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Parametric Generalization of the Modified Bernstein Operators

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Abstract. The current paper deals with the parametric modification of Bernstein operators which preserve constant and Korovkin's other test functions in limit case. The uniform convergence of the newly constructed operators is studied. Also, the rate of convergence is investigated by means of the modulus of continuity, by using functions which belong to Lipschitz class and by the help of Peetre's-$\mathcal{K}$ functionals. Finally, some numerical examples are given to illustrate the effectiveness of the newly defined operators for computing the approximation of function .

1. Introduction

Approximation theory concerns with the approximation to a target function with simpler computable and more useful functions. In 1912, Bernstein [4] defined the Bernstein operators for every bounded function on the interval $[0,1]$. In approximation problem, Bernstein operators and its generalizations are used widely because of its uncomplicated construction and effective features. In 2017, Chen et al. [8] defined a new generalization of Bernstein operators, which is called α-Bernstein operators. Here, α is a non-negative real parameter in $[0,1]$. For $\alpha = 1$, the α-Bernstein operators reduces to the classical Bernstein operators. Recently, a new modification of Bernstein operators is introduced by Usta [16] which unchanging constant test function and preserve Korovkin's other test functions t and t^2 in limit case for $s \in \mathbb{N}$ and $x \in (0,1)$ by

$$B_s^*(v;x) = \frac{1}{s}\sum_{m=0}^{s}\binom{s}{m}(m-sx)^2 x^{m-1}(1-x)^{s-m-1} v\left(\frac{m}{s}\right).$$

He proved the fundamental properties of the $B_s^*(v;x)$ operators such as

$$\begin{aligned}
B_s^*(1;x) &= 1,\\
B_s^*(t;x) &= \frac{s-2}{s}x + \frac{1}{s},\\
B_s^*(t^2;x) &= \frac{(s^2-7s+6)}{s^2}x^2 + \frac{5s-6}{s^2}x + \frac{1}{s^2}.
\end{aligned}$$

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As newsworthy studies in parametric generalizations of the well-known operators, we refer to [1], [3], [5], [7], [10], [12] and [15].
In last years, the Korovkin-type theorems were investigated by several researchers and were generalized in many different spaces such as Banach space, probability space, measurable space, abstract Banach latices, Banach algebras, etc. This theory is valuable in functional analysis, harmonic analysis, real analysis and approximation theory. In the recent year, Srivastava et al. [14] introduced and studied a set of new Korovkin-type approximation theorems for a martingale sequence over a Banach space. Also, Srivastava et al. [13] established statistical Riemann and Lebesgue integrable sequence of functions with the Korovkin-type approximation theorems by using algebraic test functions. Subsequently, Braha et al. [6] proved some properties of statistically convergent sequences and a kind of the Korovkin-type theorem.
Now, motivated by the work of [8], we define the parametric generalization of the modified Bernstein operators for every $v \in C[0,1]$ as

$$B^*_{s,\alpha}(v;x) = \sum_{m=0}^{s} p^{(\alpha)}_{s,m}(x) v\left(\frac{m}{s}\right), \quad s \in \mathbb{N} \tag{1}$$

where $s \geq 1$, $0 \leq \alpha \leq 1$, $x \in (0,1)$ and

$$\begin{aligned} p^{(\alpha)}_{1,0}(x) &= x, \ p^{(\alpha)}_{1,1}(x) = 1-x, \\ p^{(\alpha)}_{s,m}(x) &= \left\{ \frac{1}{s-1}\binom{s-2}{m}(m-(s-1)x)^2(1-\alpha)x + \frac{1}{s-1}\binom{s-2}{m-2}(m-1-(s-1)x)^2(1-\alpha)(1-x) \right. \\ &\quad \left. + \frac{1}{s}\binom{s}{m}(m-sx)^2\alpha x(1-x) \right\} x^{m-2}(1-x)^{s-m-2}, \quad s \geq 2 \end{aligned}$$

with binomial coefficients

$$\binom{s}{m} = \begin{cases} \frac{s!}{(s-m)!m!} & , \quad \text{if } 0 \leq m \leq s \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Especially,

$$\binom{s-2}{-2} = \binom{s-2}{-1} = 0.$$

We can check that for $\alpha = 1$, $B^*_{s,\alpha}(v;x)$ operators reduce to $B^*_s(v;x)$ modified Bernstein operators given by Usta [16].
After some simple calculations, we obtain the results as follows:

$$\binom{s-2}{m} = \left(1 - \frac{m}{s-1}\right)\binom{s-1}{m}, \tag{2}$$

$$\binom{s-2}{m-1} = \frac{m}{s-1}\binom{s-1}{m}. \tag{3}$$

We will use these two results in the proof of following theorem.

