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Conformal Semi-Slant Riemannian Maps from Almost Hermitian Manifolds onto Riemannian Manifolds

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Abstract. In this study, we define the notion of conformal semi-slant Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds as a generalization of conformal semi-slant submersions. We give examples for this type maps. We study integrability conditions of distributions. In addition, we apply pluriharmonic maps to investigate being horizontally homothetic map. Moreover, we examine that under which cases, the distributions can define totally geodesic foliations.

1. Introduction

Firstly, the concept of submersion was introduced by O'Neill [11] and Gray [8]. Then, this concept was studied in various types [6] as a semi-invariant [17], a slant [15], a semi-slant [13], etc [20, 23]. Then, this concept generalized to the notion of Riemannian map by Fischer [7]. Riemannian maps between Riemannian manifolds are generalization of isometric immersions and Riemannian submersions. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{dim(M_1), dim(M_2)\}$. Then the tangent bundle TM_1 of M_1 has the following decomposition:

$$TM_1 = kerF_* \oplus (kerF_*)^{\perp}.$$

Since $rankF < min\{dim(M_1), dim(M_2)\}$, always we have $(rangeF_*)^{\perp}$. In this way, tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (rangeF_*) \oplus (rangeF_*)^{\perp}.$$

A smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^m, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F^h_{*p_1} : (kerF_{*p_1})^{\perp} \longrightarrow (rangeF_*)$ is a linear isometry. Hence a Riemannian map satisfies the equation

$$g_1(X, Y) = g_2(F_*(X), F_*(Y)) \tag{1}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. So that isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $kerF_* = \{0\}$ and $(rangeF_*)^{\perp} = \{0\}$ [7].

Moreover, Şahin and the others introduced any other types of Riemannian maps [12, 14, 16, 18]. After this studies, especially Akyol, Şahin and Yanan searched conformality case of this type submersions [1–4]

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and Riemannian maps [24, 25]. We say that $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rankF_{*p} \leq min\{m, n\}$ and F_{*p} maps the horizontal space $(ker(F_{*p})^{\perp})$ conformally onto $range(F_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(F_{*p}(X), F_{*p}(Y)) = \lambda^2(p) g_M(X, Y) \tag{2}$$

for $X, Y \in \Gamma((ker(F_{*p})^{\perp})$. Also F is called conformal Riemannian if F is conformal Riemannian at each $p \in M$ [19]. Here, λ is the dilation of F at a point $p \in M$ and it is a continuous function as $\lambda : M \to [0, \infty)$.

An even-dimensional Riemannian manifold (M, g_M, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1, 1)$ on M such that $J^2 = -I$ where I denotes the identity transformation of TM and

$$g_M(X, Y) = g_M(JX, JY), \forall X, Y \in \Gamma(TM). \tag{3}$$

Let (M, g_M, J) is an almost Hermitian manifold and its Levi-Civita connection is ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.

$$(\nabla_X J)Y = 0, \tag{4}$$

we say M is a Kähler manifold [27].

Let (M, g_M, J) is an almost Hermitian manifold and (N, g_N) is a Riemannian manifold. A Riemannian map $F : (M, g_M, J) \longrightarrow (N, g_N)$ is called a semi-slant Riemannian map if there is a distribution $\mathcal{D}_1 \subset kerF_*$ such that

$$kerF_* = \mathcal{D}_1 \oplus \mathcal{D}_2, J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $kerF_*$. We call the angle θ a semi-slant angle [12].

Therefore, we will study conformal semi-slant Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds as a generalization of conformal semi-slant submersions which includes semi-slant submersions. We know that conformal semi-slant Riemannian maps include conformal invariant Riemannian maps, conformal anti-invariant Riemannian maps [21], conformal semi-invariant Riemannian maps [22] and conformal slant Riemannian maps [26]. Geometric properties were investigated and examples were given for this type maps. Also, several conditions for conformal semi-slant Riemannian maps to be horizontally homothetic maps were obtained by using the notion of pluriharmonic maps. Moreover, certain geodesicity conditions for conformal semi-slant Riemannian maps were obtained.

2. Preliminaries

In this section, we give several definitions and results to be used throughout the study for conformal semi-slant Riemannian maps. Let $F : (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds. The second fundamental form of F is defined by

$$(\nabla F_*)(X, Y) = \overset{N}{\nabla}{}^F_X F_*(Y) - F_*(\overset{M}{\nabla}_X Y) \tag{5}$$

for $X, Y \in \Gamma(TM)$. The second fundamental form ∇F_* is symmetric [9]. Recall that F is said to be totally geodesic map if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$.

Then we define O'Neill's tensor fields $\mathcal{T}$ and $\mathcal{A}$ for Riemannian submersions as

$$\mathcal{A}_X Y = h\overset{M}{\nabla}_{hX} vY + v\overset{M}{\nabla}_{hX} hY, \tag{6}$$

$$\mathcal{T}_X Y = h\overset{M}{\nabla}_{vX} vY + v\overset{M}{\nabla}_{vX} hY, \tag{7}$$

for $X, Y \in \Gamma(TM)$ with the Levi-Civita connection $\overset{M}{\nabla}$ of g_M [11]. As usual, we denote by v and h the projections on the vertical distribution $kerF_*$ and the horizontal distribution $(kerF_*)^{\perp}$, respectively. For any

$X \in \Gamma(TM)$, $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. Also, $\mathcal{T}$ is vertical, $\mathcal{T}_X = \mathcal{T}_{vX}$, and $\mathcal{A}$ is horizontal, $\mathcal{A}_X = \mathcal{A}_{hX}$. Note that the tensor field $\mathcal{T}$ is symmetric on the vertical distribution [11]. Additionally, from (6) and (7) we have

$$\overset{M}{\nabla}_U V = \mathcal{T}_U V + \hat{\nabla}_U V, \tag{8}$$
$$\overset{M}{\nabla}_U X = h\overset{M}{\nabla}_U X + \mathcal{T}_U X, \tag{9}$$
$$\overset{M}{\nabla}_X V = \mathcal{A}_X V + v\overset{M}{\nabla}_X V, \tag{10}$$
$$\overset{M}{\nabla}_X Y = h\overset{M}{\nabla}_X Y + \mathcal{A}_X Y \tag{11}$$

for $X, Y \in \Gamma((ker\, F_*)^\perp)$ and $U, V \in \Gamma(kerF_*)$, where $\hat{\nabla}_U V = v\overset{M}{\nabla}_U V$ [6].

If a vector field X on M is related to a vector field X' on N, we say X is a projectable vector field. If X is both a horizontal and a projectable vector field, we say X is a basic vector field on M. From now on, when we mention a horizontal vector field, we always consider a basic vector field [5].

