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Solving Sylvester Equation with Complex Symmetric Semi-Definite Positive Coefficient Matrices

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Abstract. Combination of real and imaginary parts (CRI) works well for solving complex symmetric linear systems. This paper develops a generalization of CRI method to determine the solution of Sylvester matrix equation. We show that this, regardless of condition, converges to solution of the Sylvester equation. At the end we test the new scheme by solving a numerical example.

1. Introduction

Algebraic Sylvester matrix equations are observed in many areas from different regions such as, control theory and many other branches of engineering [12–14, 16, 17, 34].

The so-called bilinear control system can be described by the following state-space

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \sum_{j=1}^{m} \mathcal{N}_j \mathbf{x}(t)\mathbf{u}_j(t) + \mathcal{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \tilde{\mathcal{C}}\mathbf{x}(t), \ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{1}$$

where t is the time variable, $\mathbf{x}(t) \in \mathbb{C}^n$, $\mathbf{u}(t) = [\mathbf{u}_1(t), ..., \mathbf{u}_m(t)]^T \in \mathbb{C}^m$ and $\mathbf{y}(t) \in \mathbb{C}^n$ are the stable, input and output vectors, respectively. Also $\mathcal{B}(t) \in \mathbb{C}^{n\times m}$, $\tilde{\mathcal{C}}$, $\mathcal{A} \in \mathbb{C}^{n\times n}$. Reachability and observability are two important issues for system (1), such that the reachability is defined by

$$\mathcal{R} = \sum_{k=1}^{\infty} \int_0^{\infty} ... \int_0^{\infty} \mathcal{R}_k \mathcal{R}_k^T \, dt_1 ... dt_k,$$

that is the solution of Eq. (2), where

$$\mathcal{R}_1 = e^{\mathcal{A}t_1}\mathcal{B} \qquad \text{and} \qquad \mathcal{R}_k(t_1, ..., t_k) = e^{\mathcal{A}t_k}[\mathcal{N}_1\mathcal{R}_{k-1}, ..., \mathcal{N}_m\mathcal{R}_{k-1}], \qquad k = 2, 3, ...$$

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Also the observability is the solution of the dual equation for

$$\mathcal{A}\mathcal{Y}+\mathcal{Y}\mathcal{A}^T+\sum_{j=1}^{m}\mathcal{N}_j\mathcal{Y}\mathcal{N}_j^T=\tilde{C}^T\tilde{C},$$

where $\mathcal{Y}\in\mathbb{C}^{n\times n}$ must be determined. Some useful paper about matrix equations can be found in [9–11, 18, 22–24, 26, 28, 30, 31, 35–37, 37, 40].

Here we focus on the Sylvester matrix equation of the form

$$\mathcal{A}\mathcal{Z}+\mathcal{Z}\mathcal{B}=C, \tag{2}$$

where $\mathcal{A}$ and $\mathcal{B}$ are complex matrices of the form $\mathcal{A}=\mathcal{W}+i\mathcal{T}\in\mathbb{C}^{m\times m}$, $\mathcal{B}=\mathcal{U}+i\mathcal{V}\in\mathbb{C}^{n\times n}$, where $\mathcal{W}$, $\mathcal{T}$, $\mathcal{U}$ and $\mathcal{V}$ are real symmetric positive semi-definite matrices and $i=\sqrt{-1}$. If there is no common eigenvalues of $\mathcal{A}$ and $-\mathcal{B}$, then Eq. (2) has a unique solution [2]. This fact can be proved by using the Kronecker sum. Eq. (2) can be transformed to problem

$$\mathbf{A}\mathbf{z}=\mathbf{c}, \tag{3}$$

where $\mathbf{A}=\mathcal{I}_n\otimes\mathcal{A}+\mathcal{B}^T\otimes\mathcal{I}_m$, $\mathbf{c}=\mathbf{vec}(C)$ and $\mathbf{z}=\mathbf{vec}(\mathcal{Z})$, where $\otimes$ is Kronecker product, $\mathcal{I}_n$ is identity matrix of dimension $n\times n$ and for any matrix $\mathcal{A}=(\mathbf{a}_1,...,\mathbf{a}_n)$ with the columns $\mathbf{a}_k$, $\mathbf{vec}(\mathcal{A})$ is an operator such that $\mathbf{vec}(\mathcal{A})=(\mathbf{a}_1^T,...,\mathbf{a}_n^T)^T\in\mathbb{C}^{mn}$. Obtaining the solution of equation (2), by solving linear system (3) is not a suitable method and it can have a computational cost, since the dimension of problem (3) may be very large. We can solve Eq. (2) by the use of direct methods such as Bartels-Stewart [1] and the Hessenberg-Schur methods [21]. But for solving efficiently Sylvester matrix Eq. (2), iterative methods can be used. In [2] Bai proposed HSS approach for solving Eq. (2).

Authors of [41] generalized the method of Bai [2] by introducing the MHSS iterative method for solving Sylvester equations. Authors of [20] applied PMHSS approach for solving Eq. (2). Salkuyeh and Bastani [33] introduced two-parameter generalized Hermitian and skew-Hermitian splitting (TGHSS) iteration method. Dehghan and Shirilord [15] introduced two parameters in MHSS method to obtain a generalized MHSS (GMHSS) iteration method. Hence for different values of the parameters in GMHSS scheme, we obtain different methods. Authors of [15] show that there is least one region $\Omega\in\mathbb{R}^2$, where GMHSS iterative scheme is convergent. For more work on the HSS method see [4–8].