Theorem 1.1. *The parametric generalization of the modified Bernstein operators can be expressed as*

$$\begin{aligned} B^*_{s,\alpha}(v;x) &= (1-\alpha)\sum_{m=0}^{s-1} u\left(\frac{m}{s}\right)\binom{s-1}{m}\frac{(m-(s-1)x)^2}{s-1}x^{m-1}(1-x)^{s-m-2} \\ &\quad + \alpha\sum_{m=0}^{s} v\left(\frac{m}{s}\right)\binom{s}{m}\frac{(m-sx)^2}{s}x^{m-1}(1-x)^{s-m-1}, \end{aligned}$$

where

$$u\left(\frac{m}{s}\right) = v\left(\frac{m}{s}\right)\left(1 - \frac{m}{s-1}\right) + v\left(\frac{m+1}{s}\right)\frac{m}{s-1}.$$

Proof. Let's express Eqn. (1) as

$$\begin{aligned}
B^*_{s,\alpha}(v;x) &= \sum_{m=0}^{s} v\left(\frac{m}{s}\right)\left\{\frac{1}{s-1}\binom{s-2}{m}(m-(s-1)x)^2(1-\alpha)x + \frac{1}{s-1}\binom{s-2}{m-2}(m-1-(s-1)x)^2(1-\alpha)(1-x)\right.\\
&\quad \left.+\frac{1}{s}\binom{s}{m}(m-sx)^2\alpha x(1-x)\right\} x^{m-2}(1-x)^{s-m-2}\\
&= (1-\alpha)\left(\sum_{m=0}^{s} v\left(\frac{m}{s}\right)\frac{1}{s-1}\binom{s-2}{m}(m-(s-1)x)^2 x^{m-1}(1-x)^{s-m-2}\right.\\
&\quad \left.+\sum_{m=0}^{s} v\left(\frac{m}{s}\right)\frac{1}{s-1}\binom{s-2}{m-2}(m-1-(s-1)x)^2 x^{m-2}(1-x)^{s-m-1}\right)\\
&\quad +\alpha\sum_{m=0}^{s} v\left(\frac{m}{s}\right)\binom{s}{m}\frac{(m-sx)^2}{s}x^{m-1}(1-x)^{s-m-1}\\
&= (1-\alpha)(b_1+b_2)+\alpha\sum_{m=0}^{s} v\left(\frac{m}{s}\right)\binom{s}{m}\frac{(m-sx)^2}{s}x^{m-1}(1-x)^{s-m-1},
\end{aligned}$$

where b_1 and b_2 are

$$\begin{aligned}
b_1 &= \sum_{m=0}^{s} v\left(\frac{m}{s}\right)\binom{s-2}{m}\frac{(m-(s-1)x)^2}{s-1}x^{m-1}(1-x)^{s-m-2},\\
b_2 &= \sum_{m=0}^{s} v\left(\frac{m}{s}\right)\binom{s-2}{m-2}\frac{(m-1-(s-1)x)^2}{s-1}x^{m-2}(1-x)^{s-m-1}.
\end{aligned}$$

Here, we can easily see that the terms for $m=s$ in b_1 and $m=0$ in b_2 are equal to zero. Thusly, we can write

$$\begin{aligned}
b_1 &= \sum_{m=0}^{s-1} v\left(\frac{m}{s}\right)\binom{s-2}{m}\frac{(m-(s-1)x)^2}{s-1}x^{m-1}(1-x)^{s-m-2},\\
b_2 &= \sum_{m=1}^{s} v\left(\frac{m}{s}\right)\binom{s-2}{m-2}\frac{(m-1-(s-1)x)^2}{s-1}x^{m-2}(1-x)^{s-m-1}.
\end{aligned}$$

By using Eqn. (2), we have

$$\begin{aligned}
b_1 &= \sum_{m=0}^{s-1} v\left(\frac{m}{s}\right)\binom{s-2}{m}\frac{(m-(s-1)x)^2}{s-1}x^{m-1}(1-x)^{s-m-2}\\
&= \sum_{m=0}^{s-1} v\left(\frac{m}{s}\right)\left(1-\frac{m}{s-1}\right)\binom{s-1}{m}\frac{(m-(s-1)x)^2}{s-1}x^{m-1}(1-x)^{s-m-2}. \qquad (4)
\end{aligned}$$