On the other hand, let $F : (M^m, g_{_M}) \longrightarrow (N^n, g_{_N})$ be a conformal Riemannian map between Riemannian manifolds. Then, we have

$$(\nabla F_*)(X, Y)\,|_{rangeF_*} = X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) - g_M(X, Y)F_*(grad(\ln\lambda)), \tag{12}$$

where $X, Y \in \Gamma((kerF_*)^\perp)$. Hence from (12), we obtain $\overset{N}{\nabla}{}^F_X F_*(Y)$ as

$$\overset{N}{\nabla}{}^F_X F_*(Y) = F_*(h\overset{M}{\nabla}_X Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) - g_M(X, Y)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(X, Y) \tag{13}$$

where $(\nabla F_*)^\perp(X, Y)$ is the component of $(\nabla F_*)(X, Y)$ on $(rangeF_*)^\perp$ for $X, Y \in \Gamma((kerF_*)^\perp)$ [21, 22]. Here, F is said to be horizontally homothetic map if $h(grad(\ln\lambda)) = 0$ [5].

Now, a map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) = 0 \tag{14}$$

for $X, Y \in \Gamma(TM)$ [10].

Lastly, we remark some relations on semi-slant Riemannian maps which will be same for conformal semi-slant Riemannian maps. Let $F : (M, g_{_M}, J) \longrightarrow (N, g_{_N})$ be a semi-slant Riemannian map from a Kähler manifold to a Riemannian manifold with the semi-slant angle θ. Then we obtain

$$\phi^2 X = -\cos^2\theta.X \tag{15}$$

for $X \in \mathcal{D}_2$. If the tensor ω is parallel, then we get

$$\mathcal{T}_{\phi X}\phi X = -\cos^2\theta.\mathcal{T}_X X \tag{16}$$

for $X \in \mathcal{D}_2$ [12].

3. Conformal Semi-slant Riemannian Maps

In this section, we will define the notion of conformal semi-slant Riemannian maps and give examples. Then, some useful results will be given used in forward calculations.

Definition 3.1. *Let (M, g_M, J) is an almost Hermitian manifold and (N, g_N) is a Riemannian manifold. A conformal Riemannian map $F : (M, g_M, J) \longrightarrow (N, g_N)$ is called a conformal semi-slant Riemannian map if there is a distribution $\mathcal{D}_1 \subset kerF_*$ such that*

$$kerF_* = \mathcal{D}_1 \oplus \mathcal{D}_2, J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $kerF_*$. We call the angle θ a semi-slant angle.

Then for $U \in \Gamma(kerF_*)$, we get

$$U = \tilde{P}U + \tilde{Q}U, \tag{17}$$

where $\tilde{P}$ and $\tilde{Q}$ are projections from $kerF_*$ onto $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. For $U \in \Gamma(kerF_*)$, we get

$$JU = \Phi U + \psi U, \tag{18}$$

where $\Phi U \in \Gamma(kerF_*)$ and $\psi U \in \Gamma((kerF_*)^\perp)$. For $X \in \Gamma((kerF_*)^\perp)$, we have

$$JX = BX + CX, \tag{19}$$

where $BX \in \Gamma(kerF_*)$ and $CX \in \Gamma((kerF_*)^\perp)$. Lastly, we have

$$(kerF_*)^\perp = \psi\mathcal{D}_2 \oplus \mu \tag{20}$$

where μ is the orthogonal complement of $\psi\mathcal{D}_2$ in $(kerF_*)^\perp$. μ is an invariant distribution under J. From equations (17) - (20), we get followings:

$$\Phi\mathcal{D}_1 = \mathcal{D}_1, \quad \psi\mathcal{D}_1 = 0, \quad \Phi\mathcal{D}_2 \subset \mathcal{D}_2, \quad B((kerF_*)^\perp) = \mathcal{D}_2. \tag{21}$$

From now on, we will call this type maps *CSSRM* for convenience. Now, we will give examples for *CSSRM*.

Example 3.2. *Define a map* $F : (\mathbb{R}^8, g_8, J) \longrightarrow (\mathbb{R}^5, g_5)$ *by*

$$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = e(x_5, \gamma, \frac{x_7 - x_8}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, x_6)$$

where γ is a constant. We have the horizontal and the vertical distributions, respectively, as:

$$(kerF_*)^\perp = span\{X_1 = e\frac{\partial}{\partial x_5}, X_2 = \frac{e}{\sqrt{2}}(\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8}), X_3 = \frac{e}{\sqrt{2}}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}), X_4 = e\frac{\partial}{\partial x_6}\}$$

and

$$kerF_* = span\{V_1 = \frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_4 = \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}\}.$$

Hence, F is a conformal Riemannian map with $\lambda = e$ and $0 < rankF_* = 4 \le min\{dim(\mathbb{R}^8), dim(\mathbb{R}^5)\}$. The complex structure J on $\mathbb{R}^8$ as follows $(-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7)$ where $a_i \in \mathbb{R}, i = 1, 2, ..., 8$. Now, we get

$$J(V_1) = V_2, \quad J(V_3) = -\frac{\sqrt{2}}{e}X_3, \quad J(V_4) = -\frac{\sqrt{2}}{e}X_2, \quad J(X_1) = X_4, \quad J(X_2) = \frac{e}{\sqrt{2}}V_4, \quad J(X_3) = \frac{e}{\sqrt{2}}V_3. \tag{22}$$

We obtain from (22) that $\mathcal{D}_1 = span\{V_1, V_2\}$, $\mathcal{D}_2 = span\{V_3, V_4\}$, $\psi\mathcal{D}_2 = span\{X_2, X_3\}$ and $\mu = span\{X_1, X_4\}$. For $V_3, V_4 \in \mathcal{D}_2$, by using

$$V_i.J(V_i) = \cos\theta\|V_i\|\|J(V_i)\|, \quad i = 3, 4$$

we obtain semi-slant angle $\theta = \frac{\pi}{2}$. Therefore, F is a CSSRM with $\lambda = e$, $rankF_* = 4$ and semi-slant angle $\theta = \frac{\pi}{2}$.

In a similar way, we have another example.

Example 3.3. *Define a map* $F : (\mathbb{R}^{10}, g_{10}, J) \longrightarrow (\mathbb{R}^5, g_5)$ *by*

$$F(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = \pi(x_5, \gamma, x_7\cos\alpha - x_8\sin\alpha, x_1, -x_2)$$

where γ is a constant. The map F is a CSSRM such that

$$\mathcal{D}_1 = span\{V_1 = \frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_4}, V_4 = \frac{\partial}{\partial x_9}, V_5 = \frac{\partial}{\partial x_{10}}\}, \quad \mathcal{D}_2 = span\{V_3 = \frac{\partial}{\partial x_6}, V_6 = \sin\alpha\frac{\partial}{\partial x_7} + \cos\alpha\frac{\partial}{\partial x_8}\},$$

$$\psi\mathcal{D}_2 = span\{X_1 = \pi\frac{\partial}{\partial x_5}, X_2 = \pi(\cos\alpha\frac{\partial}{\partial x_7} - \sin\alpha\frac{\partial}{\partial x_8})\}, \quad \mu = span\{X_3 = \pi\frac{\partial}{\partial x_1}, X_4 = -\pi\frac{\partial}{\partial x_2}\}$$

with $\lambda = \pi$, $rankF_* = 4$ and semi-slant angle $\theta = \frac{\pi}{2}$.

Proposition 3.4. *Let F be a CSSRM from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the slant distribution $\mathcal{D}_2$ is integrable if and only if*

$$\Phi^2\{\hat{\nabla}_{U_2}U_1 - \hat{\nabla}_{U_1}U_2\} \in \Gamma(\mathcal{D}_2)$$

for $U_1, U_2 \in \Gamma(\mathcal{D}_2)$ and $V \in \Gamma(\mathcal{D}_1)$.