In the following we will summarize the iterative method CRI for approximating the output of linear systems. The focus in the current manuscript is on the following problem to illustrate this approach.

$$\mathcal{A}\mathbf{z}=\mathbf{c}, \tag{4}$$

where $\mathcal{A}\in\mathbb{C}^{n\times n}$ and $\mathbf{z},\mathbf{c}\in\mathbb{C}^n$. Suppose $\mathcal{F},\mathcal{G}\in\mathbb{R}^{n\times n}$ are two real, symmetric and positive semi-definite matrices. Moreover suppose $\mathcal{A}=\mathcal{F}+i\mathcal{G}$. Then CRI method [39] can be expressed as follows.

1.1. *CRI method [39]:*

For a given initial approximation $\mathbf{z}_{(0)}\in\mathbb{C}^n$, we obtain next iterate $\mathbf{z}_{(j+1)}$ from:

$$\begin{cases}(\alpha\mathcal{G}+\mathcal{F})\mathbf{z}_{(j+1/2)}=(\alpha-i)\mathcal{G}\mathbf{z}_{(j)}+\mathbf{c},\\ (\alpha\mathcal{F}+\mathcal{G})\mathbf{z}_{(j+1)}=(\alpha+i)\mathcal{W}\mathbf{z}_{(j+1/2)}-i\mathbf{c}, \qquad j=0,1,2,...,\end{cases} \tag{5}$$

where $\alpha>0$. Based on the introduction of an additional parameter, authors of [27] applied a generalisation of CRI method for solving complex symmetric linear systems. In the following, a generalization of CRI method (5) will be applied to solve large sparse complex Sylvester matrix Eq. (2).

2. A generalization of CRI method

In the first stage write Eq. (2) as

$$\mathcal{W}\mathcal{Z} + \mathcal{Z}\mathcal{U} = -i\mathcal{Z}\mathcal{V} - i\mathcal{T}\mathcal{Z} + C. \tag{6}$$

Assume that $\alpha > 0$ is an arbitrary number. Then, adding $\alpha\mathcal{T}\mathcal{Z}$ and $\alpha\mathcal{Z}\mathcal{V}$ to both sides of the above relation yields

$$(\alpha\mathcal{T} + \mathcal{W})\mathcal{Z} + \mathcal{Z}(\alpha\mathcal{V} + \mathcal{U}) = (\alpha - i)[\mathcal{T}\mathcal{Z} + \mathcal{Z}\mathcal{V}] + C. \tag{7}$$

On the other hand multiplying both sides of (6) by $-i$ and then, adding $\beta\mathcal{W}\mathcal{X}$ and $\beta\mathcal{Z}\mathcal{U}$ to both sides of it yield:

$$(\beta\mathcal{W} + \mathcal{T})\mathcal{Z} + \mathcal{Z}(\beta\mathcal{U} + \mathcal{V}) = (\beta + i)[\mathcal{W}\mathcal{Z} + \mathcal{Z}\mathcal{U}] - iC. \tag{8}$$

Now by considering relations (7) and (8) we arrive at the following method to solve Eq. (2).

2.1. *The GCRI Procedure for Solving Sylvester Matrix Eq.* (2)

Compute $\mathcal{Z}_{(k+1)} \in \mathbb{C}^{m\times n}$ for $k = 0, 1, 2, ...$ by using the following procedure:

$$\begin{cases}(\alpha\mathcal{T} + \mathcal{W})\mathcal{Z}_{(k+\frac{1}{2})} + \mathcal{Z}_{(k+\frac{1}{2})}(\alpha\mathcal{V} + \mathcal{U}) = (\alpha - i)\left[\mathcal{T}\mathcal{Z}_{(k)} + \mathcal{Z}_{(k)}\mathcal{V}\right] + C, \\ (\beta\mathcal{W} + \mathcal{T})\mathcal{Z}_{(k+1)} + \mathcal{Z}_{(k+1)}(\beta\mathcal{U} + \mathcal{V}) = (\beta + i)\left[\mathcal{W}\mathcal{Z}_{(k+\frac{1}{2})} + \mathcal{Z}_{(k+\frac{1}{2})}\mathcal{U}\right] - iC,\end{cases} \tag{9}$$

where $\alpha,\ \beta > 0$ are constant and $\mathcal{Z}_{(0)} \in \mathbb{C}^{m\times n}$ is an initial guess. It can be easily seen that $\alpha\mathcal{W} + \mathcal{T}, \alpha\mathcal{U} + \mathcal{V}, \beta\mathcal{T} + \mathcal{W}$ and $\beta\mathcal{V} + \mathcal{U}$ are symmetric positive definite. Therefore, the two half-steps of this method can be effectively solved using fast and direct algorithms.

Here we introduce the convergence analysis of new iteration method (9). Suppose the matrices $\mathcal{W}$ and $\mathcal{T}$ are semi-positive definite, so before introducing the convergence theorem of the new method, we should pay attention to useful information about these matrices. To do so first recall the following lemma [39].

Lemma 2.1. *Let $\mathcal{W} \in \mathbb{R}^{n\times n}$ and $\mathcal{T} \in \mathbb{R}^{n\times n}$ be symmetric positive semi-definite matrices satisfying $\boldsymbol{null}(\mathcal{W}) \cap \boldsymbol{null}(\mathcal{T}) = \{\boldsymbol{0}\}$, where $\boldsymbol{null}(A)$ denotes null space of any matrix A. Then there exists a nonsingular matrix $\mathcal{P} \in \mathbb{R}^{n\times n}$ such that*

$$\mathcal{W} = \mathcal{P}^T\mathcal{D}_\mathcal{W}\mathcal{P}, \qquad \mathcal{T} = \mathcal{P}^T\mathcal{D}_\mathcal{T}\mathcal{P},$$

where $\mathcal{D}_\mathcal{W} = \boldsymbol{Diag}(\mu_1, ..., \mu_n)$, $\mathcal{D}_\mathcal{T} = \boldsymbol{Diag}(\lambda_1, ..., \lambda_n)$, λ_l and μ_l satisfy

$$\mu_l + \lambda_l = 1, \qquad \lambda_l \geq 0,\ \mu_l \geq 0,\ l = 1, ..., n.$$

Based on [2, Theorem 2.1] and [39, Theorem 2.1], the following theorem can be used to analyze the convergence of the new method.

