



Approximation by α -Baskakov–Jain Type Operators

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Abstract. In this manuscript, we consider the Baskakov-Jain type operators involving two parameters α and τ . Some approximation results concerning the weighted approximation are discussed. Also, we find a quantitative Voronovskaja type asymptotic theorem and Grüss Voronovskaya type approximation theorem for these operators. Some numerical examples to illustrate the approximation of these operators to certain functions are also given.

1. Introduction

Aral and Erbay [5] introduced Baskakov operators based on $\alpha \in [0, 1]$ as follows:

$$\mathcal{B}_m^{(\alpha)}(u; z) = \sum_{i=0}^{\infty} b_{m,i}^{(\alpha)}(z) u\left(\frac{i}{m}\right), \quad z \in [0, \infty), \quad (1)$$

where

$$b_{m,i}^{(\alpha)}(z) = \frac{z^{i-1}}{(1+z)^{m+i-1}} \left[\frac{\alpha z}{(1+z)} \binom{m+i-1}{i} - (1-\alpha)(1+z) \binom{m+i-3}{i-2} + (1-\alpha)z \binom{m+i-1}{i} \right].$$

For the special case $\alpha = 1$, the operators $\mathcal{B}_m^{(\alpha)}(f; z)$ reduces to the Baskakov operators [6]. Gupta [14] presented a general family of Durrmeyer type operators and derived some direct results. Kajla and Agrawal [22] defined Durrmeyer type modification of Szász operator involving Charlier polynomials. Kajla et al. [25] considered a Durrmeyer type generalization of the operators (1) and gave the uniform convergence results. In 2018, Gupta and Srivastava [20] presented a general sequence of operators which preserve linear functions. In [16], general estimates of the differences of the linear positive operators with different fundamental basis and their derivatives were obtained. In [36], Srivastava et al. gave the link between approximation theory and summability methods via four-dimensional infinite matrices.

2020 *Mathematics Subject Classification.* 26A15, 41A25, 41A35.

Keywords. Order of approximation, integral operators.

Received: 02 June 2021; Revised: 31 October 2021; Accepted: 09 November 2021

Communicated by Hari M. Srivastava

Dedicated to professor H. M. Srivastava on the occasion of his 80th birthday

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Srivastava et al. [35] proposed the idea of relatively modular deferred-weighted statistical convergence and statistically as well as relatively modular deferred-weighted summability for a double sequence of functions. In 2020, Mohiuddine et al. [30] Baskakov-Durrmeyer type operators based on the parameters and studied quantitative approximation results. Very recently, Mohiuddine et al. [27] introduced Stancu-Kantorovich variant of the operators (1) and studied the direct results. For article related to such type of a study we refer the reader to (cf. [2–4, 7–9, 11–13, 15, 17–19, 21–24, 26, 28, 29, 31–34] etc.) and reference therein.

Let $\tau > 0$ and $\alpha \in [0, 1]$. For $\gamma > 0$ and $C_\gamma[0, \infty) := \{u \in C[0, \infty) : |u(v)| \leq N_u e^{\gamma v}, \text{ for some } N_u > 0\}$, we construct a α -Baskakov-Jain type operators as follows:

$$\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) = \sum_{i=0}^{\infty} b_{m,i}^{(\alpha)}(z) \int_0^{\infty} b_{m,i}^{\tau}(v) u(v) dv, \tag{2}$$

where $b_{m,i}^{\tau}(v) = \frac{\tau}{B\left(i+1, \frac{m}{\tau}\right)} \frac{(\tau v)^i}{(1+\tau v)^{\frac{m}{\tau}+i+1}}$ and $b_{m,i}^{(\alpha)}(z)$ is defined as above.

In this article, we will study the order of convergence of these operators in a weighted space and asymptotic type formula, quantitative and Grüss Voronovskaya type approximation theorem.

