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Some Covariant and Contravariant Fixed Point Theorems over Bipolar p-Metric Spaces and Applications

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Abstract. In this article, the concept of bipolar p-metric spaces has been introduced as a generalization of usual metric spaces, b-metric spaces and also p-metric spaces. In view of this notion we prove Banach, Reich, Bianchini and Jaggi type fixed point theorems over such spaces. Supporting examples have been given in order to examine the validity of the underlying space and in support of our fixed point theorems.

1. Introduction and Preliminaries

The end of last century had witnessed revolutionary era in the study of fixed point theory. Researchers involved in this area are mainly interested in finding various types of metric type structures and several types of mappings either contractive or expansive type in nature. Fixed point theory gains attention to the mathematical community specially to the new researchers working on functional analysis, for its numerous applications in different branches of mathematics.

Several researchers proved different types of fixed point theorems in various metric type spaces. To prove fixed point, common fixed point, coupled fixed point and proximity point theorems many authors introduced different topological structured spaces. In 2016, Mutlu and Gürdal have instigated concept of bipolar metric spaces and they have proved some contractive fixed point theorems and coupled fixed point theorems therein (see [4, 5]).

Recently Roy and Saha [9] have generalized bipolar metric spaces by introducing the concept of bipolar cone$_{tvs}$ $b-$metric space. In the same article Roy and Saha have discussed about the topology of such spaces and proved Cantor's intersection like theorem with some fixed point theorems therein (see also [1]).

In the year 2017, Parvaneh et al. [7] introduced the concept of p-metric space as a generalization of metric space and b-metric space. This space have already gained very much attention to the researchers in

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the field of fixed point theory. Several contractive type fixed point theorems involving different types of conditions have been proved in this setting (see [6, 8]).

Here we recall some required definitions.

Definition 1.1. *(b-metric space, [2, 3]) Let $\mathcal{X}$ be a nonempty set and s be a real number satisfying $s \geq 1$. A function $\rho_b : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ is a b−metric on $\mathcal{X}$ if the following conditions hold:*

1. $\rho_b(\xi, \eta) = 0$ *if and only if* $\xi = \eta$;
2. $\rho_b(\xi, \eta) = \rho_b(\eta, \xi)$ *for all* $\xi, \eta \in \mathcal{X}$;
3. $\rho_b(\xi, \zeta) \leq s[\rho_b(\xi, \eta) + \rho_b(\eta, \zeta)]$ *for all* $\xi, \eta, \zeta \in \mathcal{X}$.

The space $(\mathcal{X}, \rho_b)$ is called a b−metric space.

Definition 1.2. *(p-metric space, [7]) Let $\mathcal{X}$ be a non-empty set. A function $\rho_p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is said to be extended b-metric or p-metric if there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for all $t \geq 0$ and $\Omega^{-1}(0) = 0 = \Omega(0)$ such that for all $\xi, \eta, \zeta \in \mathcal{X}$, the following conditions hold:*

1. $\rho_p(\xi, \eta) = 0$ *if and only if* $\xi = \eta$;
2. $\rho_p(\xi, \eta) = \rho_p(\eta, \xi)$;
3. $\rho_p(\xi, \zeta) \leq \Omega\left(\rho_p(\xi, \eta) + \rho_p(\eta, \zeta)\right)$.

The pair $(\mathcal{X}, \rho_p)$ is called a p-metric space.

Definition 1.3. *[4] Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonempty sets. Suppose that a function $\rho_{bi} : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ satisfies the following conditions:*

1. $\rho_{bi}(\xi, \eta) = 0$ *if and only if* $\xi = \eta$;
2. $\rho_{bi}(\xi, \eta) = \rho_{bi}(\eta, \xi)$ *for all* $\xi, \eta \in \mathcal{X} \cap \mathcal{Y}$;
3. $\rho_{bi}(\xi_1, \eta_2) \leq \rho_{bi}(\xi_1, \eta_1) + \rho_{bi}(\xi_2, \eta_1) + \rho_{bi}(\xi_2, \eta_2)$ *for all* $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathcal{X} \times \mathcal{Y}$.

The function ρ_{bi} is called a bipolar metric on $(\mathcal{X}, \mathcal{Y})$ and the triplet $(\mathcal{X}, \mathcal{Y}, \rho_{bi})$ is called a bipolar-metric space.

Definition 1.4. *[9] Let E be a real Hausdorff topological vector space with a solid cone K and $\preceq$ be the partial ordering on E induced by K. Also let $\mathcal{X}$ and $\mathcal{Y}$ be two nonempty sets and $d_b : \mathcal{X} \times \mathcal{Y} \to K$ be a function, satisfies the following properties:*

i) $d_b(\xi, \eta) = \theta_E$ *if and only if* $\xi = \eta$;

ii) $d_b(\xi, \eta) = d_b(\eta, \xi)$ *for all* $\xi, \eta \in \mathcal{X} \cap \mathcal{Y}$;

iii) $d_b(\xi_1, \eta_2) \preceq s[d_b(\xi_1, \eta_1) + d_b(\xi_2, \eta_1) + d_b(\xi_2, \eta_2)]$ *for all* $\xi_1, \xi_2 \in \mathcal{X}$ *and* $\eta_1, \eta_2 \in \mathcal{Y}$, *where the coefficient* $s \geq 1$.

The triplet $(\mathcal{X}, \mathcal{Y}, d_b)$ is called a bipolar cone$_{tvs}$ b−metric space.

Remark 1.5. *If we consider $E = \mathbb{R}$ with the usual cone $K = [0, \infty)$ then $(\mathcal{X}, \mathcal{Y}, d_b)$ gives a bipolar b−metric space.*

2. Introduction to bipolar *p*-metric space

Let us consider two nonempty set of functions:

$\Psi = \{\Omega : [0, \infty) \to [0, \infty) : \Omega$ is strictly increasing continuous function with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for all $t \geq 0\}$ and

$\Psi^* = \{\Omega \in \Psi : \Omega^{-1}(t_1 + t_2) \leq \Omega^{-1}(t_1) + \Omega^{-1}(t_2)$ for all $t_1, t_2 \geq 0\}$.

Definition 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonempty sets and $\rho : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ be a mapping. Then ρ is said to be bipolar p-metric if there exists a function $\Omega \in \Psi$ such that ρ satisfies the following conditions:*

(i) $\rho(\xi, \eta) = 0$ *if and only if* $\xi = \eta$;

(ii) $\rho(\xi, \eta) = \rho(\eta, \xi)$ *for all* $(\xi, \eta) \in (\mathcal{X} \cap \mathcal{Y})^2$;

(iii) $\rho(\xi_1, \eta_2) \leq \Omega[\rho(\xi_1, \eta_1) + \rho(\xi_2, \eta_1) + \rho(\xi_2, \eta_2)]$ *for all* $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathcal{X} \times \mathcal{Y}$.

The triplet $(\mathcal{X}, \mathcal{Y}, \rho)$ is called a bipolar p-metric space.

Example 2.2. *(i) Let* $\mathcal{X} = [0, +\infty)$, $\mathcal{Y} = (-\infty, 0]$ *and* $\rho : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ *be given by* $\rho(\xi, \eta) = \exp(|\xi - \eta|) - 1$ *for all* $0 \leq \xi < +\infty$ *and* $-\infty < \eta \leq 0$. *Then* ρ *is a bipolar p-metric on* $(\mathcal{X}, \mathcal{Y})$ *for the function* $\Omega(t) = \exp(t) - 1$ *for all* $t \geq 0$.

(ii) Let $U_n(\mathbb{R})$ *and* $L_n(\mathbb{R})$ *be the sets of all upper and lower triangular matrices of order n respectively. Suppose* $\rho : U_n(\mathbb{R}) \times L_n(\mathbb{R}) \to [0, \infty)$ *is defined as follows:*

$$\rho(A, B) = \sinh\left(\sqrt{\sum_{i,j=1}^{n} |a_{ij} - b_{ij}|^2}\right) \tag{1}$$

for all $A = (a_{ij})_{n\times n} \in U_n(\mathbb{R})$ *and* $B = (b_{ij})_{n\times n} \in L_n(\mathbb{R})$. *Then* $(U_n(\mathbb{R}), L_n(\mathbb{R}), \rho)$ *is a bipolar p-metric space for the mapping* $\Omega(t) = \sinh(\sqrt{3}t)$ *for all* $t \geq 0$.