Also, replacing m by $m+1$ in the summation of b_2 and then using Eqn. (3), we obtain

$$\begin{aligned}
b_2 &= \sum_{m=1}^{s} v\left(\frac{m}{s}\right)\binom{s-2}{m-2}\frac{(m-1-(s-1)x)^2}{s-1}x^{m-2}(1-x)^{s-m-1}\\
&= \sum_{m=0}^{s-1} v\left(\frac{m+1}{s}\right)\binom{s-2}{m-1}\frac{(m-(s-1)x)^2}{s-1}x^{m-1}(1-x)^{s-m-2}\\
&= \sum_{m=0}^{s-1} v\left(\frac{m+1}{s}\right)\frac{m}{s-1}\binom{s-1}{m}\frac{(m-(s-1)x)^2}{s-1}x^{m-1}(1-x)^{s-m-2}. \qquad (5)
\end{aligned}$$

By adding Eqn. (4) and Eqn. (5), we achieve

$$\begin{aligned} b_1 + b_2 &= \sum_{m=0}^{s-1} \left\{ v\left(\frac{m}{s}\right)\left(1 - \frac{m}{s-1}\right) + v\left(\frac{m+1}{s}\right)\frac{m}{s-1} \right\} \binom{s-1}{m} \frac{(m-(s-1)x)^2}{s-1} x^{m-1}(1-x)^{s-m-2} \\ &= \sum_{m=0}^{s-1} u\left(\frac{m}{s}\right) \binom{s-1}{m} \frac{(m-(s-1)x)^2}{s-1} x^{m-1}(1-x)^{s-m-2}. \end{aligned}$$

As we can see

$$u\left(\frac{m}{s}\right) = v\left(\frac{m}{s}\right)\left(1 - \frac{m}{s-1}\right) + v\left(\frac{m+1}{s}\right)\frac{m}{s-1},$$

which completes the proof. □

2. Auxiliary results

Lemma 2.1. *For every $x \in (0,1)$, we write*

$$\begin{aligned} B^*_{s,\alpha}(e_0;x) &= 1, \\ B^*_{s,\alpha}(e_1;x) &= \frac{s-2}{s}x + \frac{1}{s}, \\ B^*_{s,\alpha}(e_2;x) &= \frac{(s+1-2\alpha)(s^2-7s+6)}{s^2(s-1)}x^2 + \frac{5s^2-11\alpha s+14\alpha-8}{s^2(s-1)}x + \frac{s+2-3\alpha}{s^2(s-1)}, \end{aligned}$$

where $e_m = t^m$ for $m = 0,1,2$.

Lemma 2.2. *For every $x \in (0,1)$, we have*

$$\begin{aligned} B^*_{s,\alpha}(t-x;x) &= \frac{-2x+1}{s}, \\ B^*_{s,\alpha}((t-x)^2;x) &= \frac{2-8x-s^2(-3+x)x+6x^2+s(1+2x-5x^2)-\alpha(3+(-14+11s)x+2(6-7s+s^2)x^2)}{s^2(s-1)}. \end{aligned}$$

Proof. By the help of

$$\begin{aligned} B^*_{s,\alpha}(t-x;x) &= B^*_{s,\alpha}(e_1;x) - xB^*_{s,\alpha}(e_0;x), \\ B^*_{s,\alpha}((t-x)^2;x) &= B^*_{s,\alpha}(e_2;x) - 2xB^*_{s,\alpha}(e_1;x) + x^2 B^*_{s,\alpha}(e_0;x), \end{aligned}$$

we get the desired result of the lemma. □

Let the Banach space of all continuous functions v on $[0,1]$ is denoted by $C[0,1]$ endowed with the norm

$$\|v\| = \max_{x\in(0,1)} |v(x)|.$$

Theorem 2.3. *For every $x \in (0,1)$ and $v \in C[0,1]$, we have*

$$\left\| B^*_{s,\alpha}(v;x) - v(x) \right\| \to 0 \tag{6}$$

uniformly as $s \to \infty$.