2. Basic Results

Lemma 2.1. For $z \in [0, \infty)$, the moments of the operators $\mathcal{G}_{m,\tau}^{(\alpha)}(u; z)$ are given by

- (i) $\mathcal{G}_{m,\tau}^{(\alpha)}(e_0; z) = 1;$
- (ii) $\mathcal{G}_{m,\tau}^{(\alpha)}(e_1; z) = \frac{z(m+2\alpha-2)}{(m-\tau)} + \frac{1}{(m-\tau)};$
- (iii) $\mathcal{G}_{m,\tau}^{(\alpha)}(e_2; z) = \frac{mz^2(-3+m+4\alpha)}{(m-2\tau)(m-\tau)} + \frac{z(4m+10(-1+\alpha))}{(m-2\tau)(m-\tau)} + \frac{2}{(m-2\tau)(m-\tau)};$
- (iv) $\mathcal{G}_{m,\tau}^{(\alpha)}(e_3; z) = \frac{m(1+m)z^3(-4+m+6\alpha)}{(m-3\tau)(m-2\tau)(m-\tau)} + \frac{3mz^2(-11+3m+14\alpha)}{(m-3\tau)(m-2\tau)(m-\tau)} + \frac{18z(-3+m+3\alpha)}{(m-3\tau)(m-2\tau)(m-\tau)}$
 $+ \frac{6}{(m-3\tau)(m-2\tau)(m-\tau)};$
- (v) $\mathcal{G}_{m,\tau}^{(\alpha)}(e_4; z) = \frac{m(1+m)(2+m)z^4(-5+m+8\alpha)}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{4m(1+m)z^3(-19+4m+27\alpha)}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}$
 $+ \frac{24mz^2(-13+3m+16\alpha)}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{z(96m+336(-1+\alpha))}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}$
 $+ \frac{24}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)};$
- (vi) $\mathcal{G}_{m,\tau}^{(\alpha)}(e_5; z) = \frac{m(1+m)(2+m)(3+m)z^5(-6+m+10\alpha)}{(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}$
 $+ \frac{5m(1+m)(2+m)z^4(-29+5m+44\alpha)}{(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{100m(1+m)z^3(-11+2m+15\alpha)}{(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}$
 $+ \frac{600mz^2(-5+m+6\alpha)}{(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{600z(-4+m+4\alpha)}{(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}$
 $+ \frac{120}{(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)};$
- (vii) $\mathcal{G}_{m,\tau}^{(\alpha)}(e_6; z) = \frac{m(1+m)(2+m)(3+m)(4+m)z^6(-7+m+12\alpha)}{(m-6\tau)(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}$
 $+ \frac{6m(1+m)(2+m)(3+m)z^5(-41+6m+65\alpha)}{(m-6\tau)(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{90m(1+m)(2+m)z^4(-33+5m+48\alpha)}{(m-6\tau)(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}$

$$\begin{aligned}
 &+ \frac{600m(1+m)z^3(-25+4m+33\alpha)}{(m-6\tau)(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{1800mz^2(-17+3m+20\alpha)}{(m-6\tau)(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} \\
 &+ \frac{z(4320m+19440(-1+\alpha))}{(m-6\tau)(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{720}{(m-6\tau)(m-5\tau)(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}.
 \end{aligned}$$

Lemma 2.2. From Lemma 2.1, we have

$$\begin{aligned}
 \mathcal{G}_{m,\tau}^{(\alpha)}((v-z); z) &= \frac{z(\tau+2\alpha-2)}{(m-\tau)} + \frac{1}{(m-\tau)}; \mathcal{G}_{m,\tau}^{(\alpha)}((v-z)^2; z) = \frac{z^2(m+\tau(-8+2\tau+m+8\alpha))}{(m-2\tau)(m-\tau)} + \frac{2z(-5+2\tau+m+5\alpha)}{(m-2\tau)(m-\tau)} + \\
 &\frac{2}{(m-2\tau)(m-\tau)}; \\
 \mathcal{G}_{m,\tau}^{(\alpha)}((v-z)^4; z) &= \\
 &\frac{z^4(24\tau^4-10m+3m^2+16m\alpha+2\tau m(-32+3m+48\alpha)+\tau^2 m(-8+3m+80\alpha)+2\tau^3(-96+23m+96\alpha))}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} \\
 &+ \frac{z^3(96\tau^3-76m+184\tau^2 m+12m^2+720\tau^2(-1+\alpha)+108m\alpha+12\tau m(-9+m+21\alpha))}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} \\
 &+ \frac{z^2(144\tau^2-96m+204\tau m+12m^2+864\tau(-1+\alpha)+168m\alpha)}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} + \frac{z(96\tau+72m+336(-1+\alpha))}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)} \\
 &+ \frac{24}{(m-4\tau)(m-3\tau)(m-2\tau)(m-\tau)}.
 \end{aligned}$$