(iii) Let $\mathcal{L}$ *be the set of all Lebesgue measurable functions on* $[0, 1]$, *such that* $\int_0^1 |f(x)|dx < \infty$. *Now let,* $\mathcal{X} = \{f \in \mathcal{L} : f(x) \geq 0$ *for all* $x \in [0, \frac{1}{2}]$ *and* $f(x) \leq 0$ *for all* $x \in (\frac{1}{2}, 1]\}$ *and* $\mathcal{Y} = \{g \in \mathcal{L} : g(x) \leq 0$ *for all* $x \in [0, \frac{1}{2}]$ *and* $g(x) \geq 0$ *for all* $x \in (\frac{1}{2}, 1]\}$. *Let* $\rho : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ *be given by*

$$\rho(f, g) = \left[\left(1 + \int_0^1 |f(x) - g(x)|dx\right)^2 - 1\right] \text{ for all } (f, g) \in (\mathcal{X}, \mathcal{Y}). \tag{2}$$

Then $(\mathcal{X}, \mathcal{Y}, \rho)$ *is a bipolar p-metric space with the function* $\Omega(t) = (1 + t)^2 - 1$ *for all* $t \geq 0$.

Remark 2.3. *Any metric space, b-metric space (See Definition 1.1), p-metric space (See Definition 1.2), bipolar metric space (See Definition 1.3) and bipolar b-metric space (See Definition 1.4) are also bipolar p-metric space.*

Proposition 2.4. *Let* $(\mathcal{X}, \mathcal{Y}, \rho)$ *be a bipolar b-metric space with co-efficient* $\kappa \geq 1$. *Let* $\sigma(\xi, \eta) := \Gamma(\rho(\xi, \eta))$, *where* Γ *is a strictly increasing continuous function with* $t \leq \Gamma(t)$ *for all* $t \geq 0$ *and* $\Gamma(0) = 0$. *Then* σ *is a bipolar p-metric for* $\Omega(t) = \Gamma_\kappa(t) = \Gamma(\kappa t)$ *for all* $t \geq 0$.

Proof. Here we show that σ satisfies all the conditions of Definition 2.1.

(a) $\sigma(\xi, \eta) = 0$ gives $\Gamma(\rho(\xi, \eta)) = 0$. Then $\rho(\xi, \eta) = \Gamma^{-1}(0) = 0$, implies $\xi = \eta$.

(b) $\sigma(\xi, \eta) = \sigma(\eta, \xi)$ holds trivially.

(c) For all $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathcal{X} \times \mathcal{Y}$ we have,

$$\begin{aligned} \sigma(\xi_1, \eta_2) &= \Gamma(\rho(\xi_1, \eta_2)) \\ &\leq \Gamma(\kappa\{\rho(\xi_1, \eta_1) + \rho(\xi_2, \eta_1) + \rho(\xi_2, \eta_2)\}) \\ &\leq \Gamma(\kappa\{\Gamma(\rho(\xi_1, \eta_1)) + \Gamma(\rho(\xi_2, \eta_1)) + \Gamma(\rho(\xi_2, \eta_2))\}) \\ &= \Gamma(\kappa\{\sigma(\xi_1, \eta_1) + \sigma(\xi_2, \eta_1) + \sigma(\xi_2, \eta_2)\}) \\ &= \Gamma_\kappa(\{\sigma(\xi_1, \eta_1) + \sigma(\xi_2, \eta_1) + \sigma(\xi_2, \eta_2)\}). \end{aligned} \tag{3}$$

This proves our proposition. □

Definition 2.5. *i) The opposite of a bipolar p-metric space* $(\mathcal{X}, \mathcal{Y}, \rho)$ *is defined as the bipolar p-metric space* $(\mathcal{Y}, \mathcal{X}, \bar{\rho})$, *where the function* $\bar{\rho} : \mathcal{Y} \times \mathcal{X} \to [0, \infty]$ *is defined as* $\bar{\rho}(\eta, \xi) = \rho(\xi, \eta)$.
ii) Let $(\mathcal{X}_1, \mathcal{Y}_1)$ *and* $(\mathcal{X}_2, \mathcal{Y}_2)$ *be two pairs of sets.*

The function $\Lambda : \mathcal{X}_1 \cup \mathcal{Y}_1 \to \mathcal{X}_2 \cup \mathcal{Y}_2$ *is called a covariant mapping if* $\Lambda(\mathcal{X}_1) \subset \mathcal{X}_2$ *and* $\Lambda(\mathcal{Y}_1) \subset \mathcal{Y}_2$ *and we denote this as* $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1) \rightrightarrows (\mathcal{X}_2, \mathcal{Y}_2)$.

The function $\Lambda : \mathcal{X}_1 \cup \mathcal{Y}_1 \to \mathcal{X}_2 \cup \mathcal{Y}_2$ *is called a contravariant mapping if* $\Lambda(\mathcal{X}_1) \subset \mathcal{Y}_2$ *and* $\Lambda(\mathcal{Y}_1) \subset \mathcal{X}_2$ *and we denote this as* $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1) \rightleftharpoons (\mathcal{X}_2, \mathcal{Y}_2)$.

If $(\mathcal{X}_1, \mathcal{Y}_1, \rho_1)$ *and* $(\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ *are two bipolar p-metric spaces then we use the notations* $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1, \rho_1) \rightrightarrows (\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ *and* $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1, \rho_1) \rightleftharpoons (\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ *for covariant mappings and contravariant mappings respectively.*

Definition 2.6. *Let $(\mathcal{X}, \mathcal{Y}, \rho)$ be a bipolar p-metric space. A point $\zeta \in \mathcal{X} \cup \mathcal{Y}$ is said to be a left point if $\zeta \in \mathcal{X}$, a right point if $\zeta \in \mathcal{Y}$ and a central point if both hold.*

A sequence $\{\xi_n\} \subset \mathcal{X}$ is called a left sequence and a sequence $\{\eta_n\} \subset \mathcal{Y}$ is called a right sequence.

A sequence $\{\nu_n\} \subset \mathcal{X} \cup \mathcal{Y}$ is said to converge to a point ν if and only if $\{\nu_n\}$ is a left sequence, ν is a right point and $\rho(\nu_n, \nu) \to 0$ as $n \to \infty$ or $\{\nu_n\}$ is a right sequence, ν is a left point and $\rho(\nu, \nu_n) \to 0$ as $n \to \infty$.

Definition 2.7. *A sequence $\{(\xi_n, \eta_n)\} \subset \mathcal{X} \times \mathcal{Y}$ is called a bisequence. If the sequences $\{\xi_n\}$ and $\{\eta_n\}$ both converge then the bisequence $\{(\xi_n, \eta_n)\}$ is called convergent in $\mathcal{X} \times \mathcal{Y}$.*

If $\{\xi_n\}$ and $\{\eta_n\}$ both converge to a point $\nu \in \mathcal{X} \cap \mathcal{Y}$ then the bisequence $\{(\xi_n, \eta_n)\}$ is called biconvergent.

A sequence $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence if $\rho(\xi_n, \eta_m) \to 0$ whenever $n, m \to \infty$.

A bipolar p-metric space is said to be complete if every Cauchy bisequence is convergent.