Proof. Since vertical distribution is always integrable, we have $g_M([U_1, U_2], X) = 0$ for $X \in \Gamma((kerF_*)^\perp)$ and $U_1, U_2 \in \Gamma(\mathcal{D}_2)$. By using equations (3), (8), (18) and (19) we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2, V) &= g_M(J\overset{M}{\nabla}_{U_1}U_2, JV) \\ &= -g_M(\Phi B\mathcal{T}_{U_1}U_2 + \Phi^2\hat{\nabla}_{U_2}U_1, V) \end{aligned}$$

for $U_1, U_2 \in \Gamma(\mathcal{D}_2)$ and $V \in \Gamma(\mathcal{D}_1)$. Changing the roles of U_1 and U_2, we obtain

$$g_M([U_1, U_2], V) = g_M(\Phi B\{\mathcal{T}_{U_2}U_1 - \mathcal{T}_{U_1}U_2\} + \Phi^2\{\hat{\nabla}_{U_2}U_1 - \hat{\nabla}_{U_1}U_2\}, V). \tag{23}$$

Since $\mathcal{T}$ is symmetric we have $\Phi B\{\mathcal{T}_{U_2}U_1 - \mathcal{T}_{U_1}U_2\} = 0$. From (23), the proof is clear. □

Similarly, we get following proposition.

Proposition 3.5. *Let F be a CSSRM from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the complex distribution $\mathcal{D}_1$ is integrable if and only if*

$$\Phi\{\hat{\nabla}_{V_1}V_2 - \hat{\nabla}_{V_2}V_1\} \in \Gamma(\mathcal{D}_1), \quad \psi\{\hat{\nabla}_{V_1}V_2 - \hat{\nabla}_{V_2}V_1\} \in \Gamma(\mu)$$

for $V_1, V_2 \in \Gamma(\mathcal{D}_1)$ and $U \in \Gamma(\mathcal{D}_2)$.

If we take M as a Kähler manifold instead of an almost Hermitian manifold in Proposition 3.1. and Proposition 3.2., we get next propositions.

Proposition 3.6. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the slant distribution $\mathcal{D}_2$ is integrable if and only if*

$$\lambda^2\{g_M(\hat{\nabla}_{U_1}JV, \Phi U_2) - g_M(\hat{\nabla}_{U_2}JV, \Phi U_1)\} = g_N((\nabla F_*)(U_1, JV), F_*(\psi U_2)) - g_N((\nabla F_*)(U_2, JV), F_*(\psi U_1))$$

for $U_1, U_2 \in \Gamma(\mathcal{D}_2)$ and $V \in \Gamma(\mathcal{D}_1)$.

Proof. Since M is a Kähler manifold, from equations (2), (4), (5), (8), we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1}U_2, V) &= -g_M(\overset{M}{\nabla}_{U_1}V, U_2) \\ &= g_M(\overset{M}{\nabla}_{U_1}JV, JU_2) \\ &= -g_M(\mathcal{T}_{U_1}JV, \psi U_2) - g_M(\hat{\nabla}_{U_1}JV, \Phi U_2) \end{aligned}$$

for $U_1, U_2 \in \Gamma(\mathcal{D}_2)$ and $V \in \Gamma(\mathcal{D}_1)$. Changing the roles of U_1 and U_2, we obtain

$$\begin{aligned} g_M([U_1, U_2], V) &= \frac{1}{\lambda^2}g_N((\nabla F_*)(U_1, JV), F_*(\psi U_2)) - \frac{1}{\lambda^2}g_N((\nabla F_*)(U_2, JV), F_*(\psi U_1)) \\ &+ g_M(\hat{\nabla}_{U_2}JV, \Phi U_1) - g_M(\hat{\nabla}_{U_1}JV, \Phi U_2). \end{aligned} \tag{24}$$

From equation (24), the proof is clear. □

Similarly, we get following.

Proposition 3.7. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the complex distribution $\mathcal{D}_1$ is integrable if and only if*

$$\lambda^2 g_M(\hat{\nabla}_{V_2} JV_1 - \hat{\nabla}_{V_1} JV_2, \Phi U) = g_N((\nabla F_*)(V_2, JV_1) - (\nabla F_*)(V_1, JV_2), F_*(\psi U))$$

for $V_1, V_2 \in \Gamma(\mathcal{D}_1)$ and $U \in \Gamma(\mathcal{D}_2)$.

Proposition 3.8. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the horizontal distribution $(kerF_*)^\perp$ is integrable if and only if*

i- $\mathcal{A}_{X_1} CX_2 - \mathcal{A}_{X_2} CX_1 + v\overset{M}{\nabla}_{X_1} BX_2 - v\overset{M}{\nabla}_{X_2} BX_1 \in \Gamma(\mathcal{D}_2)$,

ii- $g_N((\nabla F_*)(X_1, BX_2) - (\nabla F_*)(X_2, BX_1), F_*(\psi U))$
$= \lambda^2\{g_M(h\overset{M}{\nabla}_{X_1} CX_2 - h\overset{M}{\nabla}_{X_2} CX_1, \psi U) + g_M(\mathcal{A}_{X_1} CX_2 - \mathcal{A}_{X_2} CX_1 + v\overset{M}{\nabla}_{X_1} BX_2 - v\overset{M}{\nabla}_{X_2} BX_1, \Phi U)\}$

for $X_1, X_2 \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(\mathcal{D}_1)$ and $U \in \Gamma(\mathcal{D}_2)$.

Proof. First, we will examine $0 = g_M([X_1, X_2], V)$ for $X_1, X_2 \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(\mathcal{D}_1)$. By using equations (4), (10) and (11), we have

$$g_M(\overset{M}{\nabla}_{X_1} X_2, V) = g_M(\mathcal{A}_{X_1} CX_2 + v\overset{M}{\nabla}_{X_1} BX_2, JV)$$

for $X_1, X_2 \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(\mathcal{D}_1)$. Changing the roles of X_1 and X_2, we obtain

$$g_M([X_1, X_2], V) = g_M(\mathcal{A}_{X_1} CX_2 - \mathcal{A}_{X_2} CX_1 + v\overset{M}{\nabla}_{X_1} BX_2 - v\overset{M}{\nabla}_{X_2} BX_1, JV). \tag{25}$$