Remark 2.3. We have

$$\begin{aligned}
 \lim_{m \rightarrow \infty} m \mathcal{J}_{m,\tau}^{\alpha,1}(z) &= 1+z(\tau+2\alpha-2), \\
 \lim_{m \rightarrow \infty} m \mathcal{J}_{m,\tau}^{\alpha,2}(z) &= z(2+z+\tau z), \\
 \lim_{m \rightarrow \infty} m^2 \mathcal{J}_{m,\tau}^{\alpha,4}(z) &= 3z^2(2+z+\tau z)^2, \\
 \lim_{m \rightarrow \infty} m^3 \mathcal{J}_{m,\tau}^{\alpha,6}(z) &= 15z^3(2+z+\tau z)^3,
 \end{aligned}$$

where $\mathcal{J}_{m,\tau}^{\alpha,s} := \mathcal{G}_{m,\tau}^{(\alpha)}((v-z)^s; z), s = 1, 2, 4, 6$.

3. Direct Results

Theorem 3.1. Let $u \in C_\gamma[0, \infty)$. Then $\lim_{m \rightarrow \infty} \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) = u(z)$, uniformly in each compact subset of $[0, \infty)$.

Proof. By the application of Bohman-Korovkin Result and Lemma 2.1, the proof of this theorem is direct. \square

3.1. Voronovskaja type theorem

Theorem 3.2. Let $u \in C_\gamma[0, \infty)$. If u'' exists at a point $z \in [0, \infty)$, then

$$\lim_{m \rightarrow \infty} m \left[\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) \right] = (1+z(\tau+2\alpha-2))u'(z) + \frac{1}{2}z(2+z+\tau z)u''(z).$$

Proof. From Taylor’s theorem, we have

$$u(v) = u(z) + u'(z)(v-z) + \frac{1}{2}u''(z)(v-z)^2 + \omega(v, z)(v-z)^2, \tag{3}$$

where $\lim_{v \rightarrow z} \omega(v, z) = 0$. Applying the linear operator $\mathcal{G}_{m,\tau}^{(\alpha)}$, we may write

$$\begin{aligned}
 \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) &= \mathcal{G}_{m,\tau}^{(\alpha)}((v-z); z)u'(z) + \frac{1}{2}\mathcal{G}_{m,\tau}^{(\alpha)}((v-z)^2; z)u''(z) \\
 &+ \mathcal{G}_{m,\tau}^{(\alpha)}(\omega(v, z)(v-z)^2; z).
 \end{aligned}$$

The Cauchy-Schwarz inequality implies

$$m\mathcal{G}_{m,\tau}^{(\alpha)}(\omega(v,z)(v-z)^2; z) \leq \sqrt{\mathcal{G}_{m,\tau}^{(\alpha)}(\omega^2(v,z); z)} \sqrt{m^2\mathcal{G}_{m,\tau}^{(\alpha)}((v-z)^4; z)}. \tag{4}$$

As $\omega^2(z,z) = 0$ and $\omega^2(\cdot, z) \in C_\gamma[0, \infty)$, we have

$$\lim_{m \rightarrow \infty} \mathcal{G}_{m,\tau}^{(\alpha)}(\omega^2(v,z); z) = \omega^2(z,z) = 0. \tag{5}$$

Collecting (4)-(5) and Remark 2.3, we obtain

$$\lim_{m \rightarrow \infty} m\mathcal{G}_{m,\tau}^{(\alpha)}(\omega(v,z)(v-z)^2; z) = 0. \tag{6}$$

Hence

$$\lim_{m \rightarrow \infty} m \left[\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) \right] = (1 + z(\tau + 2\alpha - 2)) u'(z) + \frac{1}{2}z(2 + z + \tau z)u''(z).$$

□

4. Weighted approximation

Let $H_\rho[0, \infty)$ denote the space of all real-valued functions on $[0, \infty)$ satisfying the condition $|u(z)| \leq N_u\rho(z)$, where $N_u > 0$ is a constant depending only on u and $\rho(z) = 1 + z^2$ is a weight function. Suppose that $C_\rho[0, \infty)$ be the space of all continuous functions in $H_\rho[0, \infty)$ endowed with the norm $\|u\|_\rho := \sup_{z \in [0, \infty)} \frac{|u(z)|}{\rho(z)}$ and

$$C_\rho^0[0, \infty) := \left\{ u \in C_\rho[0, \infty) : \lim_{z \rightarrow \infty} \frac{|u(z)|}{\rho(z)} < \infty \right\}.$$