Definition 2.8. *Let $(\mathcal{X}_1, \mathcal{Y}_1, \rho_1)$ and $(\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ be two bipolar p-metric spaces:*

i) The mapping $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1, \rho_1) \rightrightarrows (\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ is called left-continuous at a point $\xi_0 \in \mathcal{X}_1$ if for every sequence $\{\eta_n\} \subset \mathcal{Y}_1$ with $\eta_n \to \xi_0$ we have $\Lambda(\eta_n) \to \Lambda(\xi_0)$ in $(\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$.

ii) The mapping $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1, \rho_1) \rightrightarrows (\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ is called right-continuous at a point $\eta_0 \in \mathcal{Y}_1$ if for every sequence $\{\xi_n\} \subset \mathcal{X}_1$ with $\xi_n \to \eta_0$ we have $\Lambda(\xi_n) \to \Lambda(\eta_0)$ in $(\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$.

iii) The mapping $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1, \rho_1) \rightrightarrows (\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ is said to be continuous, if it is left-continuous at each point $\xi \in \mathcal{X}_1$ and right-continuous at each point $\eta \in \mathcal{Y}_1$.

iv) A contravariant mapping $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1, \rho_1) \rightleftharpoons (\mathcal{X}_2, \mathcal{Y}_2, \rho_2)$ is continuous if and only if it is continuous as a covariant map $\Lambda : (\mathcal{X}_1, \mathcal{Y}_1, \rho_1) \rightrightarrows (\mathcal{Y}_2, \mathcal{X}_2, \bar{\rho_2})$.

Proposition 2.9. *Let $(\mathcal{X}, \mathcal{Y}, \rho)$ be a bipolar p-metric space. If a central point is a limit of a sequence then it is the unique limit of this sequence.*

Proof. Let $\{\xi_n\}$ be a left sequence in $(\mathcal{X}, \mathcal{Y}, \rho)$ which converges to some $\zeta \in \mathcal{X} \cap \mathcal{Y}$. If $\eta \in \mathcal{Y}$ be a limit of this sequence then we get

$$\begin{aligned}\rho(\zeta, \eta) &\leq \Omega[\rho(\zeta, \zeta) + \rho(\xi_n, \zeta) + \rho(\xi_n, \eta)] \\ &= \Omega[\rho(\xi_n, \zeta) + \rho(\xi_n, \eta)] \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4}$$

Thus (4) shows that $\zeta = \eta$. Therefore ζ is the unique limit of $\{\xi_n\}$. In a similar way if $\{\eta_n\}$ is a right sequence in $(\mathcal{X}, \mathcal{Y}, \rho)$ which converges to $\zeta \in \mathcal{X} \cap \mathcal{Y}$ then also ζ is the unique limit of $\{\eta_n\}$. □

Proposition 2.10. *In a bipolar p-metric space $(\mathcal{X}, \mathcal{Y}, \rho)$ every convergent Cauchy bisequence is biconvergent.*

Proof. Let $\{(\xi_n, \eta_n)\}$ be a Cauchy bisequence converges to $(\xi, \eta) \in \mathcal{X} \times \mathcal{Y}$ that is $\xi_n \to \eta$ and $\eta_n \to \xi$ as $n \to \infty$. Then

$$\rho(\xi, \eta) \leq \Omega[\rho(\xi, \eta_m) + \rho(\xi_n, \eta_m) + \rho(\xi_n, \eta)] \tag{5}$$

Taking $n, m \to \infty$ in the right hand side of (5) we get $\rho(\xi, \eta) = 0$ and therefore $\xi = \eta \in \mathcal{X} \cap \mathcal{Y}$. Hence the bisequence $\{(\xi_n, \eta_n)\}$ is biconvergent. □

Proposition 2.11. *In a bipolar p-metric space $(\mathcal{X}, \mathcal{Y}, \rho)$ every biconvergent bisequence is a Cauchy bisequence.*

Proof. Let $\{(\xi_n, \eta_n)\}$ be a biconvergent bisequence which is biconvergent to some $\zeta \in \mathcal{X} \cap \mathcal{Y}$. Then

$$\begin{aligned}\rho(\xi_n, \eta_m) &\leq \Omega[\rho(\xi_n, \zeta) + \rho(\zeta, \zeta) + \rho(\zeta, \eta_m)] \\ &= \Omega[\rho(\xi_n, \zeta) + \rho(\zeta, \eta_m)] \to 0 \text{ as } n, m \to \infty. \end{aligned} \tag{6}$$

Therefore $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence. □

Proposition 2.12. *In a bipolar p-metric space $(\mathcal{X}, \mathcal{Y}, \rho)$ if a Cauchy bisequence has a convergent bisubsequence then it is also convergent.*

Proof. Let $\{(\xi_n, \eta_n)\}$ be a Cauchy bisequence which has a convergent bisubsequence $\{(\xi_{n_p}, \eta_{n_p})\}$ converging to $(\xi, \eta) \in \mathcal{X} \times \mathcal{Y}$. Then we have

$$\rho(\xi_m, \eta) \leq \Omega[\rho(\xi_m, \eta_{n_r}) + \rho(\xi_{n_r}, \eta_{n_r}) + \rho(\xi_{n_r}, \eta)] \text{ for all } m, r \in \mathbb{N}. \tag{7}$$

Taking $m, r \to \infty$ from (7) we see that $\xi_m \to \eta$. Similarly we can show that $\eta_m \to \xi$ as $m \to \infty$. Hence our proposition. □

Proposition 2.13. *Let $(\mathcal{X}, \mathcal{Y}, \rho)$ be a bipolar p-metric space and let the bisequence $\{(\xi_n, \eta_n)\}$ converges to some (ξ, η) then*

$$\Omega^{-1}(\rho(\xi, \eta)) \leq \liminf_{n\to\infty} \rho(\xi_n, \eta_n) \leq \limsup_{n\to\infty} \rho(\xi_n, \eta_n) \leq \Omega(\rho(\xi, \eta)). \tag{8}$$

Proof. Using condition (iii) of Definition 2.1 we get

$$\begin{aligned} \rho(\xi, \eta) &\leq \Omega[\rho(\xi, \eta_n) + \rho(\xi_n, \eta_n) + \rho(\xi_n, \eta)] \text{ and} \\ \rho(\xi_n, \eta_n) &\leq \Omega[\rho(\xi_n, \eta) + \rho(\xi, \eta) + \rho(\xi, \eta_n)] \text{ for all } n \geq 1. \end{aligned} \tag{9}$$

Therefore from (9) it follows that

$$\rho(\xi, \eta) \leq \Omega[\liminf_{n\to\infty} \rho(\xi_n, \eta_n)] \text{ implies } \Omega^{-1}(\rho(\xi, \eta)) \leq \liminf_{n\to\infty} \rho(\xi_n, \eta_n) \tag{10}$$

also, since $\lim_{n\to\infty} \rho(\xi, \eta_n) = 0$ and $\lim_{n\to\infty} \rho(\xi_n, \eta) = 0$, we obtain

$$\limsup_{n\to\infty} \rho(\xi_n, \eta_n) \leq \Omega(\rho(\xi, \eta)). \tag{11}$$

□

3. Some covariant and contravariant fixed point theorems

For every $\Omega \in \Psi$ we consider a subset of $(0, 1)$ denoted as Δ_Ω, which is given by

$$\begin{aligned} \Delta_\Omega = \{\lambda \in (0, 1) : \lim_{n\to\infty} \Omega^{(p+1)} \left(\sum_{i=n}^{n+p} \Omega^{-(n+p-i)}[\lambda^i \Theta] \right) = 0 \text{ for any} \\ p = 1, 2, 3, \ldots \text{ and any fixed } \Theta > 0\}. \end{aligned} \tag{12}$$

Lemma 3.1. *Let $(\mathcal{X}, \mathcal{Y}, \rho)$ be a bipolar p-metric space for some $\Omega \in \Psi^*$ and $\{(\xi_n, \eta_n)\}$ a bisequence in $(\mathcal{X}, \mathcal{Y})$. If for some $\lambda \in \Delta_\Omega$ and $M_1, M_2 \geq 0$, $\{(\xi_n, \eta_n)\}$ satisfies (i) $\rho(\xi_n, \eta_n) \leq \lambda^n M_1$ and (ii) $\rho(\xi_{n+1}, \eta_n) \leq \lambda^n M_2$ for all $n \in \mathbb{N}$ then $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence.*

Proof. If $M_1 = 0 = M_2$ then clearly the bisequence $\{(\xi_n, \eta_n)\}$ is Cauchy. So let us assume that atleast one of M_1 and M_2 is strictly greater than zero.