Proof. It can be seen clearly that

$$\lim_{s\to\infty} B^*_{s,\alpha}(e_i;x) = t^i, \qquad i = 0,1,2.$$

Thus, we conclude the proof of the theorem thanks to the result of Korovkin's theorem [11]. □

3. Rate of convergence

For $v \in C[0,1]$, the modulus of continuity is given by

$$\omega(v,\delta) := \sup_{|t-x|\leq\delta} \sup_{x\in(0,1)} |v(t) - v(x)|, \quad \delta > 0.$$

Modulus of continuity has the following property [2]

$$|v(t) - v(x)| \leq \left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta).$$

Theorem 3.1. *For every $x \in (0,1)$ and $v \in C[0,1]$,*

$$|B^*_{s,\alpha}(v;x) - v(x)| \leq 2\omega(v,\delta_s). \tag{7}$$

Here

$$\delta_s(x) = \sqrt{\frac{2 - 3\alpha + s + (-8 + 2s + 3s^2 - \alpha(-14 + 11s))x + (6 - 5s - s^2 - 2\alpha(6 - 7s + s^2))x^2}{(s-1)s^2}}.$$

Proof.

$$\begin{aligned}
\left|B^*_{s,\alpha}(v;x) - v(x)\right| &= \left|\sum_{m=0}^{s} p^{(\alpha)}_{s,m}(x) v\left(\frac{m}{s}\right) - v(x)\right| \\
\leq & \sum_{m=0}^{s} p^{(\alpha)}_{s,m}(x)\left|v\left(\frac{m}{s}\right) - v(x)\right| \\
\leq & \sum_{m=0}^{s} p^{(\alpha)}_{s,m}(x)\left(1 + \frac{1}{\delta^2}\frac{(m-sx)^2}{s^2}\right)\omega(v,\delta) \\
= & \left(1 + \frac{1}{\delta^2}\frac{2 - 3\alpha + s + (-8 + 2s + 3s^2 - \alpha(-14 + 11s))x + (6 - 5s - s^2 - 2\alpha(6 - 7s + s^2))x^2}{(s-1)s^2}\right)\omega(v,\delta).
\end{aligned}$$

If we choose

$$\delta = \delta_s = \sqrt{\frac{2 - 3\alpha + s + (-8 + 2s + 3s^2 - \alpha(-14 + 11s))x + (6 - 5s - s^2 - 2\alpha(6 - 7s + s^2))x^2}{(s-1)s^2}}$$

then we achieve that

$$|B^*_{s,\alpha}(v;x) - v(x)| \leq 2\omega\left(v, \sqrt{\frac{2 - 3\alpha + s + (-8 + 2s + 3s^2 - \alpha(-14 + 11s))x + (6 - 5s - s^2 - 2\alpha(6 - 7s + s^2))x^2}{(s-1)s^2}}\right),$$

which is the desired result. □

Just now, we investigate the rate of convergence of $B^*_{s,\alpha}(v;x)$ by using functions of Lipschitz class. Let's recall that a function $v \in Lip_K(c)$ on $(0,1)$ if the inequality

$$|v(t) - v(x)| \leq K|t-x|^c \quad ; \ \forall t, x \in (0,1) \tag{8}$$

holds.

Theorem 3.2. *Let $x \in (0,1)$, $v \in Lip_K(c)$, $0 < c \leq 1$, then we have*

$$\left|B^*_{s,\alpha}(v;x) - v(x)\right| \leq K\delta_s^c(x),$$

where

$$\delta_s(x) = \sqrt{\frac{2-3\alpha+s+(-8+2s+3s^2-\alpha(-14+11s))x+(6-5s-s^2-2\alpha(6-7s+s^2))x^2}{(s-1)s^2}}.$$

Proof. Let $x \in (0,1)$, $v \in Lip_K(c)$ and $0 < c \leq 1$. By using (8) and the linearity and monotonicity of the operators $B^*_{s,\alpha}$ we have

$$\begin{aligned}\left|B^*_{s,\alpha}(v;x) - v(x)\right| &\leq B^*_{s,\alpha}(|v(t)-v(x)|;x)\\ &\leq KB^*_{s,\alpha}(|t-x|^c;x).\end{aligned}$$

By taking $p = \frac{2}{c}$, $q = \frac{2}{2-c}$ in the Hölder inequality, we get

$$\begin{aligned}\left|B^*_{s,\alpha}(v;x) - v(x)\right| &\leq K\left\{B^*_{s,\alpha}\left((t-x)^2;x\right)\right\}^{\frac{c}{2}}\\ &\leq K\delta_s^c(x)\end{aligned}$$

immediately. If we choose

$$\delta_s(x) = \sqrt{\frac{2-3\alpha+s+(-8+2s+3s^2-\alpha(-14+11s))x+(6-5s-s^2-2\alpha(6-7s+s^2))x^2}{(s-1)s^2}}$$

the proof is completed. □

Lastly, we will give the rate of convergence of our operator $B^*_{s,\alpha}(v;x)$ by means of Peetre's-$\mathcal{K}$ functionals. First of all, we give the following lemma.