One can see (i) from (25). In a similar way, we obtain

$$\begin{aligned} g_M([X_1, X_2], U) &= g_M(\mathcal{A}_{X_1} BX_2 - \mathcal{A}_{X_2} BX_1 + h\overset{M}{\nabla}_{X_1} CX_2 - h\overset{M}{\nabla}_{X_2} CX_1, \psi U) \\ &+ g_M(v\overset{M}{\nabla}_{X_1} BX_2 - v\overset{M}{\nabla}_{X_2} BX_1 + \mathcal{A}_{X_1} CX_2 - \mathcal{A}_{X_2} CX_1, \Phi U) \end{aligned} \tag{26}$$

for $X_1, X_2 \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(\mathcal{D}_2)$. From equations (5) and (26), we get

$$\begin{aligned} g_M([X_1, X_2], U) &= -\frac{1}{\lambda^2}\{g_N((\nabla F_*)(X_1, BX_2) - (\nabla F_*)(X_2, BX_1), F_*(\psi U))\} \\ &+ g_M(h\overset{M}{\nabla}_{X_1} CX_2 - h\overset{M}{\nabla}_{X_2} CX_1, \psi U) \\ &+ g_M(\mathcal{A}_{X_1} CX_2 - \mathcal{A}_{X_2} CX_1 + v\overset{M}{\nabla}_{X_1} BX_2 - v\overset{M}{\nabla}_{X_2} BX_1, \Phi U). \end{aligned} \tag{27}$$

One can see (ii) from (27). The proof is complete. □

We already have the notion of pluriharmonic map [10] and its other cases such that if we take components from $\mathcal{D}_1$ ($\mathcal{D}_2$, μ, $(kerF_*)^\perp - kerF_*$, respectively) in (14), we say F is a $\mathcal{D}_1$-pluriharmonic map ($\mathcal{D}_2$, μ, $(kerF_*)^\perp - kerF_*$, respectively) [21, 22]. Now, we use pluriharmonicity to introduce some geometric properties.

Theorem 3.9. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any one condition below implies the second condition;*

i- *F is a $\mathcal{D}_1$-pluriharmonic map,*

ii- $C\{\mathcal{T}_{V_1} JV_2 - \mathcal{T}_{JV_1} V_2\} = \psi\{\hat{\nabla}_{JV_1} V_2 - \hat{\nabla}_{V_1} JV_2\}$

for any $V_1, V_2 \in \Gamma(\mathcal{D}_1)$.

Proof. By using notion of $\mathcal{D}_1$-pluriharmonic map and equations (5), (8) and (21), we write

$$\begin{aligned} 0 &= (\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2) \\ &= F_*(J\overset{M}{\nabla}_{V_1} JV_2) - F_*(J\overset{M}{\nabla}_{JV_1} V_2) \\ &= F_*(C\mathcal{T}_{V_1} JV_2 + \psi\hat{\nabla}_{V_1} JV_2) - F_*(C\mathcal{T}_{JV_1} V_2 + \psi\hat{\nabla}_{JV_1} V_2) \end{aligned} \tag{28}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D}_1)$. If F is a $\mathcal{D}_1$-pluriharmonic map, then we have

$$0 = F_*(C\mathcal{T}_{V_1} JV_2 + \psi\hat{\nabla}_{V_1} JV_2) - F_*(C\mathcal{T}_{JV_1} V_2 + \psi\hat{\nabla}_{JV_1} V_2).$$

Hence, one can see $C\{\mathcal{T}_{V_1} JV_2 - \mathcal{T}_{JV_1} V_2\} = \psi\{\hat{\nabla}_{JV_1} V_2 - \hat{\nabla}_{V_1} JV_2\}$. If (ii) is provided, we obtain from (28)

$$0 = (\nabla F_*)(V_1, V_2) + (\nabla F_*)(JV_1, JV_2).$$

It means F is a $\mathcal{D}_1$-pluriharmonic map for any $V_1, V_2 \in \Gamma(\mathcal{D}_1)$. The proof is complete. □

Recall that F is said to be horizontally homothetic map if $h(grad(\ln\lambda)) = 0$ [5].

Theorem 3.10. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition;*

i- F is a $\mathcal{D}_2$-pluriharmonic map,

ii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(\psi U_1, \psi U_2) = 0$,

iii- $\sin^2\theta\mathcal{T}_{U_1} U_2 + \mathcal{A}_{\psi U_2}\Phi U_1 + \mathcal{A}_{\psi U_1}\Phi U_2 = 0$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_2)$.

Proof. Since second fundamental form of a map (∇F_*) is a symmetric, from equations (5), (13) and (14), we have

$$\begin{aligned} 0 &= (\nabla F_*)(U_1, U_2) + (\nabla F_*)(JU_1, JU_2) \\ &= (\nabla F_*)(U_1, U_2) + (\nabla F_*)(\Phi U_1, \Phi U_2) + (\nabla F_*)(\psi U_2, \Phi U_1) \\ &+ (\nabla F_*)(\psi U_1, \Phi U_2) + (\nabla F_*)(\psi U_1, \psi U_2) \\ &= -F_*(\overset{M}{\nabla}_{U_1} U_2) - F_*(\overset{M}{\nabla}_{\Phi U_1}\Phi U_2) - F_*(\overset{M}{\nabla}_{\psi U_2}\Phi U_1) - F_*(\overset{M}{\nabla}_{\psi U_1}\Phi U_2) \\ &+ \psi U_1(\ln\lambda)F_*(\psi U_2) + \psi U_2(\ln\lambda)F_*(\psi U_1) - g_M(\psi U_1, \psi U_2)F_*(grad(\ln\lambda)) \\ &+ (\nabla F_*)^\perp(\psi U_1, \psi U_2) \end{aligned} \tag{29}$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_2)$. By using equations (8), (10) and (16) in (29), we have

$$\begin{aligned} 0 &= -F_*(\mathcal{T}_{U_1} U_2) + \cos^2\theta F_*(\mathcal{T}_{U_1} U_2) - F_*(\mathcal{A}_{\psi U_2}\Phi U_1 + \mathcal{A}_{\psi U_1}\Phi U_2) \\ &+ \psi U_1(\ln\lambda)F_*(\psi U_2) + \psi U_2(\ln\lambda)F_*(\psi U_1) - g_M(\psi U_1, \psi U_2)F_*(grad(\ln\lambda)) \\ &+ (\nabla F_*)^\perp(\psi U_1, \psi U_2) \\ &= -\sin^2\theta F_*(\mathcal{T}_{U_1} U_2) - F_*(\mathcal{A}_{\psi U_2}\Phi U_1 + \mathcal{A}_{\psi U_1}\Phi U_2) \\ &+ \psi U_1(\ln\lambda)F_*(\psi U_2) + \psi U_2(\ln\lambda)F_*(\psi U_1) - g_M(\psi U_1, \psi U_2)F_*(grad(\ln\lambda)) \\ &+ (\nabla F_*)^\perp(\psi U_1, \psi U_2). \end{aligned} \tag{30}$$

Now, we suppose that (i) and (iii) are provided in (30). We get

$$\begin{aligned} 0 &= \psi U_1(\ln\lambda)F_*(\psi U_2) + \psi U_2(\ln\lambda)F_*(\psi U_1) - g_M(\psi U_1, \psi U_2)F_*(grad(\ln\lambda)) \\ &+ (\nabla F_*)^\perp(\psi U_1, \psi U_2). \end{aligned} \tag{31}$$

Clearly, one can see $(\nabla F_*)^{\perp}(\psi U_1, \psi U_2) = 0$. For $\psi U_1 \in \Gamma(\psi\mathcal{D}_2)$ by using (2) in (31), we obtain

$$\begin{aligned} 0 &= \lambda^2 \psi U_1(\ln\lambda) g_M(\psi U_2, \psi U_1) + \lambda^2 \psi U_2(\ln\lambda) g_M(\psi U_1, \psi U_1) \\ &- g_M(\psi U_1, \psi U_2)\lambda^2 \psi U_1(\ln\lambda) \\ &= \lambda^2 \psi U_2(\ln\lambda) g_M(\psi U_1, \psi U_1). \end{aligned} \tag{32}$$