For any $1 \leq n < m$ we get,

$$\rho(\xi_n, \eta_m) \leq \Omega[\rho(\xi_n, \eta_n) + \rho(\xi_{n+1}, \eta_n) + \rho(\xi_{n+1}, \eta_m)]. \tag{13}$$

So,

$$\begin{aligned} \Omega^{-1}(\rho(\xi_n, \eta_m)) &\leq \rho(\xi_n, \eta_n) + \rho(\xi_{n+1}, \eta_n) + \rho(\xi_{n+1}, \eta_m) \\ &\leq \lambda^n(M_1 + M_2) + \rho(\xi_{n+1}, \eta_m). \end{aligned}$$

$$
\begin{aligned}
\Omega^{-1}(\rho(\xi_n, \eta_m)) &\leq \rho(\xi_n, \eta_n) + \rho(\xi_{n+1}, \eta_n) + \\
&\quad \Omega[\rho(\xi_{n+1}, \eta_{n+1}) + \rho(\xi_{n+2}, \eta_{n+1}) + \rho(\xi_{n+2}, \eta_m)] \\
\Rightarrow \Omega^{-2}(\rho(\xi_n, \eta_m)) &\leq \Omega^{-1}[\rho(\xi_n, \eta_n) + \rho(\xi_{n+1}, \eta_n)] + \\
&\quad \rho(\xi_{n+1}, \eta_{n+1}) + \rho(\xi_{n+2}, \eta_{n+1}) + \rho(\xi_{n+2}, \eta_m) \\
&\leq \Omega^{-1}[\lambda^n(M_1 + M_2)] + \lambda^{n+1}(M_1 + M_2) + \rho(\xi_{n+2}, \eta_m). \qquad (14)
\end{aligned}
$$

Proceeding in a similar way we get,

$$
\begin{aligned}
\Omega^{-(m-n+1)}(\rho(\xi_n, \eta_m)) &\leq \Omega^{-(m-n)}[\lambda^n(M_1 + M_2)] + \Omega^{-(m-n-1)}[\lambda^{n+1}(M_1 + M_2)] + \\
&\quad \ldots + \Omega^{-1}[\lambda^{m-1}(M_1 + M_2)] + \rho(\xi_{m+1}, \eta_m) \\
&\leq \sum_{i=n}^{m} \Omega^{-(m-i)}[\lambda^i(M_1 + M_2)]. \qquad (15)
\end{aligned}
$$

From which it follows that

$$
\rho(\xi_n, \eta_m) \leq \Omega^{(m-n+1)}\left(\sum_{i=n}^{m} \Omega^{-(m-i)}[\lambda^i(M_1 + M_2)]\right). \qquad (16)
$$

Also by a similar calculation as before we can show that for any $1 \leq m < n$

$$
\rho(\xi_n, \eta_m) \leq \Omega^{(n-m+1)}\left(\sum_{i=m}^{n} \Omega^{-(n-i)}[\lambda^i(\lambda M_1 + M_2)]\right). \qquad (17)
$$

Since $\lambda \in \Delta_\Omega$ then from (16) and (17) we can conclude that $\{(\xi_n, \eta_n)\}$ is a Cauchy bisequence in $(\mathcal{X}, \mathcal{Y})$. □

Theorem 3.2. *(Covariant Banach type fixed point theorem) Let $(\mathcal{X}, \mathcal{Y}, \rho)$ be a complete bipolar p-metric space for some $\Omega \in \Psi^*$ and $\Upsilon : (\mathcal{X}, \mathcal{Y}, \rho) \rightrightarrows (\mathcal{X}, \mathcal{Y}, \rho)$ a mapping satisfying*

$$
\rho(\Upsilon\xi, \Upsilon\eta) \leq \alpha\rho(\xi, \eta) \text{ for all } (\xi, \eta) \in \mathcal{X} \times \mathcal{Y} \text{ and} \qquad (18)
$$

for some $\alpha \in \Delta_\Omega$. Then $\Upsilon : \mathcal{X} \cup \mathcal{Y} \to \mathcal{X} \cup \mathcal{Y}$ has a unique fixed point.

Proof. Let $(\xi_0, \eta_0) \in \mathcal{X} \times \mathcal{Y}$. We construct two iterative sequences $\{\xi_n\} \subset \mathcal{X}$ and $\{\eta_n\} \subset \mathcal{Y}$ by $\xi_n = T\xi_{n-1} = T^n\xi_0$ and $\eta_n = T\eta_{n-1} = T^n\eta_0$ for all $n \in \mathbb{N}$. Now

$$
\begin{aligned}
\rho(\xi_n, \eta_n) = \rho(\Upsilon\xi_{n-1}, \Upsilon\eta_{n-1}) &\leq \alpha\rho(\xi_{n-1}, \eta_{n-1}) \\
&\leq \alpha^2\rho(\xi_{n-2}, \eta_{n-2}) \\
&\quad \vdots \\
&\leq \alpha^n\rho(\xi_0, \eta_0) \text{ for all } n \geq 1. \qquad (19)
\end{aligned}
$$

Similarly $\rho(\xi_{n+1}, \eta_n) \leq \alpha^n\rho(\xi_1, \eta_0)$ for all $n \in \mathbb{N}$. Since $\alpha \in \Delta_\Omega$, then by Lemma 3.1 it follows that $\{(\xi_n, \eta_n)\}$ is Cauchy bisequence in $(\mathcal{X}, \mathcal{Y})$. As $(\mathcal{X}, \mathcal{Y}, \rho)$ is complete then $\{(\xi_n, \eta_n)\}$ biconverges to some $\zeta \in \mathcal{X} \cap \mathcal{Y}$. Then we see that

$$
\rho(\xi_n, \Upsilon\zeta) = \rho(\Upsilon\xi_{n-1}, \Upsilon\zeta) \leq \alpha\rho(\xi_{n-1}, \zeta) \to 0 \text{ as } n \to \infty. \qquad (20)
$$

Therefore we have $\Upsilon\zeta = \zeta$. Let $\nu \in \mathcal{X}$ be another fixed point of Υ. So that $\Upsilon\nu = \nu$ and we have $\rho(\nu, \zeta) = \rho(\Upsilon\nu, \Upsilon\zeta) \leq \alpha\rho(\nu, \zeta)$, where $0 < \alpha < 1$, showing that $\nu = \zeta$. If $\nu \in Y$ then we can also we see that $\nu = \zeta$, implying that Υ has a unique fixed point in $(\mathcal{X}, \mathcal{Y}, \rho)$. □

Example 3.3. *Consider $U_n(\mathbb{R})$ and $L_n(\mathbb{R})$ as the sets of all upper and lower triangular matrices of order n respectively. Let $\rho : U_n(\mathbb{R}) \times L_n(\mathbb{R}) \to [0,\infty)$ be defined by*

$$\rho(A,B) = \sqrt{\sum_{i,j=1}^{n} |a_{ij} - b_{ij}|^2} \tag{21}$$

for all $A = (a_{ij})_{n\times n} \in U_n(\mathbb{R})$ and $B = (b_{ij})_{n\times n} \in L_n(\mathbb{R})$. Then $(U_n(\mathbb{R}), L_n(\mathbb{R}), \rho)$ is a complete bipolar p-metric space for the mapping $\Omega(t) = \sqrt{3}t$ for all $t \geq 0$. Also let $\Upsilon : (U_n(\mathbb{R}), L_n(\mathbb{R}), \rho) \rightrightarrows (U_n(\mathbb{R}), L_n(\mathbb{R}), \rho)$ be given by $\Upsilon((a_{ij})_{n\times n}) = \left(\frac{a_{ij}}{4}\right)_{n\times n}$ for all $(a_{ij})_{n\times n} \in U_n(\mathbb{R}) \cup L_n(\mathbb{R})$. Then Υ is a covariant Banach type mapping for $\alpha = \frac{1}{4}$. Now

$$\lim_{n\to\infty} \Omega^{(p+1)} \left(\sum_{i=n}^{n+p} \Omega^{-(n+p-i)}[\alpha^i \Theta] = 0 \right)$$

implies

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{3}}\right)^{n-1} \left(\frac{\sqrt{3}}{4}\right)^{n} \left(\frac{1-\left(\frac{\sqrt{3}}{4}\right)^{p+1}}{1-\left(\frac{\sqrt{3}}{4}\right)} \right) \Theta = 0,$$

for any fixed $\Theta > 0$. Therefore $\frac{1}{4} \in \Delta_\Omega$. So all the conditions of Theorem 3.2 are satisfied and $O_{n\times n}$ is the unique fixed point of Υ, where $O_{n\times n}$ is the null matrix of order n.