Lemma 3.3. *For $x \in (0,1)$ and $v \in C[0,1]$, we have*

$$|B^*_{s,\alpha}(v;x)| \leq \|v\|. \tag{9}$$

Proof. For $B^*_{s,\alpha}$ we have

$$\begin{aligned}|B^*_{s,\alpha}(v;x)| &= \left|\sum_{m=0}^{s} p^{(\alpha)}_{s,m}(x)v\left(\frac{m}{s}\right)\right|\\ &\leq \sum_{m=0}^{s} p^{(\alpha)}_{s,m}(x)\left|v\left(\frac{m}{s}\right)\right|\\ &\leq \|v\|B^*_{s,\alpha}(1;x)\\ &= \|v\|.\end{aligned}$$

□

And then, we recall the properties of Peetre's-$\mathcal{K}$ functionals. $C^2[0,1]$ is the space of the functions v, for which v, v' and v'' are continuous on $[0,1]$. Now, we define classical Peetre's-$\mathcal{K}$ functional as follows:

$$\mathcal{K}(v,\delta) := \inf_{u\in C^2[0,1]}\{\|v-u\| + \delta\,\|u''\|\}$$

and second modulus of smoothness of the function is defined by

$$\omega_2(v,\delta) := \sup_{0<h<\delta}\ \sup_{x,x+2h\in(0,1)} |v(x+2h) - 2v(x+h) + v(x)|,$$

where $\delta > 0$. By [9], it is known that for $A > 0$

$$\mathcal{K}(v,\delta) \leq A\omega_2(v, \sqrt{\delta}).$$

Theorem 3.4. *Let $x \in (0,1)$ and $v \in C[0,1]$. Then we have for all $s \in \mathbb{N}$, there exists a positive constant M such that*

$$|B^{*}_{s,\alpha}(v;x) - v(x)| \leq A\omega_2(v, \alpha_s(x)) + 2\omega(v, \beta_s(x)),$$

where

$$\alpha_s(x) = \sqrt{\frac{3-6\alpha+3s+(-12+28\alpha-22\alpha s+6s^2)x+(8-24\alpha-6s+28\alpha s-2s^2-4\alpha s^2)x^2}{2(s-1)s^2}}$$

and

$$\beta_s(x) = \left|\frac{1-2x}{s}\right|.$$

Proof. Define an auxiliary operator $B^{**}_s : C[0,1] \to C[0,1]$ by

$$B^{**}_s(u;x) = B^{*}_{s,\alpha}(u;x) - u\left(\frac{(s-2)x+1}{s}\right) + u(x). \tag{10}$$

From Lemma 2.1, we have

$$\begin{aligned} B^{**}_s(1;x) &= 1, \\ B^{**}_s(t-x;x) &= B^{*}_{s,\alpha}((t-x);x) - \left(\frac{(s-2)x+1}{s} - x\right) + x - x \\ &= \left(\frac{1-2x}{s}\right) - \left(\frac{(s-2)x+1}{s} - x\right) + x - x \\ &= 0. \end{aligned} \tag{11}$$

For a given function $u \in C^2[0,1]$, we have by the Taylor expansion that

$$u(t) = u(x) + (t-x)u'(x) + \int_x^t (t-m)u''(m)dm, \quad t \in (0,1). \tag{12}$$

Applying B^{**}_s operator to both sides of the Eqn. (12), we get

$$\begin{aligned} B^{**}_s(u;x) &= B^{**}_s\left(u(x) + (t-x)u'(x) + \int_x^t (t-m)u''(m)dm; x\right) \\ &= u(x) + B^{**}_s((t-x)u'(x);x) + B^{**}_s\left(\int_x^t (t-m)u''(m)dm; x\right). \end{aligned}$$

So,

$$B^{**}_s(u;x) - u(x) = u'(x)B^{**}_s((t-x);x) + B^{**}_s\left(\int_x^t (t-m)u''(m)dm; x\right).$$