In (32), we have $\psi U_2(\ln\lambda) = 0$. It means λ is a constant on $\psi\mathcal{D}_2$. For $Y \in \Gamma(\mu)$ by using equations (2), (20) in (31), we obtain

$$\begin{aligned} 0 &= \lambda^2 \psi U_1(\ln\lambda) g_M(\psi U_2, Y) + \lambda^2 \psi U_2(\ln\lambda) g_M(\psi U_1, Y) \\ &- g_M(\psi U_1, \psi U_2)\lambda^2 Y(\ln\lambda) \\ &= -\lambda^2 Y(\ln\lambda) g_M(\psi U_1, \psi U_2). \end{aligned} \tag{33}$$

In (33), we have $Y(\ln\lambda) = 0$ with $\psi U_1 = \psi U_2$. It means λ is a constant on μ. So, we say λ is a constant on $(kerF_*)^{\perp}$. Therefore, F is a horizontally homothetic map and $(\nabla F_*)^{\perp}(\psi U_1, \psi U_2) = 0$. Suppose that (i) and (ii) are provided in (30). So, from (30), we obtain

$$0 = -\sin^2\theta F_*(\mathcal{T}_{U_1}U_2) - F_*(\mathcal{A}_{\psi U_2}\Phi U_1 + \mathcal{A}_{\psi U_1}\Phi U_2)$$

which gives the proof of (iii). Therefore, if (ii) and (iii) are provided in (30), easily we obtain

$$0 = (\nabla F_*)(U_1, U_2) + (\nabla F_*)(JU_1, JU_2)$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_2)$. So, F is a $\mathcal{D}_2$-pluriharmonic map. The proof is complete. □

Theorem 3.11. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any one condition below implies the second condition;*

i- F is a μ-pluriharmonic map,

ii- F is a horizontally homothetic map and $(\nabla F_*)^{\perp}(Y_1, Y_2) = 0$

for any $Y_1, Y_2 \in \Gamma(\mu)$.

Proof. Firstly, suppose that (i) is provided. By using equations (12) and (14), we get

$$\begin{aligned} 0 &= (\nabla F_*)(Y_1, Y_2) + (\nabla F_*)(JY_1, JY_2) \\ &= Y_1(\ln\lambda)F_*(Y_2) + Y_2(\ln\lambda)F_*(Y_1) - g_M(Y_1, Y_2)F_*(grad(\ln\lambda)) \\ &+ JY_1(\ln\lambda)F_*(JY_2) + JY_2(\ln\lambda)F_*(JY_1) - g_M(JY_1, JY_2)F_*(grad(\ln\lambda)) \\ &+ (\nabla F_*)^{\perp}(Y_1, Y_2) + (\nabla F_*)^{\perp}(JY_1, JY_2) \end{aligned} \tag{34}$$

for any $Y_1, Y_2 \in \Gamma(\mu)$. Since μ is invariant under J we can take $Y_1 = JY_2$ and $Y_2 = JY_1$ in (34). We obtain

$$0 = 2\{Y_1(\ln\lambda)F_*(Y_2) + Y_2(\ln\lambda)F_*(Y_1) - g_M(Y_1, Y_2)F_*(grad(\ln\lambda)) + (\nabla F_*)^{\perp}(Y_1, Y_2)\}. \tag{35}$$

One can see $(\nabla F_*)^{\perp}(Y_1, Y_2) = 0$ in (35). Lastly, we have

$$0 = 2\{Y_1(\ln\lambda)F_*(Y_2) + Y_2(\ln\lambda)F_*(Y_1) - g_M(Y_1, Y_2)F_*(grad(\ln\lambda))\}. \tag{36}$$

Now, for any $Y_1 \in \Gamma(\mu)$ in (36), we obtain

$$\begin{aligned} 0 &= 2\{Y_1(\ln\lambda)\lambda^2 g_M(Y_2, Y_1) + Y_2(\ln\lambda)\lambda^2 g_M(Y_1, Y_1) \\ &- g_M(Y_1, Y_2)\lambda^2 Y_1(\ln\lambda)\} \\ &= 2\lambda^2 Y_2(\ln\lambda) g_M(Y_1, Y_1). \end{aligned} \tag{37}$$

In (37), we have $Y_2(\ln\lambda) = 0$. It means λ is a constant on μ. For $\psi U \in \Gamma(\psi\mathcal{D}_2)$ from equations (2) and (20) in (36), we obtain

$$\begin{aligned} 0 &= 2\{Y_1(\ln\lambda)\lambda^2 g_M(Y_2,\psi U) + Y_2(\ln\lambda)\lambda^2 g_M(Y_1,\psi U) \\ &- g_M(Y_1,Y_2)\lambda^2\psi U(\ln\lambda)\} \\ &= -2\lambda^2\psi U(\ln\lambda) g_M(Y_1,Y_2). \end{aligned} \tag{38}$$

In (38), we have $\psi U(\ln\lambda) = 0$ with $Y_1 = Y_2$. It means λ is a constant on $\psi\mathcal{D}_2$. So, we say λ is a constant on $(kerF_*)^\perp$. Therefore, F is a horizontally homothetic map and $(\nabla F_*)^\perp(Y_1,Y_2) = 0$. Clearly, if F is a horizontally homothetic map and $(\nabla F_*)^\perp(Y_1,Y_2) = 0$ we obtain (i) from (34). □

Theorem 3.12. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then any two conditions below imply the third condition;*

i- F is a $\{(kerF_*)^\perp - kerF_*\}$-pluriharmonic map,

ii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(CX,\psi U) = 0$,

iii- $\mathcal{A}_X U + \mathcal{A}_{\psi U} BX + \mathcal{T}_{BX}\Phi U + \mathcal{A}_{CX}\Phi U = 0$

for any $U \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$.

Proof. Now, by using symmetry property of second fundamental form of a map (∇F_*) and from equations (14), (18) and (19), we get

$$\begin{aligned} 0 &= (\nabla F_*)(X,U) + (\nabla F_*)(JX,JU) \\ &= -F_*(\overset{M}{\nabla}_X U) + (\nabla F_*)(\psi U, BX) + (\nabla F_*)(BX,\Phi U) + (\nabla F_*)(CX,\psi U) + (\nabla F_*)(CX,\Phi U) \end{aligned} \tag{39}$$

for any $U \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$. By using (8), (9) and (12) in (39), we obtain

$$\begin{aligned} 0 &= -F_*(\mathcal{A}_X U + \mathcal{A}_{\psi U} BX + \mathcal{T}_{BX}\Phi U + \mathcal{A}_{CX}\Phi U) \\ &+ CX(\ln\lambda)F_*(\psi U) + \psi U(\ln\lambda)F_*(CX) \\ &- g_M(CX,\psi U)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(CX,\psi U). \end{aligned} \tag{40}$$