Theorem 3.4. *(Contravariant Reich type fixed point theorem) Let $(\mathcal{X}, \mathcal{Y}, \rho)$ be a complete bipolar p-metric space for some $\Omega \in \Psi^*$ and $\Upsilon : (\mathcal{X}, \mathcal{Y}, \rho) \rightleftharpoons (\mathcal{X}, \mathcal{Y}, \rho)$ a mapping satisfying*

$$\rho(\Upsilon\eta, \Upsilon\xi) \leq \alpha\rho(\xi,\eta) + \beta\rho(\xi, \Upsilon\xi) + \gamma\rho(\Upsilon\eta, \eta) \tag{22}$$

for all $(\xi,\eta) \in \mathcal{X} \times \mathcal{Y}$, where $\alpha, \beta, \gamma \geq 0$ such that $\alpha+\beta+\gamma < 1$ and $\left(\frac{\alpha+\gamma}{1-\beta}\right)\left(\frac{\alpha+\beta}{1-\gamma}\right) \in \Delta_\Omega$. Then $\Upsilon : \mathcal{X} \cup \mathcal{Y} \to \mathcal{X} \cup \mathcal{Y}$ has a unique fixed point, provided that $\gamma t < \Omega^{-1}(t)$ for all $t > 0$.

Proof. Let $\xi_0 \in \mathcal{X}$ be arbitrary. For any non-negative integer n, we define $\eta_n = \Upsilon\xi_n$ and $\xi_{n+1} = \Upsilon\eta_n$. Then we have,

$$\begin{aligned} \rho(\xi_n, \eta_n) = \rho(\Upsilon\eta_{n-1}, \Upsilon\xi_n) &\leq \alpha\rho(\xi_n, \eta_{n-1}) + \beta\rho(\xi_n, \Upsilon\xi_n) + \gamma\rho(\Upsilon\eta_{n-1}, \eta_{n-1}) \\ &= (\alpha+\gamma)\rho(\xi_n, \eta_{n-1}) + \beta\rho(\xi_n, \eta_n) \text{ for all } n \geq 1. \end{aligned} \tag{23}$$

Therefore $\rho(\xi_n, \eta_n) \leq \left(\frac{\alpha+\gamma}{1-\beta}\right)\rho(\xi_n, \eta_{n-1})$ for all $n \in \mathbb{N}$. Also we get,

$$\begin{aligned} \rho(\xi_n, \eta_{n-1}) &= \rho(\Upsilon\eta_{n-1}, \Upsilon\xi_{n-1}) \leq \alpha\rho(\xi_{n-1}, \eta_{n-1}) + \beta\rho(\xi_{n-1}, \Upsilon\xi_{n-1}) \\ &\quad + \gamma\rho(\Upsilon\eta_{n-1}, \eta_{n-1}) \\ &= (\alpha+\beta)\rho(\xi_{n-1}, \eta_{n-1}) + \gamma\rho(\xi_n, \eta_{n-1}) \text{ for all } n \geq 1. \end{aligned} \tag{24}$$

Thus $\rho(\xi_n, \eta_{n-1}) \leq \left(\frac{\alpha+\beta}{1-\gamma}\right)\rho(\xi_{n-1}, \eta_{n-1})$ for all $n \in \mathbb{N}$. So from the above two inequalities we get,

$$\rho(\xi_n, \eta_n) \leq \lambda^n \rho(\xi_0, \eta_0) \text{ and } \rho(\xi_{n+1}, \eta_n) \leq \lambda^n \left(\frac{\alpha+\beta}{1-\gamma}\right)\rho(\xi_0, \eta_0), \tag{25}$$

for all $n \geq 0$, where $\lambda = \left(\frac{\alpha+\gamma}{1-\beta}\right)\left(\frac{\alpha+\beta}{1-\gamma}\right)$. Therefore by the Lemma 3.1 it follows that $\{(\xi_n, \eta_n)\}$ is Cauchy bisequence in $(\mathcal{X}, \mathcal{Y})$. By the completeness of $(\mathcal{X}, \mathcal{Y}, \rho)$, $\{(\xi_n, \eta_n)\}$ biconverges to some $\zeta \in \mathcal{X} \cap \mathcal{Y}$. Then we see that

$$\rho(\Upsilon\zeta, \Upsilon\xi_n) \leq \alpha\rho(\xi_n, \zeta) + \beta\rho(\xi_n, \eta_n) + \gamma\rho(\Upsilon\zeta, \zeta) \text{ for all } n \geq 1. \tag{26}$$

Moreover we have,

$$\begin{aligned}\rho(\Upsilon\zeta,\zeta) &\leq \Omega[\rho(\Upsilon\zeta,\Upsilon\xi_n)+\rho(\xi_n,\eta_n)+\rho(\xi_n,\zeta)]\\ &\leq \Omega[\alpha\rho(\xi_n,\zeta)+\beta\rho(\xi_n,\eta_n)+\gamma\rho(\Upsilon\zeta,\zeta)+\rho(\xi_n,\eta_n)+\rho(\xi_n,\zeta)],\end{aligned} \tag{27}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ we obtain $\rho(\Upsilon\zeta,\zeta) \leq \Omega[\gamma\rho(\Upsilon\zeta,\zeta)]$. If $\Upsilon\zeta \neq \zeta$ then $\rho(\Upsilon\zeta,\zeta) \leq \Omega[\gamma\rho(\Upsilon\zeta,\zeta)] < \rho(\Upsilon\zeta,\zeta)$, a contradiction. Hence ζ is a fixed point of Υ.

Now if ϑ and ν are two fixed points of Υ then $\vartheta,\nu \in \mathcal{X}\cap\mathcal{Y}$ we have

$$\rho(\vartheta,\nu) = \rho(\Upsilon\vartheta,\Upsilon\nu) \leq \alpha\rho(\nu,\vartheta)+\beta\rho(\nu,\Upsilon\nu)+\gamma\rho(\Upsilon\vartheta,\vartheta) < \rho(\vartheta,\nu).$$

This shows that $\rho(\vartheta,\nu)=0$ that is $\vartheta=\nu$. Hence Υ has a unique fixed point in $(\mathcal{X},\mathcal{Y},\rho)$. □

Example 3.5. *Let $\mathcal{X}=[0,1]$, $\mathcal{Y}=[1,2]$ and $\rho:\mathcal{X}\times\mathcal{Y}\to[0,\infty)$ be given by $\rho(\xi,\eta)=|\xi-\eta|^2$ for all $(\xi,\eta)\in\mathcal{X}\times\mathcal{Y}$. Then $(\mathcal{X},\mathcal{Y},\rho)$ is a complete bipolar p-metric space for the mapping $\Omega(t)=3t$ for all $t\geq 0$. Now let $\Upsilon:(\mathcal{X},\mathcal{Y},\rho)\rightrightarrows(\mathcal{X},\mathcal{Y},\rho)$ be given by $\Upsilon(\nu)=\frac{(\sqrt{2}+1)-\nu}{\sqrt{2}}$ for all $\nu\in\mathcal{X}\cup\mathcal{Y}$.*

Then Υ is a contravariant Reich type mapping for $\alpha=\frac{1}{2}$ and $\beta=0=\gamma$. Now

$$\lim_{n\to\infty}\Omega^{(p+1)}\left(\sum_{i=n}^{n+p}\Omega^{-(n+p-i)}\left[\left(\frac{1}{4}\right)^i\Theta\right]\right)=0$$

implies

$$\lim_{n\to\infty}\left(\frac{1}{3}\right)^{n-1}\left(\frac{3}{4}\right)^n\left(\frac{1-\left(\frac{3}{4}\right)^{p+1}}{1-\left(\frac{3}{4}\right)}\right)\Theta=0,$$

for any fixed $\Theta>0$. Therefore $\frac{1}{4}\in\Delta_\Omega$. So all the conditions of Theorem 3.4 are satisfied and $\nu=1$ is the unique fixed point of Υ.

