega+1)$ is not normal. $\square$

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