Theorem 2.2. *Let $\mathcal{A} = \mathcal{W} + i\mathcal{T} \in \mathbb{C}^{m\times m}$ and $\mathcal{B} = \mathcal{U} + i\mathcal{V} \in \mathbb{C}^{n\times n}$, where $\mathcal{W}, \mathcal{T}, \mathcal{U}$ and $\mathcal{V}$ are real symmetric positive semi-definite matrices and let $\alpha,\ \beta > 0$. Denote*

$$\mathcal{Q} = \mathcal{I}_n \otimes \mathcal{W} + \mathcal{U} \otimes \mathcal{I}_m \in \mathbb{R}^{nm\times nm}, \qquad \mathcal{R} = \mathcal{I}_n \otimes \mathcal{T} + \mathcal{V} \otimes \mathcal{I}_m \in \mathbb{R}^{nm\times nm}. \tag{10}$$

Define

$$\Omega_1 = \{(\alpha, \beta) | -1 + \sqrt{1 + \alpha^2} < \beta < \alpha\},$$

and

$$\Omega_2 = \{(\alpha, \beta) | -1 + \sqrt{1 + \beta^2} < \alpha < \beta\}.$$

Then the iteration matrix of CRI method (9) *is*

$$\Pi(\alpha,\beta) = \sqrt{(\alpha^2+1)(\beta^2+1)}(\beta\mathcal{Q}+\mathcal{R})^{-1}\mathcal{Q}(\alpha\mathcal{R}+\mathcal{Q})^{-1}\mathcal{R}, \tag{11}$$

and the spectral radius of the matrix $\Pi(\alpha,\beta)$ *satisfies*

$$\rho(\Pi(\alpha,\beta)) \leq \begin{cases} \theta_1(\alpha,\beta) := \dfrac{\alpha^2+1}{(\beta+1)^2} < 1, & \forall\,(\alpha,\beta)\in\Omega_1, \\ \\ \theta_2(\alpha,\beta) := \dfrac{\beta^2+1}{(\alpha+1)^2} < 1, & \forall\,(\alpha,\beta)\in\Omega_2, \end{cases} \tag{12}$$

then the CRI iteration (9) *converges unconditionally to the unique exact solution* $\mathcal{Z}_* \in \mathbb{C}^{m\times n}$ *of Eq.* (2) *for any initial guess* $\mathcal{Z}_{(0)}$.

Proof. By using Kronecker product, we can write scheme (9) in the following form:

$$\begin{cases} [\mathcal{I}_n\otimes(\alpha\mathbf{T}+\mathcal{W})+(\alpha\mathbf{V}+\mathcal{U})^T\otimes\mathcal{I}_m]\,\mathbf{z}_{(k+\frac{1}{2})} = (\alpha-i)\,[\mathcal{I}_n\otimes\mathbf{T}+\mathcal{V}^T\otimes\mathcal{I}_m]\,\mathbf{z}_{(k)}+\mathbf{c}, \\ [\mathcal{I}_n\otimes(\beta\mathbf{W}+\mathbf{T})+(\beta\mathbf{U}+\mathcal{V})^T\otimes\mathcal{I}_m]\,\mathbf{z}_{(k+1)} = (\beta+i)\,[\mathcal{I}_n\otimes\mathcal{W}+\mathcal{U}^T\otimes\mathcal{I}_m]\,\mathbf{z}_{(k+\frac{1}{2})}-i\mathbf{c}, \end{cases} \tag{13}$$

where $\mathbf{c} = \mathbf{vec}(C)$ and $\mathbf{z} = \mathbf{vec}(\mathcal{Z})$. Note that

$$\mathcal{I}_n\otimes(\alpha\mathbf{T}+\mathcal{W})+(\alpha\mathbf{V}+\mathcal{U})^T\otimes\mathcal{I}_m = \alpha(\mathcal{I}_n\otimes\mathcal{T}+\mathcal{V}\otimes\mathcal{I}_m)+(\mathcal{I}_n\otimes\mathcal{W}+\mathcal{U}\otimes\mathcal{I}_m) = \alpha\mathcal{R}+\mathcal{Q},$$

and

$$\mathcal{I}_n\otimes(\beta\mathbf{W}+\mathbf{T})+(\beta\mathbf{U}+\mathcal{V})^T\otimes\mathcal{I}_m = \beta(\mathcal{I}_n\otimes\mathcal{W}+\mathcal{U}\otimes\mathcal{I}_m)+(\mathcal{I}_n\otimes\mathcal{T}+\mathcal{V}\otimes\mathcal{I}_m) = \beta\mathcal{Q}+\mathcal{R},$$

where $\mathcal{R}$ and $\mathcal{Q}$ are defined in (10). Then Eq. (13) can be rewritten as

$$\begin{cases} (\alpha\mathcal{R}+\mathcal{Q})\mathbf{z}_{(k+\frac{1}{2})} = (\alpha-i)\mathcal{R}\mathbf{z}_{(k)}+\mathbf{c}, \\ (\beta\mathcal{Q}+\mathcal{R})\mathbf{z}_{(k+1)} = (\beta+i)\mathcal{Q}\mathbf{z}_{(k+\frac{1}{2})}-i\mathbf{c}. \end{cases} \tag{14}$$