Theorem 3.6. *(Contravariant Reich-Bianchini type fixed point theorem) Let $(\mathcal{X},\mathcal{Y},\rho)$ be a complete bipolar p-metric space for some $\Omega\in\Psi^*$ and $\Upsilon:(\mathcal{X},\mathcal{Y},\rho)\rightleftharpoons(\mathcal{X},\mathcal{Y},\rho)$ a mapping satisfying*

$$\rho(\Upsilon\eta,\Upsilon\xi)\leq\delta\max\{\rho(\xi,\eta),\rho(\xi,\Upsilon\xi),\rho(\Upsilon\eta,\eta)\}, \tag{28}$$

for all $(\xi,\eta)\in\mathcal{X}\times\mathcal{Y}$, where $\delta\in[0,1)$ such that $\delta^2\in\Delta_\Omega$. Then $\Upsilon:\mathcal{X}\cup\mathcal{Y}\to\mathcal{X}\cup\mathcal{Y}$ has a unique fixed point, provided that $\delta t<\Omega^{-1}(t)$ for all $t>0$.

Proof. The proof is similar to the proof of Theorem 3.4 and therefore we omit the proof. □

Theorem 3.7. *(Contravariant Jaggi type fixed point theorem) Let $(\mathcal{X},\mathcal{Y},\rho)$ be a complete bipolar p-metric space for some $\Omega\in\Psi^*$ and $\Upsilon:(\mathcal{X},\mathcal{Y},\rho)\rightleftharpoons(\mathcal{X},\mathcal{Y},\rho)$ a mapping satisfying*

$$\rho(\Upsilon\eta,\Upsilon\xi)\leq\mu_1\rho(\xi,\eta)+\mu_2\frac{\rho(\xi,\Upsilon\xi)\rho(\Upsilon\eta,\eta)}{\rho(\xi,\eta)}\text{ for all }(\xi,\eta)\in\mathcal{X}\times\mathcal{Y} \tag{29}$$

with $\xi\neq\eta$ and for $\mu_1,\mu_2\geq 0$, $\mu_1+\mu_2<1$ such that $\left(\frac{\mu_1}{1-\mu_2}\right)^2\in\Delta_\Omega$. Then $\Upsilon:\mathcal{X}\cup\mathcal{Y}\to\mathcal{X}\cup\mathcal{Y}$ has a unique fixed point, provided that Υ is continuous in $(\mathcal{X},\mathcal{Y})$.

Proof. We construct the iterative sequence $\{(\xi_n,\eta_n)\}$ as in Theorem 3.4. If either $\xi_n=\eta_n$ for some $n\geq 1$ or $\xi_{n+1}=\eta_n$ for some $n\geq 0$ then Υ has atleast one fixed point in $\mathcal{X}\cup\mathcal{Y}$. So without loss of generality we assume that $\xi_n \neq \eta_n$ and $\xi_{n+1} \neq \eta_n$ for all positive integer n. Now,

$$\begin{aligned}\rho(\xi_n,\eta_n) &= \rho(\Upsilon\eta_{n-1},\Upsilon\xi_n)\\ &\leq \mu_1\rho(\xi_n,\eta_{n-1}) + \mu_2\frac{\rho(\xi_n,\Upsilon\xi_n)\rho(\Upsilon\eta_{n-1},\eta_{n-1})}{\rho(\xi_n,\eta_{n-1})}\\ &= \mu_1\rho(\xi_n,\eta_{n-1}) + \mu_2\frac{\rho(\xi_n,\eta_n)\rho(\xi_n,\eta_{n-1})}{\rho(\xi_n,\eta_{n-1})}\\ &= \mu_1\rho(\xi_n,\eta_{n-1}) + \mu_2\rho(\xi_n,\eta_n) \text{ for all } n \geq 1. \end{aligned} \tag{30}$$

Therefore for any $n \in \mathbb{N}$, $\rho(\xi_n,\eta_n) \leq \frac{\mu_1}{1-\mu_2}\rho(\xi_n,\eta_{n-1}) = \mu\rho(\xi_n,\eta_{n-1})$. Also

$$\begin{aligned}\rho(\xi_{n+1},\eta_n) &= \rho(\Upsilon\eta_n,\Upsilon\xi_n)\\ &\leq \mu_1\rho(\xi_n,\eta_n) + \mu_2\frac{\rho(\xi_n,\Upsilon\xi_n)\rho(\Upsilon\eta_n,\eta_n)}{\rho(\xi_n,\eta_n)}\\ &= \mu_1\rho(\xi_n,\eta_n) + \mu_2\frac{\rho(\xi_n,\eta_n)\rho(\xi_{n+1},\eta_n)}{\rho(\xi_n,\eta_n)}\\ &= \mu_1\rho(\xi_n,\eta_n) + \mu_2\rho(\xi_{n+1},\eta_n) \text{ for all } n \geq 1, \text{ i.e.} \end{aligned} \tag{31}$$

$\rho(\xi_{n+1},\eta_n) \leq \frac{\mu_1}{1-\mu_2}\rho(\xi_n,\eta_n) = \mu\rho(\xi_n,\eta_n)$. So from the previous two inequalities we get

$$\begin{aligned}\rho(\xi_n,\eta_n) &\leq (\mu^2)^n\rho(\xi_0,\eta_0) \text{ and}\\ \rho(\xi_{n+1},\eta_n) &\leq (\mu^2)^n\mu\rho(\xi_0,\eta_0) \text{ for all } n \geq 0,\end{aligned} \tag{32}$$

Therefore by the Lemma 3.1 it follows that $\{(\xi_n,\eta_n)\}$ is Cauchy bisequence in $(\mathcal{X},\mathcal{Y})$. By the completeness of $(\mathcal{X},\mathcal{Y},\rho)$, $\{(\xi_n,\eta_n)\}$ biconverges to some $\zeta \in \mathcal{X}\cap\mathcal{Y}$. Since Υ is continuous it follows that $\Upsilon\xi_n = \eta_n \to \Upsilon\zeta$ as $n \to \infty$ and therefore $\Upsilon\zeta = \zeta$.

Now if ϑ and ν are two distinct fixed points of Υ then $\vartheta,\nu \in \mathcal{X}\cap\mathcal{Y}$ and we have $\rho(\vartheta,\nu) = \rho(\Upsilon\vartheta,\Upsilon\nu) \leq \mu_1\rho(\nu,\vartheta) + \mu_2\frac{\rho(\nu,\Upsilon\nu)\rho(\Upsilon\vartheta,\vartheta)}{\rho(\nu,\vartheta)} = \mu_1\rho(\nu,\vartheta) < \rho(\nu,\vartheta)$, a contradiction. Hence Υ has a unique fixed point in $(\mathcal{X},\mathcal{Y},\rho)$. $\square$

4. An application to Ulam-Hyers stability

Let $(\mathcal{X},\mathcal{Y},\rho)$ be a bipolar p-metric space for some $\Omega \in \Psi$ and $\Upsilon : (\mathcal{X},\mathcal{Y},\rho) \rightleftharpoons (\mathcal{X},\mathcal{Y},\rho)$ be a given mapping. Let us consider the fixed point equation

$$\Upsilon\xi = \xi, \quad \xi \in \mathcal{X}\cap\mathcal{Y} \tag{33}$$

and for some $\epsilon > 0$

$$\rho(\eta,\Upsilon\eta) < \epsilon \text{ for } \eta \in \mathcal{X} \text{ or } \rho(\Upsilon\eta,\eta) < \epsilon \text{ for } \eta \in \mathcal{Y}. \tag{34}$$