Theorem 4.1. For each $u \in C_\rho^0[0, \infty)$ and $r > 0$, we have

$$\lim_{m \rightarrow \infty} \sup_{z \in [0, \infty)} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)|}{(1 + z^2)^{1+r}} = 0.$$

Proof. For any fixed $z_0 > 0$, there holds the relation

$$\begin{aligned} \sup_{z \in [0, \infty)} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)|}{(1 + z^2)^{1+r}} &\leq \sup_{z \leq z_0} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)|}{(1 + z^2)^{1+r}} + \sup_{z > z_0} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)|}{(1 + z^2)^{1+r}} \\ &\leq \sup_{z \leq z_0} \left\{ |\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)| \right\} + \sup_{z > z_0} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(u; z)|}{(1 + z^2)^{1+r}} \\ &\quad + \sup_{z > z_0} \frac{|u(z)|}{(1 + z^2)^{1+r}}. \end{aligned}$$

Since $|u(v)| \leq \|u\|_\rho(1 + v^2), \forall v \geq 0$

$$\begin{aligned} \sup_{z \in [0, \infty)} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)|}{(1 + z^2)^{1+r}} &\leq \left\| \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) \right\|_{C[0, z_0]} + \|u\|_\rho \sup_{z > z_0} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(1 + v^2; z)|}{(1 + z^2)^{1+r}} \\ &\quad + \sup_{z > z_0} \frac{\|u\|_\rho}{(1 + z^2)^r} \\ &= I_1 + I_2 + I_3, \text{ say.} \end{aligned} \tag{7}$$

Now, in view of Theorem 3.1, for a given $\epsilon > 0, \exists m_1 \in \mathbb{N}$ such that

$$I_1 = \left\| \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) \right\|_{C[0, z_0]} < \frac{\epsilon}{3}, \text{ for all } m \geq m_1. \tag{8}$$

Since $\lim_{m \rightarrow \infty} \sup_{z > z_0} \frac{\mathcal{G}_{m,\tau}^{(\alpha)}(1 + v^2; z)}{1 + z^2} = 1$, it follows that there exists $m_2 \in \mathbb{N}$ such that

$$\sup_{z > z_0} \frac{\mathcal{G}_{m,\tau}^{(\alpha)}(1 + v^2; z)}{1 + z^2} \leq \frac{(1 + z_0^2)^r}{\|u\|_\rho} \cdot \frac{\epsilon}{3} + 1, \text{ for all } m \geq m_2.$$

Hence,

$$\begin{aligned} I_2 &= \|u\|_\rho \sup_{z > z_0} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(1 + v^2; z)|}{(1 + z^2)^{1+r}} \leq \frac{\|u\|_\rho}{(1 + z_0^2)^r} \sup_{z > z_0} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(1 + v^2; z)|}{1 + z^2} \\ &\leq \frac{\|u\|_\rho}{(1 + z_0^2)^r} + \frac{\epsilon}{3}, \text{ for all } m \geq m_2. \end{aligned} \tag{9}$$

Choose z_0 to be so large that

$$\frac{\|u\|_\rho}{(1 + z_0^2)^r} < \frac{\epsilon}{6},$$

then

$$I_3 = \sup_{z > z_0} \frac{\|u\|_\rho}{(1 + z^2)^r} \leq \frac{\|u\|_\rho}{(1 + z_0^2)^r} < \frac{\epsilon}{6}. \tag{10}$$

Let $m_0 = \max\{m_1, m_2\}$, then by combining (8-10)

$$\sup_{z \in [0, \infty)} \frac{|\mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)|}{(1 + z^2)^{1+r}} < \epsilon, \text{ for all } m \geq m_0.$$