It is clear that, scheme (14) is the CRI method [39] for solving Eq. (4), with $\mathcal{A} = \mathcal{Q}+i\mathcal{R}$. Suppose that $\lambda^{\mathcal{Q}}_{p,q}$, $\lambda^{\mathcal{R}}_{p,q}$, $\lambda^{\mathcal{W}}_{p}$, $\lambda^{\mathcal{T}}_{p}$, $\lambda^{\mathcal{U}}_{p}$ and $\lambda^{\mathcal{V}}_{q}$ denote the eigenvalues of $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{W}$, $\mathcal{T}$, $\mathcal{U}$ and $\mathcal{V}$ ($p = 1,\dots,m,\ q = 1,\dots,n$), respectively. Since

$$\lambda^{\mathcal{Q}}_{p,q} = \lambda^{\mathcal{W}}_{p}+\lambda^{\mathcal{U}}_{q} \geq 0, \qquad \lambda^{\mathcal{R}}_{p,q} = \lambda^{\mathcal{T}}_{p}+\lambda^{\mathcal{V}}_{q} \geq 0, \qquad p = 1,\dots,m,\ \ q = 1,\dots,n,$$

then $\mathcal{Q}$ and $\mathcal{R}$ are symmetric positive semi-definite matrices. On the other hand we assume that Eq. (2) has a unique solution, therefore the matrix

$$\mathcal{I}\otimes\mathcal{A}+\mathcal{B}^T\otimes\mathcal{I} = \mathcal{I}\otimes(\mathcal{W}+i\mathcal{T})+(\mathcal{U}+i\mathcal{V})^T\otimes\mathcal{I} = \mathcal{Q}+i\mathcal{R} = \mathcal{A},$$

is nonsingular, this yields $\mathrm{null}(\mathcal{R})\cap\mathrm{null}(\mathcal{Q}) = \{\mathbf{0}\}$, hence according to Lemma 2.1, there exists a nonsingular matrix $\mathcal{P}\in\mathbb{R}^{nm\times nm}$ such that

$$\mathcal{Q} = \mathcal{P}^T\mathcal{D}_{\mathcal{Q}}\mathcal{P}, \qquad \mathcal{R} = \mathcal{P}^T\mathcal{D}_{\mathcal{R}}\mathcal{P}, \tag{15}$$

where $\mathcal{D}_{\mathcal{Q}} = \mathrm{Diag}(\mu^{\mathcal{Q}}_1,\dots,\mu^{\mathcal{Q}}_{nm})$ and $\mathcal{D}_{\mathcal{R}} = \mathrm{Diag}(\eta^{\mathcal{R}}_1,\dots,\eta^{\mathcal{R}}_{nm})$ are diagonal $\eta^{\mathcal{R}}_l$ and $\mu^{\mathcal{Q}}_l$ satisfy

$$\mu^{\mathcal{Q}}_l+\eta^{\mathcal{R}}_l = 1, \qquad \eta^{\mathcal{R}}_l \geq 0,\ \mu^{\mathcal{Q}}_l \geq 0,\ l = 1,\dots,nm.$$

Remove $\mathbf{z}_{(k+\frac{1}{2})}$ from (14) to obtain $\mathbf{z}_{(k+1)} = \Pi(\alpha,\beta)\mathbf{z}_{(k)} + \mathcal{K}(\alpha,\beta;\mathbf{c})$, where $\Pi(\alpha,\beta)$ is iteration matrix for new method (9) and is defined in (11) and $\mathcal{K}(\alpha,\beta;\mathbf{c})$ is a $nm \times 1$ vector. We know that GCRI procedure (9) is convergent if $\rho(\Pi(\alpha,\beta)) < 1$. But

$$
\begin{aligned}
&(\beta\mathcal{Q}+\mathcal{R})^{-1}\mathcal{Q}(\alpha\mathcal{R}+\mathcal{Q})^{-1}\mathcal{R}\\
&= (\beta\mathcal{P}^T\mathcal{D}_{\mathcal{Q}}\mathcal{P}+\mathcal{P}^T\mathcal{D}_{\mathcal{R}}\mathcal{P})^{-1}\mathcal{P}^T\mathcal{D}_{\mathcal{Q}}\mathcal{P}(\alpha\mathcal{P}^T\mathcal{D}_{\mathcal{R}}\mathcal{P}+\mathcal{P}^T\mathcal{D}_{\mathcal{Q}}\mathcal{P})^{-1}\mathcal{P}^T\mathcal{D}_{\mathcal{R}}\mathcal{P}\\
&= \mathcal{P}^{-1}(\beta\mathcal{D}_{\mathcal{Q}}+\mathcal{D}_{\mathcal{R}})^{-1}\mathcal{D}_{\mathcal{Q}}(\alpha\mathcal{D}_{\mathcal{R}}+\mathcal{D}_{\mathcal{Q}})^{-1}\mathcal{D}_{\mathcal{R}}\mathcal{P}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\rho(\Pi(\alpha,\beta)) &= |(\beta+i)(\alpha-i)|\rho(\mathcal{P}^{-1}(\beta\mathcal{D}_{\mathcal{Q}}+\mathcal{D}_{\mathcal{R}})^{-1}\mathcal{D}_{\mathcal{Q}}(\alpha\mathcal{D}_{\mathcal{R}}+\mathcal{D}_{\mathcal{Q}})^{-1}\mathcal{D}_{\mathcal{R}}\mathcal{P})\\
&= \sqrt{(\alpha^2+1)(\beta^2+1)}\rho((\beta\mathcal{D}_{\mathcal{Q}}+\mathcal{D}_{\mathcal{R}})^{-1}\mathcal{D}_{\mathcal{Q}}(\alpha\mathcal{D}_{\mathcal{R}}+\mathcal{D}_{\mathcal{Q}})^{-1}\mathcal{D}_{\mathcal{R}})\\
&= \sqrt{(\alpha^2+1)(\beta^2+1)}\max_{0\le\mu_l^{\mathcal{Q}},\eta_l^{\mathcal{R}}\le 1}\left\{\frac{\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}}{\left(\beta\mu_l^{\mathcal{Q}}+\eta_l^{\mathcal{R}}\right)\left(\alpha\eta_l^{\mathcal{R}}+\mu_l^{\mathcal{Q}}\right)}\right\}\\
&= \sqrt{(\alpha^2+1)(\beta^2+1)}\max_{0\le\mu_l^{\mathcal{Q}},\eta_l^{\mathcal{R}}\le 1}\left\{\frac{\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}}{(\alpha\beta+1)\,\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}+\alpha\left(\eta_l^{\mathcal{R}}\right)^2+\beta\left(\mu_l^{\mathcal{Q}}\right)^2}\right\}.
\end{aligned}
$$