Suppose that (i) and (iii) are provided in (40). So, we have

$$\begin{aligned} 0 &= CX(\ln\lambda)F_*(\psi U) + \psi U(\ln\lambda)F_*(CX) - g_M(CX,\psi U)F_*(grad(\ln\lambda)) \\ &+ (\nabla F_*)^\perp(CX,\psi U). \end{aligned} \tag{41}$$

Clearly, we see $(\nabla F_*)^\perp(CX,\psi U) = 0$. Lastly, we have

$$0 = CX(\ln\lambda)F_*(\psi U) + \psi U(\ln\lambda)F_*(CX) - g_M(CX,\psi U)F_*(grad(\ln\lambda)). \tag{42}$$

By using (2) in (42) and for $CX \in \Gamma((kerF_*)^\perp)$, we obtain

$$\begin{aligned} 0 &= CX(\ln\lambda)\lambda^2 g_M(\psi U, CX) + \psi U(\ln\lambda)\lambda^2 g_M(CX,CX) \\ &- g_M(CX,\psi U)\lambda^2 CX(\ln\lambda) \\ &= \lambda^2\psi U(\ln\lambda) g_M(CX,CX). \end{aligned} \tag{43}$$

In (43), we have $\psi U(\ln\lambda) = 0$. For $\psi U \in \Gamma((kerF_*)^\perp)$ in (42) from (2), we obtain

$$\begin{aligned} 0 &= CX(\ln\lambda)\lambda^2 g_M(\psi U,\psi U) + \psi U(\ln\lambda)\lambda^2 g_M(CX,\psi U) \\ &- g_M(CX,\psi U)\lambda^2\psi U(\ln\lambda) \\ &= \lambda^2 CX(\ln\lambda) g_M(\psi U,\psi U). \end{aligned} \tag{44}$$

In (44), we have $CX(\ln\lambda) = 0$. Because of $\psi U(\ln\lambda) = 0$ and $CX(\ln\lambda) = 0$, λ is a constant on $(kerF_*)^\perp$. Therefore, F is a horizontally homothetic map and $(\nabla F_*)^\perp(CX, \psi U) = 0$. If (i) and (ii) are provided in (40), we obtain

$$0 = -F_*(\mathcal{A}_X U + \mathcal{A}_{\psi U} BX + \mathcal{T}_{BX}\Phi U + \mathcal{A}_{CX}\Phi U).$$

So, we get the proof of (iii). If (ii) and (iii) are provided in (40), we easily see

$$0 = (\nabla F_*)(X, U) + (\nabla F_*)(JX, JU)$$

for any $U \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$. Hence, F is a $\{(kerF_*)^\perp - kerF_*\}$-pluriharmonic map. □

4. Totally Geodesic Distributions

In this section, we give some conditions for distributions to be define totally geodesic foliation in M.

Theorem 4.1. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the slant distribution $\mathcal{D}_2$ defines a totally geodesic foliation in M if and only if*

i- $-\lambda^2 g_M(\hat{\nabla}_{U_1}\Phi U_2, \Phi V) = g_N((\nabla F_*)(U_1, \Phi V), F_*(\psi U_2))$

ii- $\lambda^2\{g_M(\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi U_2, BX) + g_M(h\overset{M}{\nabla}_{U_1}\psi U_2, CX)\} = g_N((\nabla F_*)(U_1, \Phi U_2), F_*(CX))$

are provided for any $U_1, U_2 \in \Gamma(\mathcal{D}_2)$, $V \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma((kerF_*)^\perp)$.

Proof. If the slant distribution $\mathcal{D}_2$ defines a totally geodesic foliation in M, the equations $g_M(\overset{M}{\nabla}_{U_1} U_2, V)$ and $g_M(\overset{M}{\nabla}_{U_1} U_2, X)$ must be vanished for any $U_1, U_2 \in \Gamma(\mathcal{D}_2)$, $V \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma((kerF_*)^\perp)$. Firstly, by using equations (4), (8), (9), (18) and (21), we have

$$g_M(\overset{M}{\nabla}_{U_1} U_2, V) = g_M(\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi U_2, \Phi V)$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_2)$ and $V \in \Gamma(\mathcal{D}_1)$. Since $\mathcal{T}$ is an anti-symmetric tensor field, we have $g_M(\mathcal{T}_{U_1}\psi U_2, \Phi V) = -g_M(\mathcal{T}_{U_1}\Phi V, \psi U_2)$. In addition, we know $(\nabla F_*)(U_1, \Phi V) = -F_*(\mathcal{T}_{U_1}\Phi V)$. Using equation (2), we get

$$g_M(\overset{M}{\nabla}_{U_1} U_2, V) = g_M(\hat{\nabla}_{U_1}\Phi U_2, \Phi V) + \frac{1}{\lambda^2} g_N((\nabla F_*)(U_1, \Phi V), F_*(\psi U_2)). \tag{45}$$

We obtain (i) from equation (45). Now, in a similar way, we have

$$g_M(\overset{M}{\nabla}_{U_1} U_2, X) = g_M(\mathcal{T}_{U_1}\psi U_2 + h\overset{M}{\nabla}_{U_1}\psi U_2, CX) + g_M(\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi U_2, BX)$$

for any $U_1, U_2 \in \Gamma(\mathcal{D}_2)$ and $X \in \Gamma((kerF_*)^\perp)$. We know $(\nabla F_*)(U_1, \Phi U_2) = -F_*(\mathcal{T}_{U_1}\Phi U_2)$. Using equation (2), we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1} U_2, X) &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(U_1, \Phi U_2), F_*(CX)) + g_M(h\overset{M}{\nabla}_{U_1}\psi U_2, CX) \\ &+ g_M(\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi U_2, BX). \end{aligned} \tag{46}$$

We obtain (ii) from (46). □

Theorem 4.2. *Let F be a CSSRM from a Kähler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the complex distribution $\mathcal{D}_1$ defines a totally geodesic foliation in M if and only if*

i- $\tilde{P}\{\Phi\{\hat{\nabla}_{V_1} BX + \mathcal{T}_{V_1} CX\}\} = 0$

ii- $-\lambda^2 g_M(\hat{\nabla}_{V_1}\Phi U, \Phi V_2) = g_N((\nabla F_*)(V_1, \Phi V_2), F_*(\psi U))$

are provided for any $V_1, V_2 \in \Gamma(\mathcal{D}_1)$, $U \in \Gamma(\mathcal{D}_2)$ and $X \in \Gamma((kerF_*)^\perp)$.