□

In the following we study a quantitative Voronoskaja type result for the operators $\mathcal{G}_{m,\tau}^{(\alpha)}$ for functions u in the weighted space $C_\rho[0, \infty)$. Āspir [37], considered the weighted modulus of continuity $\Omega(u; \sigma)$ as follows:

$$\Omega(u; \sigma) = \sup_{0 \leq h < \sigma, z \in [0, \infty)} \frac{|u(z + h) - u(z)|}{(1 + h^2)(1 + z^2)} \tag{11}$$

for $u \in C_\rho[0, \infty)$. From [37], if $u \in C_\rho^0[0, \infty)$, then $\Omega(\cdot; \sigma)$ has the properties

$$\lim_{\sigma \rightarrow 0} \Omega(u; \sigma) = 0$$

and

$$\Omega(u; \lambda\sigma) \leq 2(1 + \lambda)(1 + \sigma^2)\Omega(u; \sigma), \quad \lambda > 0. \tag{12}$$

From the equations (11)-(12) and $u \in C_\rho^0[0, \infty)$, we can write

$$\begin{aligned} |u(v) - u(z)| &\leq (1 + (v - z)^2)(1 + z^2)\Omega(u; |v - z|) \\ &\leq 2\left(1 + \frac{|v - z|}{\sigma}\right)(1 + \sigma^2)\Omega(u; \sigma)(1 + (v - z)^2)(1 + z^2). \end{aligned} \tag{13}$$

Theorem 4.2. Let $u \in C_\rho^0[0, \infty)$ such that $u'(z), u''(z) \in C_\rho^0[0, \infty)$. Then for sufficiently large m and each $z \in [0, \infty)$,

$$\left| m \left\{ \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) - u'(z)\mathcal{G}_{m,\tau}^{(\alpha)}((v - z); z) - \frac{u''(z)}{2!}\mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) \right\} \right| = O(1)\Omega(u''; \sqrt{1/m}).$$

Proof. Applying Taylor’s formula, we have

$$\begin{aligned} u(v) &= u(z) + u'(z)(v - z) + \frac{u''(\beta)}{2!}(v - z)^2 \\ &= u(z) + u'(z)(v - z) + \frac{u''(z)}{2!}(v - z)^2 + h_2(v, z), \end{aligned} \tag{14}$$

where β is a number between z and v , we have

$$h_2(v, z) = \frac{u''(\beta) - u''(z)}{2!}(v - z)^2. \tag{15}$$

Using the property (13) of the weighted modulus of continuity, we may write

$$\begin{aligned} |u''(\beta) - u''(z)| &\leq (1 + (\beta - z)^2)(1 + z^2)\Omega(u''; |\beta - z|) \\ &\leq (1 + (v - z)^2)(1 + z^2)\Omega(u''; |v - z|) \\ &\leq 2(1 + (v - z)^2)(1 + z^2) \left(1 + \frac{|v - z|}{\sigma}\right) (1 + \sigma^2)\Omega(u''; \sigma), \end{aligned} \tag{16}$$

but

$$\left(1 + \frac{|v - z|}{\sigma}\right)(1 + (v - z)^2) \leq \begin{cases} 2(1 + \sigma^2), & |v - z| < \sigma, \\ 2\frac{(v - z)^4}{\sigma^4}(1 + \sigma^2), & |v - z| \geq \sigma, \end{cases}$$

i.e.

$$\left(1 + \frac{|v - z|}{\sigma}\right)(1 + (v - z)^2) \leq 2\left(1 + \frac{(v - z)^4}{\sigma^4}\right)(1 + \sigma^2). \tag{17}$$

Collecting the equations (15)-(17) and choosing $0 < \sigma < 1$, we find that

$$|h_2(v, z)| \leq 2(1 + \sigma^2)^2(1 + z^2)\Omega(u''; \sigma) \left(1 + \frac{(v - z)^4}{\sigma^4}\right)(v - z)^2. \tag{18}$$

Operating the operator $\mathcal{G}_{m,\tau}^{(\alpha)}$ and Lemma 2.2 on both sides of (14), we obtain

$$\left| \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) - u'(z)\mathcal{G}_{m,\tau}^{(\alpha)}(v - z; z) - \frac{u''(z)}{2!}\mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) \right| \leq \mathcal{G}_{m,\tau}^{(\alpha)}(|h_2(v, z)|; z). \tag{19}$$