In [39] it was proved that when $\alpha = \beta > 0$ then $\rho(\Pi(\alpha,\beta)) = \rho(\Pi(\alpha,\alpha)) < 1$. Now suppose that $\alpha \neq \beta$ and $(\alpha,\beta) \in \Omega_1$, then $1+\alpha^2 < (\beta+1)^2$ and $\sqrt{(\alpha^2+1)(\beta^2+1)} < \alpha^2+1$. Moreover

$$
(\alpha\beta+1)\,\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}+\alpha\left(\eta_l^{\mathcal{R}}\right)^2+\beta\left(\mu_l^{\mathcal{Q}}\right)^2 \ge \left(\beta^2+1\right)\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}+\beta\left(\left(\eta_l^{\mathcal{R}}\right)^2+\left(\mu_l^{\mathcal{Q}}\right)^2\right) \ge (\beta+1)^2\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}.
$$

Hence

$$
\rho(\Pi(\alpha,\beta)) \le \theta_1(\alpha,\beta) = \frac{\alpha^2+1}{(\beta+1)^2} < \frac{\alpha^2+1}{\alpha^2+1} = 1.
$$

If $\alpha \neq \beta$ and $(\alpha,\beta) \in \Omega_2$, then $1+\beta^2 < (\alpha+1)^2$ and $\sqrt{(\alpha^2+1)(\beta^2+1)} < \beta^2+1$. Moreover

$$
(\alpha\beta+1)\,\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}+\alpha\left(\eta_l^{\mathcal{R}}\right)^2+\beta\left(\mu_l^{\mathcal{Q}}\right)^2 \ge \left(\alpha^2+1\right)\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}+\alpha\left(\left(\eta_l^{\mathcal{R}}\right)^2+\left(\mu_l^{\mathcal{Q}}\right)^2\right) \ge (\alpha+1)^2\mu_l^{\mathcal{Q}}\eta_l^{\mathcal{R}}.
$$

Thus

$$
\rho(\Pi(\alpha,\beta)) \le \theta_2(\alpha,\beta) = \frac{\beta^2+1}{(\alpha+1)^2} < \frac{\beta^2+1}{\beta^2+1} = 1.
$$

The recent result definitely shows that GCRI method converges to the unique solution of Eq. (2) for any $(\alpha,\beta) \in \Omega_1 \cup \Omega_2$ and any initial guess. □

3. Numerical Results

Consider the equation $\mathcal{A}\mathcal{Z}+\mathcal{Z}\mathcal{B}=\mathcal{C}$, with

$$
\mathcal{T} = \mathcal{I}\otimes\mathcal{V}+\mathcal{V}\otimes\mathcal{I}, \qquad \text{and} \qquad \mathcal{W} = 10(\mathcal{I}\otimes\mathcal{V}_c+\mathcal{V}_c\otimes\mathcal{I})+9(\mathbf{e}_1\mathbf{e}_m^T+\mathbf{e}_m\mathbf{e}_1^T)\otimes\mathcal{I},
$$

where

$$
\mathcal{V} = \mathrm{Tri}(-1,2,-1) = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}_{m\times m},
$$

and $\mathcal{V}_c = \mathcal{V} - \mathbf{e}_1\mathbf{e}_m^T - \mathbf{e}_m\mathbf{e}_1^T \in \mathbb{R}^{m\times m}$, $\mathbf{e}_1 = (1, 0, 0, ..., 0)^T \in \mathbb{R}^m$, $\mathbf{e}_m = (0, 0, ..., 1)^T \in \mathbb{R}^m$ and $\mathcal{B} = \mathcal{A}$. Therefore, the dimension of the matrices $\mathcal{W}$, $\mathcal{T}$, $\mathcal{U}$ and $\mathcal{V}$ will be $n = m^2$. Also right hand side matrix C is such that $\mathcal{Z}_* = (z_{i,j})$ with

$$z_{i,j} = \exp\left[-\left(x_i^2 + y_j^2\right)\right], \qquad i, j = 1, 2, ..., n, \tag{16}$$

can be exact solution, where $x_i = -1 + 2(i-1)/(n-1)$ and $y_j = -1 + 2(j-1)/(n-1)$, $\quad i, j = 1, 2, ..., n$.

In numerical results initial guess is taken $\mathcal{X}_{(0)} = O$ (zero matrix) and the stopping criteria for outer iterations is

$$\|C - \mathcal{A}\mathcal{X}_{(k)} - \mathcal{X}_{(k)}\mathcal{B}\|_F / \|C\|_F \leq 5 \times 10^{-6}.$$

The optimal parameters for PMHSS, CRI and GCRI methods are tabulated in Table 1. Also some numerical results such as, running time to seconds, iterations and logarithm of residual error are listed in Table 2. According to Table 2, we see that the number of iterations of these methods has not changed much with increasing the dimension of the problem, which shows that these methods are effective when increasing the dimension of the problem.