Any point $\eta \in \mathcal{X}\cup\mathcal{Y}$ which satisfies the above equation (34) is called an $\epsilon-$solution of the mapping Υ. We say that the fixed point problem (33) is Ulam-Hyers stable in a bipolar p-metric space if there exists a function $\chi : [0,+\infty) \to [0,+\infty)$ with $\chi(t) > 0$ for all $t > 0$ such that for each $\epsilon > 0$ and an $\epsilon-$solution $\eta \in \mathcal{X}\cup\mathcal{Y}$, there exists a solution ξ of the fixed point equation (33) such that

$$\rho(\eta,\xi) < \chi(\epsilon) \text{ or } \rho(\xi,\eta) < \chi(\epsilon). \tag{35}$$

Theorem 4.1. *Let $(\mathcal{X},\mathcal{Y},\rho)$ be a complete bipolar p-metric space for some $\Omega \in \Psi^*$ and $\Upsilon : (\mathcal{X},\mathcal{Y},\rho) \rightleftharpoons (\mathcal{X},\mathcal{Y},\rho)$ a mapping satisfying*

$$\rho(\Upsilon\eta,\Upsilon\xi) \leq \alpha\rho(\xi,\eta) \text{ for all } (\xi,\eta) \in \mathcal{X}\times\mathcal{Y} \text{ and} \tag{36}$$

for $0 < \alpha < 1$ with $\alpha^2 \in \Delta_\Omega$. If the function $\left(\Omega^{-1} - \alpha I\right)^{-1} (\equiv \chi) : [0,+\infty) \to [0,+\infty)$ exists and strictly increasing (I is the identity mapping on $[0,+\infty)$) then the fixed point equation (33) of Υ is Ulam-Hyers stable.

Proof. Theorem 3.4 shows that Υ has a unique fixed point in $\mathcal{X} \cap \mathcal{Y}$, that is the fixed point equation (33) of Υ has a unique solution say ξ. Let $\epsilon > 0$ be arbitrary and η be an $\epsilon-$solution with $\eta \in \mathcal{X}$ that is $\rho(\eta, \Upsilon\eta) < \epsilon$.

Since Υ satisfies the contractive condition (36) therefore

$$\begin{aligned} \rho(\eta, \xi) &\leq \Omega[\rho(\eta, \Upsilon\eta) + \rho(\xi, \Upsilon\eta) + \rho(\xi, \xi)] \\ &= \Omega[\rho(\eta, \Upsilon\eta) + \rho(\Upsilon\xi, \Upsilon\eta)] \\ &= \Omega[\rho(\eta, \Upsilon\eta) + \alpha\rho(\eta, \xi)] \\ \Rightarrow \Omega^{-1}(\rho(\eta, \xi)) - \alpha\rho(\eta, \xi) &\leq \rho(\eta, \Upsilon\eta) < \epsilon. \end{aligned} \tag{37}$$

Therefore $\rho(\eta, \xi) < \left(\Omega^{-1} - \alpha I\right)^{-1}(\epsilon) = \chi(\epsilon)$. Similarly we can show that if η be an $\epsilon-$solution with $\eta \in \mathcal{Y}$ then also $\rho(\xi, \eta) < \left(\Omega^{-1} - \alpha I\right)^{-1}(\epsilon) = \chi(\epsilon)$. Hence the fixed point equation (33) of Υ is Ulam-Hyers stable. □

Further, let us give a numerical example to put in evidence the utility of the previous given theorem.

Example 4.2. *Let us consider the bipolar p-metric space $(\mathcal{X}, \mathcal{Y}, \rho)$ and the contravariant mapping Υ defined in Example 3.5. Here Υ has the unique fixed point 1 in $\mathcal{X} \cap \mathcal{Y}$. Now let $\xi \in [0, 1]$ be an $\epsilon-$solution of the mapping Υ. Then $\rho(\xi, \Upsilon\xi) < \epsilon$ that is $\frac{(\sqrt{2}+1)^2}{2}|\xi - 1|^2 < \epsilon$. From which it follows that $\rho(\xi, 1) = |\xi - 1|^2 < \chi(\epsilon)$, where $\chi(t) = \frac{2}{(\sqrt{2}+1)^2} t$ for all $t \geq 0$. Similarly for an $\epsilon-$solution $\eta \in [1, 2]$ of the mapping Υ we can show that $\rho(1, \eta) < \chi(\epsilon)$. Hence the fixed point problem of Υ is Ulam-Hyers stable.*

5. Application to Electric Circuit Differential Equation

It is not a novelty the fact that fixed point theory provides interesting results to prove the existence and uniqueness of a solution of integral, differential or fractional equations, used in modelling of the real phenomena. This section is devoted to an application of one of our main fixed point theorem for proving the existence and uniqueness of a solution for the electric circuit equation, given in the second-order differential equation form.

Let us consider a series electric circuit which contain a resistor ($\mathcal{R}$, Ohms) a capacitor ($\mathcal{C}$, Faradays), an inductor ($\mathcal{L}$, Henries) a voltage ($\mathcal{V}$, Volts) and an electromotive force ($\mathcal{E}$, Volts), as in the following scheme, Figure 1.

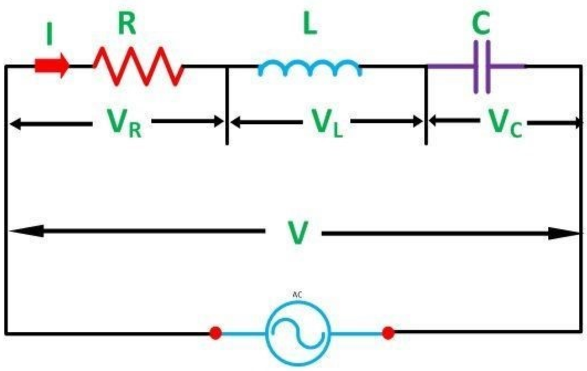

Figure 1: Series RLC

Considering the definition of the intensity of electric current $\mathcal{I} = \frac{dq}{dt}$, where q denote the electric charge and t-the time, let us recall the following usually formulas

- $\mathcal{V} = \mathcal{I}\mathcal{R}$;

- $\mathcal{V} = \frac{q}{C}$;
- $\mathcal{V} = \mathcal{L}\frac{d\mathcal{I}}{dt}$.

Since in a series circuit there is only one current flowing, then $\mathcal{I}$ have the same value in the entire circuit. Kirchhoff's Voltage Law is the second of his fundamental laws we can use for circuit analysis. His voltage law states that for a closed loop series path the algebraic sum of all the voltages around any closed loop in a circuit is equal to zero. The Kirchhoff's Voltage Law states: "The algebraic sum of all the voltages around any closed loop in a circuit is equal to zero."

The main idea of the Kirchhoff's Voltage Law is that as you move around a closed loop/circuit, you will end up back to where you started in the circuit. Therefore you back to the same initial potential without voltage losses around the loop. Therefore, any voltage drop around the loop must be equal to any voltage source encountered along the way. Mathematical expression of this consequence of the Kirchhoff's Voltage Law is: "the sum of voltage rises across any loops is equal to the sum of voltage drops across that loop". Then we have the following relation:

$$\mathcal{I}\mathcal{R} = \frac{q}{C} + \mathcal{L}\frac{d\mathcal{I}}{dt} = \mathcal{V}(t).$$

We can write this voltage equation in the parameters of a second-order differential equation as follows.

$$\mathcal{L}\frac{d^2q}{dt^2} + \mathcal{R}\frac{dq}{dt} + \frac{q}{C} = \mathcal{V}(t), \text{ with the boundary conditions, } q(0) = 0, q'(0) = 0. \tag{38}$$

where $C = \frac{4\mathcal{L}}{\mathcal{R}^2}$ and $\tau = \frac{\mathcal{R}}{2\mathcal{L}}$ - the nondimensional time for the resonance case in Physics.
The Green function associated with equation (38) is the following:

$$\mathcal{G}(t,s) = \begin{cases} -se^{-\tau(s-t)}, & \text{if } 0 \le s \le t \le 1; \\ -te^{-\tau(s-t)}, & \text{if } 0 \le t \le s \le 1. \end{cases}$$

In this conditions, the differential problem (38) can be written as the following integral equation.

$$\xi(t) = \int_0^t \mathcal{G}(t,s) f(s, \xi(s)) ds, \text{ where } t \in [0,1] \tag{39}$$

and $f(s,\cdot) : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a monotone nondecreasing mapping for all $s \in [0,1]$.