Applying Remark 2.3 and using equation (18), we may write

$$\begin{aligned} \mathcal{G}_{m,\tau}^{(\alpha)}(|h_2(v, z)|; z) &\leq 2(1 + \sigma^2)^2(1 + z^2)\Omega(u''; \sigma)\mathcal{G}_{m,\tau}^{(\alpha)}\left(\left((v - z)^2 + \frac{(v - z)^6}{\sigma^4}\right); z\right) \\ &= 2(1 + \sigma^2)^2(1 + z^2)\Omega(u''; \sigma)\left(\mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) + \frac{1}{\sigma^4}\mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^6; z)\right) \\ &= 2(1 + \sigma^2)^2(1 + z^2)\Omega(u''; \sigma)\left(O(1/m) + \frac{1}{\sigma^4}O(1/m^3)\right). \end{aligned}$$

By taking $\sigma = \sqrt{1/m}$, we obtain

$$m\mathcal{G}_{m,\tau}^{(\alpha)}(|h_2(v, z)|; z) = O(1)\Omega(u''; \sqrt{1/m}). \tag{20}$$

Using (19) and (20), we find that

$$\left| m \left\{ \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z)u'(z)\mathcal{G}_{m,\tau}^{(\alpha)}(v - z; z) - \frac{u''(z)}{2!}\mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) \right\} \right| = O(1)\Omega(u''; \sqrt{1/m}), \text{ as } m \rightarrow \infty.$$

□

5. Grüss Voronovskaya Type Theorem

Theorem 5.1. Let u, w and $uw \in C_0^0[0, \infty)$ such that $u', w', (uw)', u'', w''$ and $(uw)'' \in C_0^0[0, \infty)$. Then, for each $z \in [0, \infty)$,

$$\lim_{m \rightarrow \infty} m \left\{ \mathcal{G}_{m,\tau}^{(\alpha)}((uw); z) - \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) \mathcal{G}_{m,\tau}^{(\alpha)}(w; z) \right\} = u'(z)w'(z)z(2 + z + \tau z).$$

Proof. Since $(uw)(z) = u(z)w(z)$, $(uw)'(z) = u'(z)w(z) + u(z)w'(z)$ and $(uw)''(z) = u''(z)w(z) + 2u'(z)w'(z) + u(z)w''(z)$, we get

$$\begin{aligned} & \mathcal{G}_{m,\tau}^{(\alpha)}((uw); z) - \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) \mathcal{G}_{m,\tau}^{(\alpha)}(w; z) \\ &= \left\{ \mathcal{G}_{m,\tau}^{(\alpha)}((uw); z) - u(z)w(z) - (uw)'(z) \mathcal{G}_{m,\tau}^{(\alpha)}(v - z; z) - \frac{(uw)''(z)}{2!} \mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) \right\} \\ & \quad - w(z) \left\{ \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) - u(z) - u'(z) \mathcal{G}_{m,\tau}^{(\alpha)}(v - z; z) - \frac{u''(z)}{2!} \mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) \right\} \\ & \quad - \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) \left\{ \mathcal{G}_{m,\tau}^{(\alpha)}(w; z) - w(z) - w'(z) \mathcal{G}_{m,\tau}^{(\alpha)}(v - z; z) - \frac{w''(z)}{2!} \mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) \right\} \\ & \quad + \frac{1}{2!} \mathcal{G}_{m,\tau}^{(\alpha)}((v - z)^2; z) \left\{ u(z)w''(z) + 2u'(z)w'(z) - w''(z) \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) \right\} \\ & \quad + \mathcal{G}_{m,\tau}^{(\alpha)}(v - z; z) \left\{ u(z)w'(z) - w'(z) \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) \right\}. \end{aligned}$$

From Theorems 3.1 and 4.2 and Lemma 2.2, we find that

$$\lim_{m \rightarrow \infty} m \left\{ \mathcal{G}_{m,\tau}^{(\alpha)}((uw); z) - \mathcal{G}_{m,\tau}^{(\alpha)}(u; z) \mathcal{G}_{m,\tau}^{(\alpha)}(w; z) \right\} = u'(z)w'(z)z(2 + z + \tau z).$$

This completes the proof. \square

6. Numerical Examples

Example 6.1. The convergence of $\mathcal{G}_{m,\tau}^{(\alpha)}(u; z)$ operators is illustrated in Figure 1, where $u(z) = z^3(1 + 2z)^3$, $z \in [0, 1]$, $\alpha = 0.9$ $\tau = 0.5$ and $m = 15, 16, 17, 18, 19$. We observed that when the values of m are increasing, the graph of operators $\mathcal{G}_{m,\tau}^{(\alpha)}(u; z)$ are going to the graph of the function u .