Also according to Table 2, the number of iterations and CPU time for GCRI method (9) is less than CRI and PMHSS methods, which shows the fast convergence of new method.

In Figure 1 we can see that (almost) locations of the optimal parameters for GCRI method when the sizes of matrices $\mathcal{W}$ and $\mathcal{T}$ are 64×64. Moreover at Figure 2 we can observe (almost) locations of the optimal parameters for PMHSS and CRI methods for $n = 64$.

We plotted the dispersion of the eigenvalues of iteration matrices in Fig. 3. According to this figure, the modulus of the eigenvalues of the iteration matrices for PMHSS and CRI schemes are large, which affects the modulus of the spectral radius of the iteration matrices. On the contrary, this figure shows the higher speed of GCRI method.

Figure 4 shows the logarithm of the residual error against the iterations number. The result of this graph is that the CRI method is faster than the PMHSS method.

In Figure 5 we plotted approximate solutions for imaginary and real parts. As one can see, by increasing the number of iterations, the solution obtained from GCRI method converges to the exact solution (16). In addition, it is observed that the imaginary part of the solution converges to the zero matrix, and this is in accordance with predetermined solution (16).

4. Conclusion

This paper studies an iterative method for solving complex Sylvester equation. A detailed convergence analysis was provided for the new method. We proved that on the region $\Omega_1 \cup \Omega_2$, the spectral radius of the matrix $\Pi(\alpha, \beta)$ is less than one. Finally, we tried to support the theoretical results discussed in this article via examining two numerical examples. We see that for large size of coefficient matrices two CRI and GCRI methods have similar rate of convergence.

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Table 1: The values of optimal parameters.

$n \times n$	64×64	100×100	400×400	900×900
PMHSS method				
α^*	0.65	0.69	0.70	0.73
CRI method				
α^*	1	1	1	1
GCRI method				
α^*	0.3	0.3	0.8	1
β^*	4	4	1.5	1.2

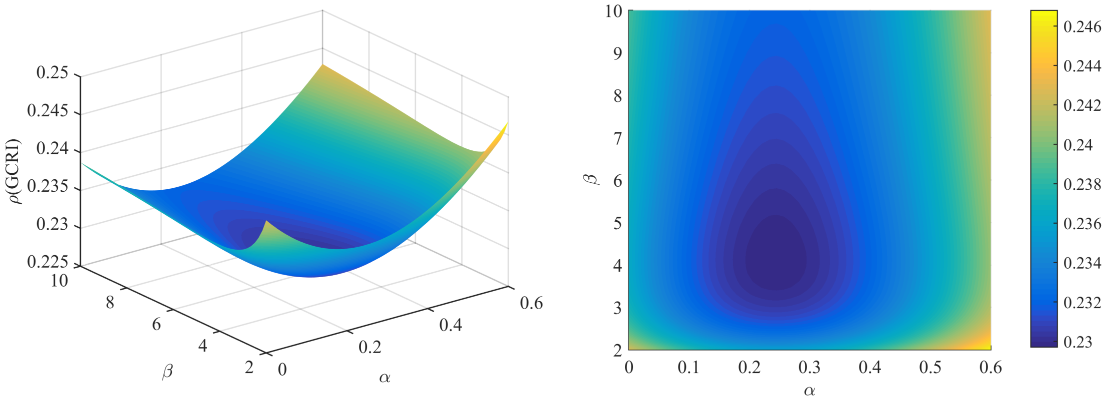

Figure 1: The almost locations of the optimal parameters for GCRI method.

Table 2: The comparison of iteration number, CPU time and residual error.

$n \times n$	64×64	100×100	400×400	900×900
PMHSS method				
IT	32	32	31	31
Time	0.2561	0.6684	52.5192	610.1070
$R(X^{(k)})$	-3.7472	-3.6493	-3.2091	-3.0157
CRI method				
IT	16	17	20	20
Time	0.1262	0.4431	31.3413	384.2159
$R(X^{(k)})$	-3.8081	-3.7094	-3.2001	-2.9972
GCRI method				
IT	12	14	18	19
Time	0.1027	0.3154	31.3413	365.0932
$R(X^{(k)})$	-3.4639	-3.5806	-3.5258	-2.3902

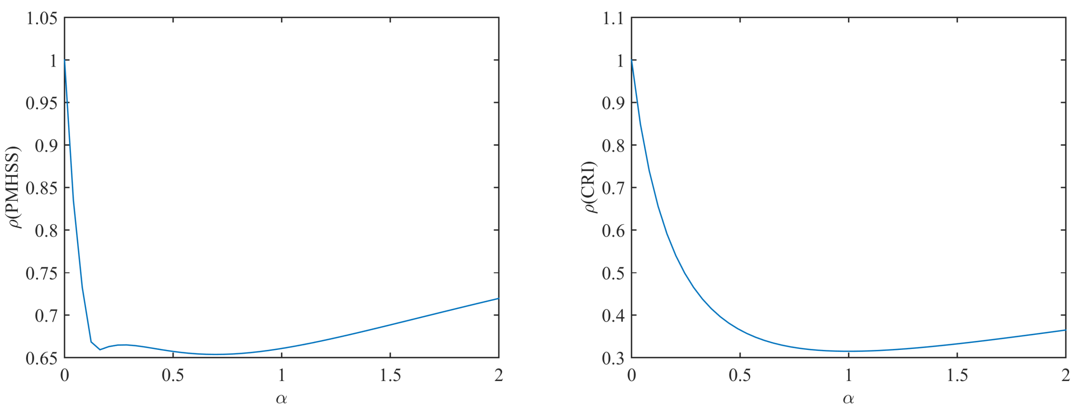

Figure 2: The almost locations of the optimal parameters for PMHSS and CRI methods.