Proof. If the complex distribution $\mathcal{D}_1$ defines a totally geodesic foliation in M, the equations $g_M(\overset{M}{\nabla}_{V_1}V_2, X)$ and $g_M(\overset{M}{\nabla}_{V_1}V_2, U)$ must be vanished for any $V_1, V_2 \in \Gamma(\mathcal{D}_1)$, $X \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(\mathcal{D}_2)$. By using equations (4), (8), (9), (18) and (19), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{V_1}V_2, X) &= -g_M(\overset{M}{\nabla}_{V_1}JX, JV_2) \\ &= -g_M(\hat{\nabla}_{V_1}BX, JV_2) - g_M(\mathcal{T}_{V_1}CX, JV_2) \\ &= g_M(\Phi\hat{\nabla}_{V_1}BX + \Phi\mathcal{T}_{V_1}CX, V_2) \end{aligned} \tag{47}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D}_1)$ and $X \in \Gamma((kerF_*)^\perp)$. We obtain (i) from (47). In a similar way, we have from equation (21)

$$\begin{aligned} g_M(\overset{M}{\nabla}_{V_1}V_2, U) &= -g_M(\overset{M}{\nabla}_{V_1}JU, JV_2) \\ &= -g_M(\hat{\nabla}_{V_1}\Phi U, \Phi V_2) - g_M(\mathcal{T}_{V_1}\psi U, \Phi V_2) \end{aligned} \tag{48}$$

for any $V_1, V_2 \in \Gamma(\mathcal{D}_1)$ and $U \in \Gamma(\mathcal{D}_2)$. Since $\mathcal{T}$ is an anti-symmetric tensor field and by using equation (5), we have $-g_M(\mathcal{T}_{V_1}\psi U, \Phi V_2) = g_M(\mathcal{T}_{V_1}\Phi V_2, \psi U)$ and $(\nabla F_*)(V_1, \Phi V_2) = -F_*(\mathcal{T}_{V_1}\Phi V_2)$. Hence, we obtain from (2)

$$g_M(\overset{M}{\nabla}_{V_1}V_2, U) = -g_M(\hat{\nabla}_{V_1}\Phi U, \Phi V_2) - \frac{1}{\lambda^2}g_N((\nabla F_*)(V_1, \Phi V_2), F_*(\psi U)). \tag{49}$$

We obtain (ii) from (49). □

Theorem 4.3. *Let F be a CSSRM from a Kähler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then the horizontal distribution* $(kerF_*)^\perp$ *defines a totally geodesic foliation in M if and only if*

$$\lambda^2\{g_M(h\overset{M}{\nabla}_{X_1}\psi\tilde{Q}W, CX_2) + g_M(v\overset{M}{\nabla}_{X_1}\Phi W + \mathcal{A}_{X_1}\psi\tilde{Q}W, BX_2)\} = g_N((\nabla F_*)(X_1, \Phi W), F_*(CX_2))$$

for any $X_1, X_2 \in \Gamma((kerF_*)^\perp)$ *and* $W \in \Gamma(kerF_*)$*.*

Proof. Here, we examine $0 = g_M(\overset{M}{\nabla}_{X_1}X_2, W)$ for any $X_1, X_2 \in \Gamma((kerF_*)^\perp)$ and $W \in \Gamma(kerF_*)$. By using equations (4) and (17), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{X_1}X_2, W) &= -g_M(\overset{M}{\nabla}_{X_1}\tilde{P}W + \tilde{Q}W, X_2) \\ &= -g_M(\overset{M}{\nabla}_{X_1}J\tilde{P}W, JX_2) - g_M(\overset{M}{\nabla}_{X_1}J\tilde{Q}W, JX_2) \end{aligned}$$

for any $X_1, X_2 \in \Gamma((kerF_*)^\perp)$ and $W \in \Gamma(kerF_*)$. Using equations (18), (19) and (21), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{X_1}X_2, W) &= -g_M(\mathcal{A}_{X_1}\Phi\tilde{P}W, CX_2) - g_M(v\overset{M}{\nabla}_{X_1}\Phi\tilde{P}W, BX_2) \\ &- g_M(\mathcal{A}_{X_1}\Phi\tilde{Q}W + h\overset{M}{\nabla}_{X_1}\psi\tilde{Q}W, CX_2) \\ &- g_M(v\overset{M}{\nabla}_{X_1}\Phi\tilde{Q}W + \mathcal{A}_{X_1}\psi\tilde{Q}W, BX_2). \end{aligned}$$

Since $\Phi\{\tilde{P}W + \tilde{Q}W\} = \Phi W$ and $(\nabla F_*)(X_1, \Phi W) = -F_*(\mathcal{A}_{X_1}\Phi W)$ we obtain,

$$\begin{aligned} g_M(\overset{M}{\nabla}_{X_1}X_2, W) &= -g_M(v\overset{M}{\nabla}_{X_1}\Phi W + \mathcal{A}_{X_1}\psi\tilde{Q}W, BX_2) \\ &- g_M(h\overset{M}{\nabla}_{X_1}\psi\tilde{Q}W, CX_2) - g_M(\mathcal{A}_{X_1}\Phi W, CX_2) \\ &= -g_M(v\overset{M}{\nabla}_{X_1}\Phi W + \mathcal{A}_{X_1}\psi\tilde{Q}W, BX_2) - g_M(h\overset{M}{\nabla}_{X_1}\psi\tilde{Q}W, CX_2) \\ &+ \frac{1}{\lambda^2}g_N((\nabla F_*)(X_1, \Phi W), F_*(CX_2)). \end{aligned} \tag{50}$$

We complete the proof from (50). □

Theorem 4.4. *Let F be a CSSRM from a Kähler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then the vertical distribution* $kerF_*$ *defines a totally geodesic foliation in M if and only if*

i- $-\lambda^2 g_M(h\overset{M}{\nabla}_{U_1} JX, \psi U_2) = g_N((\nabla F_*)(U_1, \Phi U_2), F_*(JX))$

ii- $\lambda^2\{g_M(h\overset{M}{\nabla}_{U_1}\psi U_2, CY) + g_M(\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi U_2, BY)\} = g_N((\nabla F_*)(U_1, \Phi U_2), F_*(CY))$

are provided for any $U_1, U_2 \in \Gamma(kerF_*)$*,* $X \in \Gamma(\mu)$ *and* $Y \in \Gamma(\psi\mathcal{D}_2)$*.*

Proof. Since μ is an invariant distribution and vertical tensor field $\mathcal{T}$ is an anti-symmetric tensor field by using equations (4), (9) and (18), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1} U_2, X) &= -g_M(\mathcal{T}_{U_1} JX, \Phi U_2) - g_M(h\overset{M}{\nabla}_{U_1} JX, \psi U_2) \\ &= g_M(\mathcal{T}_{U_1}\Phi U_2, JX) - g_M(h\overset{M}{\nabla}_{U_1} JX, \psi U_2) \end{aligned}$$

for any $U_1, U_2 \in \Gamma(kerF_*)$ and $X \in \Gamma(\mu)$. We know that $(\nabla F_*)(U_1, \Phi U_2) = -F_*(\mathcal{T}_{U_1}\Phi U_2)$, so we obtain

$$g_M(\overset{M}{\nabla}_{U_1} U_2, X) = -\frac{1}{\lambda^2} g_N((\nabla F_*)(U_1, \Phi U_2), F_*(JX)) - g_M(h\overset{M}{\nabla}_{U_1} JX, \psi U_2). \tag{51}$$

We obtain (i) from (51). In a similar way, from equations (8), (9), (18) and (19), we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{U_1} U_2, Y) &= g_M(\mathcal{T}_{U_1}\Phi U_2 + h\overset{M}{\nabla}_{U_1}\psi U_2, CY) + g_M(\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi U_2, BY) \\ &= -\frac{1}{\lambda^2} g_N((\nabla F_*)(U_1, \Phi U_2), F_*(CY)) + g_M(h\overset{M}{\nabla}_{U_1}\psi U_2, CY) \\ &+ g_M(\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi U_2, BY) \end{aligned} \tag{52}$$

for any $U_1, U_2 \in \Gamma(kerF_*)$ and $Y \in \Gamma(\psi\mathcal{D}_2)$. Therefore, we obtain (ii) from (52). The proof is complete. □