Let $\mathcal{X} = (\mathfrak{C}[0,1], [0,+\infty))$ be the set of all continuous functions defined on $[0,1]$ with values in the interval $[0,+\infty)$ and $\mathcal{Y} = (\mathfrak{C}[0,1], (-\infty,0])$ be the set of all continuous functions defined on $[0,1]$ with values in the interval $(-\infty,0]$.

The triple $(\mathcal{X},\mathcal{Y},\rho)$ is a complete bipolar p-metric space with respect to the bipolar p-metric $\rho : \mathcal{X} \times \mathcal{Y} \to [0,\infty]$ defined by

$$\rho(\xi,\eta) = \|\xi - \eta\|_\infty = \exp(\sup_{t\in[0,1]} |\xi(t) - \eta(t)|) - 1, \text{ for all } (\xi(t),\eta(t)) \in (\mathcal{X},\mathcal{Y}),$$

for $\Omega \in \Psi^*$ defined as $\Omega \in \Psi^*$ as $\Omega(t) = \exp(t) - 1$ for all $t \ge [0,1]$.

Further, let us give the main result of this section.

Theorem 5.1. *Let $\Upsilon : (\mathcal{X},\mathcal{Y},\rho) \rightrightarrows (\mathcal{X},\mathcal{Y},\rho)$ be a mapping such that the following assertions hold:*

(*i*) $\mathcal{G} : [0,1]^2 \to [0,\infty)$ *is a continuous function;*

(*ii*) $f(s,\cdot) : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a monotone nondecreasing function for all* $s \in [0,1]$.*;*

(*iii*) *there exists* $\alpha \in \Delta_\Omega$ *such that, for all* $(t,s) \in [0,1]^2$ *and* $(\xi,\eta) \in (\mathcal{X},\mathcal{Y})$*, we have the inequality:*

$$|f(t,\xi) - f(t,\eta)| \leq |\xi(t) - \eta(t)| - \alpha,$$

where $\Delta_\Omega \in (0,1)$ *such that*

$$\Delta_\Omega = \{\lambda \in (0,1) : \lim_{n\to\infty} \Omega^{(p+1)} \left(\sum_{i=n}^{n+p} \Omega^{-(n+p-i)}[\lambda^i \Theta] \right) = 0 \text{ for any } p = 1,2,3,\dots \text{ and any fixed } \Theta > 0\}.$$

Then the voltage differential equation (38) has a unique solution.

Proof. Let us define the function $\Upsilon : (\mathcal{X},\mathcal{Y},\rho) \rightrightarrows (\mathcal{X},\mathcal{Y},\rho)$ by

$$\Upsilon\xi(t) = \int_0^t \mathcal{G}(t,s) f(s,\xi(s)) ds.$$

Then, we should proof the function Υ respect all the conditions of the Theorem 3.2; hence $\Upsilon : \mathcal{X} \cup \mathcal{Y} \to \mathcal{X} \cup \mathcal{Y}$ admits a unique fixed point. Then, there exists a unique solution for the differential problem (38).

We have the following estimations

$$\begin{aligned}
\left|\Upsilon\xi(t) - \Upsilon\eta(t)\right|^2 &= \left| \int_0^t \mathcal{G}(t,s) f(s,\xi(s)) ds - \int_0^t \mathcal{G}(t,s) f(s,\eta(s)) ds \right|^2 \\
&\leq \left[\int_0^t \mathcal{G}(t,s) \left| f(s,\xi(s)) - f(s,\eta(s)) \right| ds \right]^2 \\
&\leq \left[\int_0^t \mathcal{G}(t,s) \left(\left|\xi(t) - \eta(t)\right| - \alpha \right) ds \right]^2 \\
&\leq \left(\left|\xi(t) - \eta(t)\right| - \alpha \right)^2 \left(\int_0^t \mathcal{G}(t,s) ds \right)^2.
\end{aligned}$$

Taking the supremum on both sides in the previous inequality we get

$$\sup_{t\in[0,1]} \left|\Upsilon\xi(t) - \Upsilon\eta(t)\right|^2 \leq \sup_{t\in[0,1]} \left(\left|\xi(t) - \eta(t)\right| - \alpha \right)^2.$$

Obviously, the following inequality is true

$$\sup_{t\in[0,1]} \left|\Upsilon\xi(t) - \Upsilon\eta(t)\right| \leq \sup_{t\in[0,1]} \left|\xi(t) - \eta(t)\right| - \alpha.$$

Applying the exponential function on both sides we get

$$\exp(\sup_{t\in[0,1]} \left|\Upsilon\xi(t) - \Upsilon\eta(t)\right|) \leq \frac{1}{e^\alpha} \exp(\sup_{t\in[0,1]} \left|\xi(t) - \eta(t)\right|).$$

Decreasing 1 on both sides we get

$$\exp(\sup_{t\in[0,1]}\left|\Upsilon\xi(t)-\Upsilon\eta(t)\right|)-1 \le \frac{1}{e^{\alpha}}\exp(\sup_{t\in[0,1]}\left|\xi(t)-\eta(t)\right|)-1$$
$$\le \frac{1}{e^{\alpha}}\left[\exp(\sup_{t\in[0,1]}\left|\xi(t)-\eta(t)\right|)-1\right],$$

which means

$$\rho(\Upsilon\xi,\Upsilon\eta)\le \frac{1}{e^{\alpha}}\rho(\xi,\eta).$$

Since $\frac{1}{e^{\alpha}} \le \alpha \in \Delta_{\Omega}$ then all the conditions of Theorem 3.2 are true. Thus, the differential voltage equation (38) has a unique solution.

□

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Author's contributions

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References

[1] D. Bajović, Z. D. Mitrović and M. Saha, Remark on contraction principle in cone$_{tvs}$ b-metric spaces, *J. Anal.*, (2020), https://doi.org/10.1007/s41478-020-00261-x.
[2] I. A. Bakhtin, The contraction mapping principle in quasi-metric spaces, *Funct. Anal.*, **30**, Ulyanovsk. Gos. Ped. Inst, Ulyanovsk (1989), 26-37.
[3] S. Czerwik, Contraction mappings in $b-$metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **1** (1993), 5-11.
[4] A. Mutlu, U. Gürdal, Bipolar metric spaces and some fixed point theorems, *J. Nonlinear Sci. Appl.*, **9** (2016), 5362-5373.
[5] A. Mutlu, K. Özkan and U. Gürdal, Coupled fixed point theorems on bipolar metric spaces, *European J. Pure Appl. Math.*, **10**(4) (2017), 655-667.
[6] K. Roy, S. Panja, M. Saha and V. Parvaneh, An extended b-metric type space and related fixed point theorems with an application to nonlinear integral equations, *Advances in Mathematical Physics*, Volume 2020, Article ID 8868043, 7 pages, https://doi.org/10.1155/2020/8868043.
[7] V. Parvaneh, A. Dinmohammadi and Z. Kadelburg, Coincidence point results for weakly α-admissible pairs in extended b-metric spaces, *J. Math. Anal.* **8** (2017), 74-89.
[8] V. Parvaneh and S. J. H. Ghoncheh, Fixed points of $(\psi,\phi)_{\Omega}$-contractive mappings in ordered p-metric spaces, *Global Analysis and Discrete Mathematics*, **4**(1) (2020), 15-29.
[9] K. Roy, M. Saha, Generalized contractions and fixed point theorems over bipolar cone$_{tvs}$ $b-$metric spaces with an application to homotopy theory, *Mat. Vesnik*, **72**(4) (2020), 281-296.