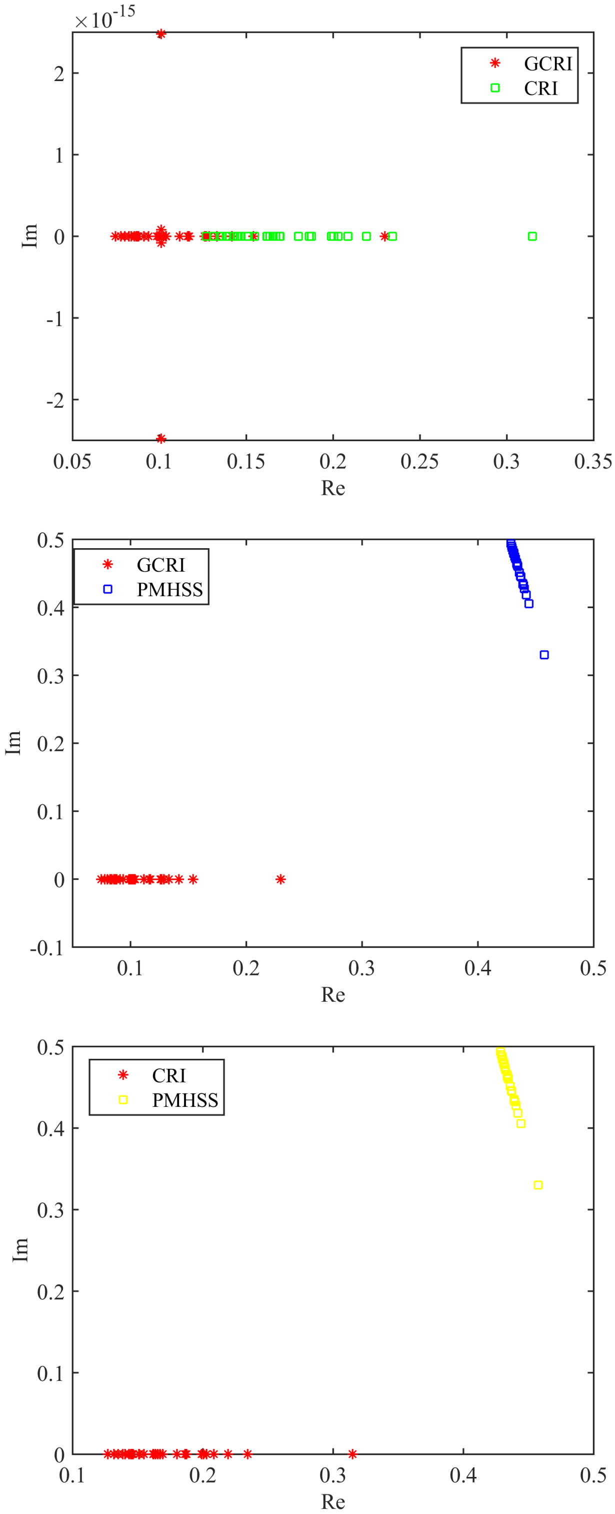

Figure 3: The eigenvalue distribution of the iteration matrices.