Example 6.2. The convergence of $\mathcal{G}_{m,\tau}^{(\alpha)}(u; z)$ operators is illustrated in Figure 2, where $u(z) = z^2(1 + 5z)^2$, $z \in [0, 1]$, $\alpha = 0.9$ $\tau = 0.5$ and $m = 15, 16, 17, 18, 19$. It is seen that when the values of m are increasing, the graph of operators $\mathcal{G}_{m,\tau}^{(\alpha)}(u; z)$ are going to the graph of the function u .

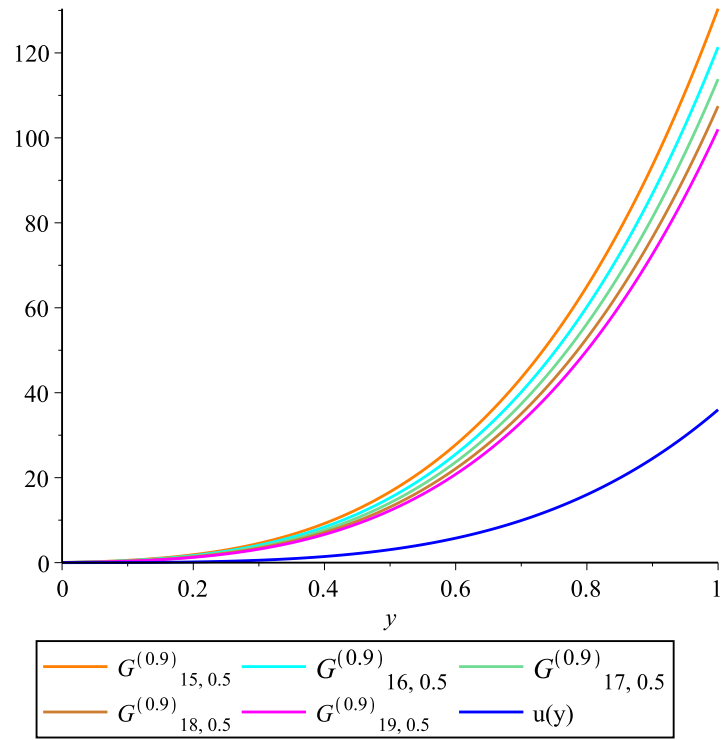


Figure 1: Approximation Process

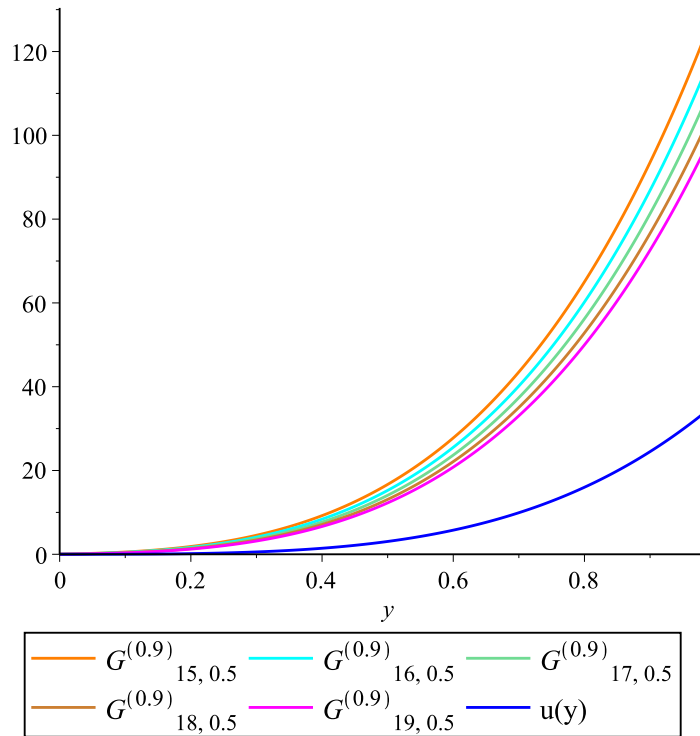


Figure 2: Approximation Process

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