Theorem 4.5. *Let F be a CSSRM from a Kähler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then F defines totally geodesic map if and only if*

i- F is a horizontally homothetic map and $(\nabla F_*)^\perp(X_1, X_2) = 0$

ii- $C\{\mathcal{T}_{U_1}\Phi U_2 + h\overset{M}{\nabla}_{U_1}\psi\tilde{Q}U_2\} + \psi\{\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi\tilde{Q}U_2\} = 0$

iii- $C\{\mathcal{A}_{X_1}\Phi U_1 + h\overset{M}{\nabla}_{X_1}\psi U_1\} + \psi\{v\overset{M}{\nabla}_{X_1}\Phi U_1 + \mathcal{A}_{X_1}\psi U_1\} = 0$

are provided for any $X_1, X_2 \in \Gamma((kerF_*)^\perp)$ *and* $U_1, U_2 \in \Gamma(kerF_*)$*.*

Proof. By using notion of totally geodesic map with respect to second fundamental form of a map for $E, G \in \Gamma(TM)$ we have $(\nabla F_*)(E, G) = 0$. Firstly, we want to show (i). From equations (5), (12) and (13), we have

$$\begin{aligned} (\nabla F_*)(X_1, X_2) &= X_1(\ln\lambda)F_*(X_2) + X_2(\ln\lambda)F_*(X_1) \\ &- g_M(X_1, X_2)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(X_1, X_2) \end{aligned}$$

for any $X_1, X_2 \in \Gamma((kerF_*)^\perp)$. Clearly, we can see $(\nabla F_*)^\perp(X_1, X_2) = 0$ since $(\nabla F_*)^\perp$ is a component of $(rangeF_*)^\perp$. So, we have

$$\begin{aligned} 0 &= X_1(\ln\lambda)F_*(X_2) + X_2(\ln\lambda)F_*(X_1) \\ &- g_M(X_1, X_2)F_*(grad(\ln\lambda)). \end{aligned} \tag{53}$$

For any $X_1 \in \Gamma((kerF_*)^\perp)$ from (2), we obtain in (53)

$$\begin{aligned} 0 &= \lambda^2 X_1(\ln\lambda) g_M(X_2, X_1) + \lambda^2 X_2(\ln\lambda) g_M(X_1, X_1) \\ &- \lambda^2 X_1(\ln\lambda) g_M(X_1, X_2) \\ 0 &= \lambda^2 X_2(\ln\lambda) g_M(X_1, X_1). \end{aligned} \tag{54}$$

We have $X_2(\ln\lambda) = 0$ from (54). It means λ is a constant on horizontal distribution. Therefore, F is a horizontally homothetic map and $(\nabla F_*)^\perp(X_1, X_2) = 0$. Similarly, by using (4), (5), (17), (18) and (21) we have

$$\begin{aligned} (\nabla F_*)(U_1, U_2) &= F_*(J\overset{M}{\nabla}_{U_1} JU_2) \\ &= F_*(J\overset{M}{\nabla}_{U_1} J\tilde{P}U_2 + J\overset{M}{\nabla}_{U_1} J\tilde{Q}U_2) \\ &= F_*(J\mathcal{T}_{U_1} J\tilde{P}U_2 + J\hat{\nabla}_{U_1} J\tilde{P}U_2) \\ &+ F_*(J\mathcal{T}_{U_1}\Phi\tilde{Q}U_2 + J\hat{\nabla}_{U_1}\Phi\tilde{Q}U_2) \\ &+ F_*(J\mathcal{T}_{U_1}\psi\tilde{Q}U_2 + Jh\overset{M}{\nabla}_{U_1}\psi\tilde{Q}U_2) \end{aligned} \tag{55}$$

for any $U_1, U_2 \in \Gamma(kerF_*)$. By using equations (18) and (19) in (55), we obtain

$$\begin{aligned} 0 &= F_*(C\mathcal{T}_{U_1}\Phi\tilde{P}U_2 + \psi\hat{\nabla}_{U_1}\Phi\tilde{P}U_2) \\ &+ F_*(C\mathcal{T}_{U_1}\Phi\tilde{Q}U_2 + \psi\hat{\nabla}_{U_1}\Phi\tilde{Q}U_2) \\ &+ F_*(\psi\mathcal{T}_{U_1}\psi\tilde{Q}U_2 + Ch\overset{M}{\nabla}_{U_1}\psi\tilde{Q}U_2). \end{aligned} \tag{56}$$

Since $\Phi\{\tilde{P}U_2 + \tilde{Q}U_2\} = \Phi U_2$ in (56), we obtain

$$\begin{aligned} 0 &= F_*(C\mathcal{T}_{U_1}\Phi U_2 + \psi\hat{\nabla}_{U_1}\Phi U_2) \\ &+ F_*(\psi\mathcal{T}_{U_1}\psi\tilde{Q}U_2 + Ch\overset{M}{\nabla}_{U_1}\psi\tilde{Q}U_2). \end{aligned} \tag{57}$$

We obtain from equation (57)

$$C\{\mathcal{T}_{U_1}\Phi U_2 + h\overset{M}{\nabla}_{U_1}\psi\tilde{Q}U_2\} + \psi\{\hat{\nabla}_{U_1}\Phi U_2 + \mathcal{T}_{U_1}\psi\tilde{Q}U_2\} = 0.$$

Lastly, by using uations (4), (5), (11), (12), (18) and (19), we have

$$\begin{aligned} (\nabla F_*)(X_1, U_1) &= F_*(J\overset{M}{\nabla}_{X_1} JU_1) \\ &= F_*(J\overset{M}{\nabla}_{X_1}\Phi U_1 + J\overset{M}{\nabla}_{X_1}\psi U_1) \\ &= F_*(J\mathcal{A}_{X_1}\Phi U_1 + Jv\overset{M}{\nabla}_{X_1}\Phi U_1) + F_*(J\mathcal{A}_{X_1}\psi U_1 + Jh\overset{M}{\nabla}_{X_1}\psi U_1) \\ &= F_*(C\mathcal{A}_{X_1}\Phi U_1 + \psi v\overset{M}{\nabla}_{X_1}\Phi U_1) \\ &+ F_*(\psi\mathcal{A}_{X_1}\psi U_1 + Ch\overset{M}{\nabla}_{X_1}\psi U_1) \end{aligned} \tag{58}$$

for any $X_1 \in \Gamma((kerF_*)^\perp)$ and $U_1 \in \Gamma(kerF_*)$. We obtain from equation (58)

$$C\{\mathcal{A}_{X_1}\Phi U_1 + h\overset{M}{\nabla}_{X_1}\psi U_1\} + \psi\{v\overset{M}{\nabla}_{X_1}\Phi U_1 + \mathcal{A}_{X_1}\psi U_1\} = 0.$$

The proof is complete. □

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