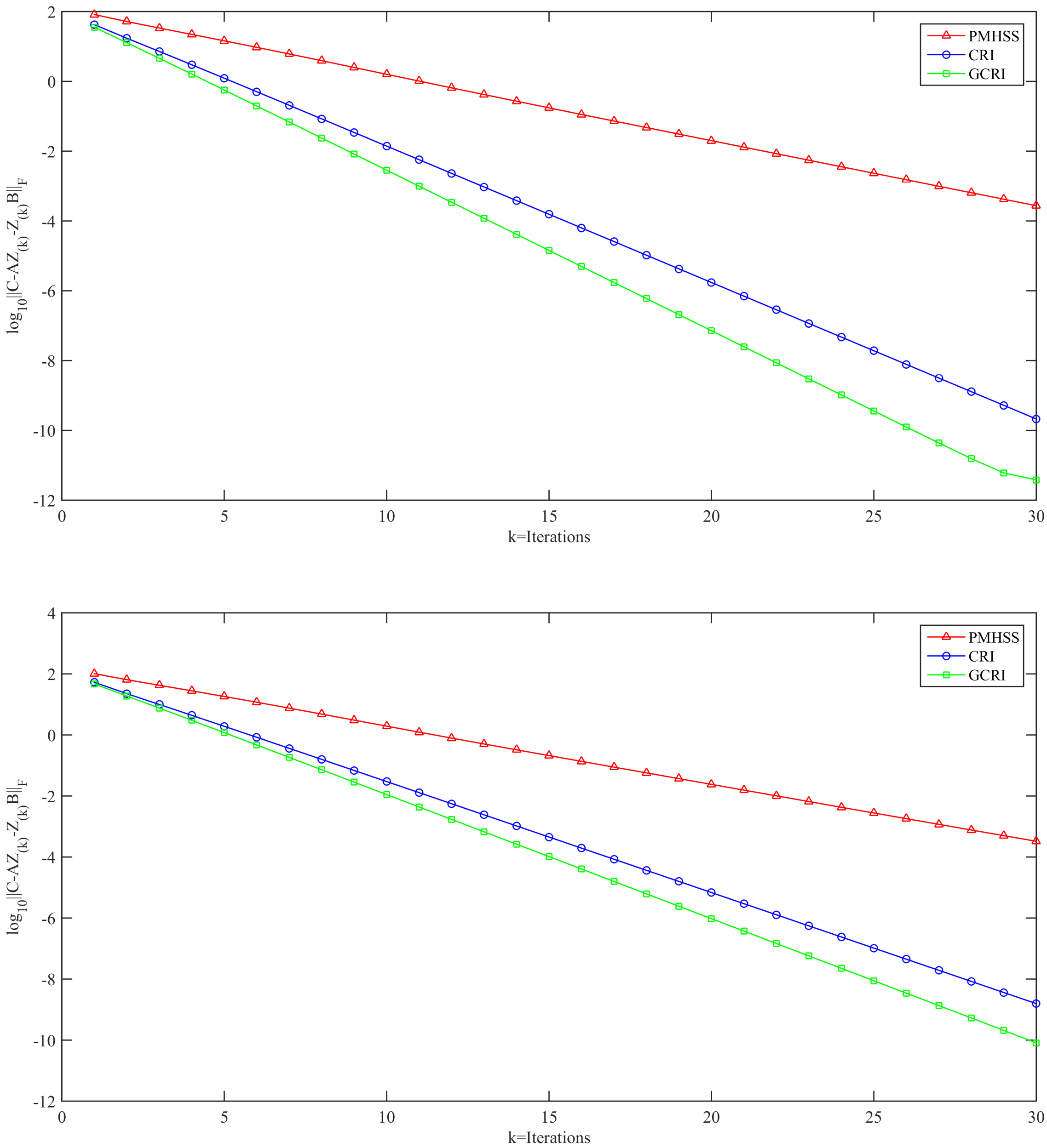

Figure 4: The logarithm of the residual error versus iteration number; Up (n = 64) and Bottom (n = 100).

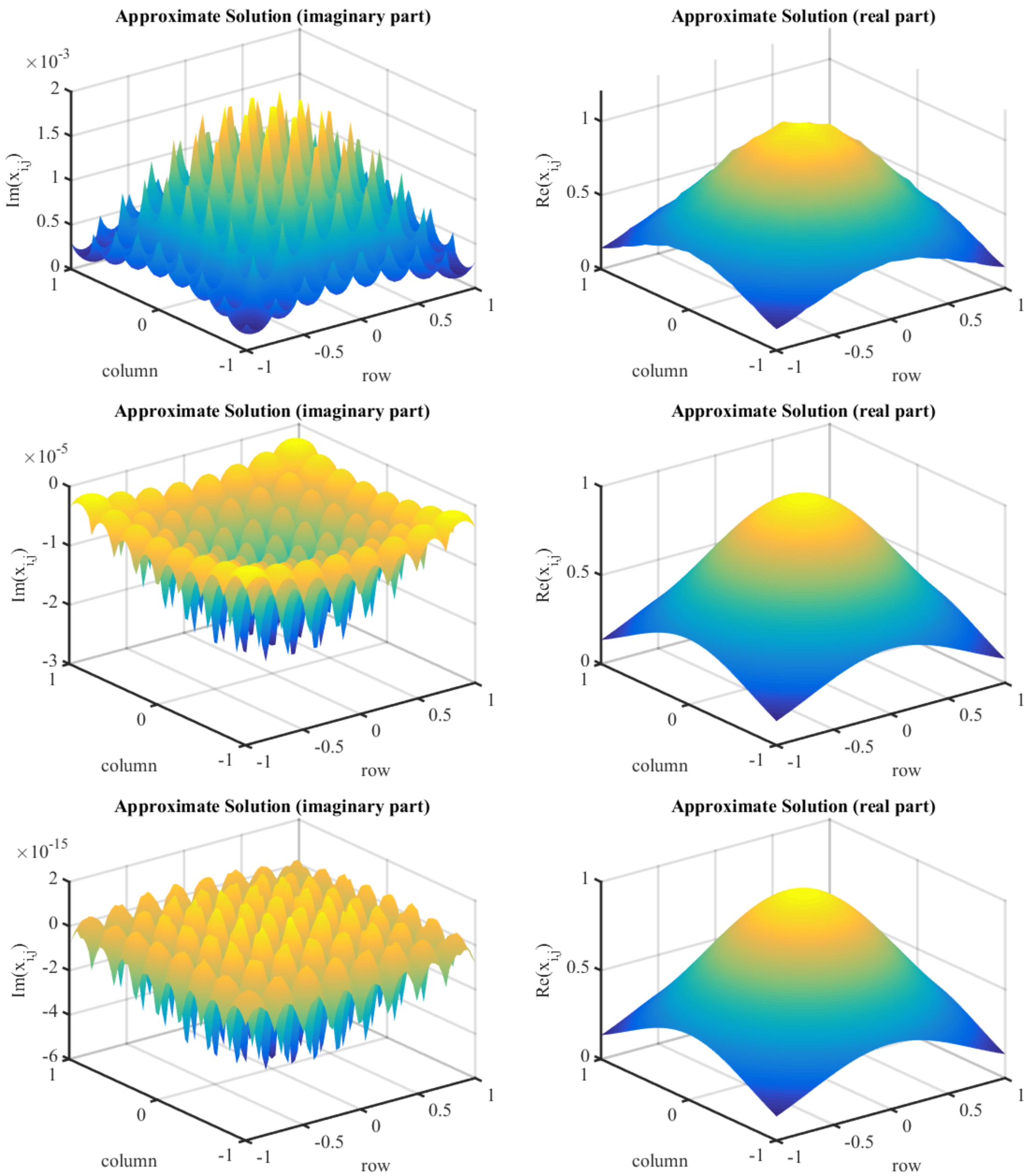

Figure 5: Approximate solutions for imaginary and real parts by GCRI method; Top (after 3 iterations); Middle (after 10 iterations); Bottom (after 30 iterations).