Using (10) and (11), we obtain

$$\begin{aligned} B^{**}_s(u;x) - u(x) &= B^{**}_s\left(\int_x^t (t-m)u''(m)dm; x\right) \\ &= B^{*}_{s,\alpha}\left(\int_x^t (t-m)u''(m)dm; x\right) - \int_x^{\frac{(s-2)x+1}{s}} \left(\frac{(s-2)x+1}{s} - m\right)u''(m)dm \\ &\quad + \int_x^x \left(\frac{(s-2)x+1}{s} - m\right)u''(m)dm. \end{aligned} \tag{13}$$

Moreover,

$$\begin{aligned}\left|\int_x^t (t-m)u''(m)dm\right| &\leq \int_x^t |t-m||u''(m)|dm \leq \|u''\| \int_x^t |t-m|dm \\ &\leq (t-x)^2\|u''\| \end{aligned} \tag{14}$$

and

$$\begin{aligned}&\left|\int_x^{\frac{(s-2)x+1}{s}} \left(\frac{(s-2)x+1}{s} - u\right)u''(m)dm\right| \leq \|u''\| \int_x^{\frac{(s-2)x+1}{s}} \left(\frac{(s-2)x+1}{s} - m\right)dm \\ &= \frac{\|u''\|}{2}\left(\left(\frac{(s-2)x+1}{s}\right)^2 - 2\frac{(s-2)x+1}{s}x + x^2\right) \\ &= \frac{\|u''\|}{2}\left(\frac{(s-2)x+1}{s} - x\right)^2. \end{aligned} \tag{15}$$

Let's rewrite (14) and (15) in the absolute value of (13). Then we obtain

$$\begin{aligned}\left|B_s^{**}(u;x) - u(x)\right| &\leq \|u''\|B_{s,\alpha}^*((t-x)^2;x) + \frac{\|u''\|}{2}\left(\frac{(s-2)x+1}{s} - x\right)^2 \\ &= \|u''\|\left(B_{s,\alpha}^*((t-x)^2;x) + \frac{1}{2}\left(\frac{(s-2)x+1}{s} - x\right)^2\right) \\ &= \|u''\|\alpha_s^2(x),\end{aligned}$$

where

$$\begin{aligned}\alpha_s(x) &= \sqrt{B_{s,\alpha}^*((t-x)^2;x) + \frac{1}{2}\left(\frac{(s-2)x+1}{s} - x\right)^2} \\ &= \sqrt{\frac{3 - 6\alpha + 3s + (-12 + 28\alpha - 22\alpha s + 6s^2)x + (8 - 24\alpha - 6s + 28\alpha s - 2s^2 - 4\alpha s^2)x^2}{2(s-1)s^2}}.\end{aligned}$$

Currently, we will find a bound for the auxiliary operator $B_s^{**}(u;x)$. In the light of the Lemma 3.3, we obtain

$$\begin{aligned}|B_s^{**}(u;x)| &= \left|B_{s,\alpha}^*(u;x) - u\left(\frac{(s-2)x+1}{s}\right) + u(x)\right| \\ &\leq |B_{s,\alpha}^*(u;x)| + \left|u\left(\frac{(s-2)x+1}{s}\right)\right| + |u(x)| \\ &\leq 3\|u\|.\end{aligned}$$

Accordingly,

$$\begin{aligned}|B_{s,\alpha}^*(v;x) - v(x)| &= \left|B_s^{**}(v;x) - v(x) + v\left(\frac{(s-2)x+1}{s}\right) - v(x) + u(x) - u(x) + B_s^{**}(u;x) - B_s^{**}(u;x)\right| \\ &\leq |B_s^{**}(v-u;x) - (v-u)(x)| + |B_s^{**}(u;x) - u(x)| + \left|v\left(\frac{(s-2)x+1}{s}\right) - v(x)\right| \\ &\leq 4\|v-u\| + \|u''\|\alpha_s^2(x) + \omega(v,\beta_s(x))\left(\frac{\left|\frac{(s-2)x+1}{s} - x\right|}{\beta_s(x)} + 1\right) \\ &\leq 4\left(\|v-u\| + \|u''\|\alpha_s^2(x)\right) + 2\omega\left(v, \left|\frac{(s-2)x+1}{s} - x\right|\right),\end{aligned} \tag{16}$$

where

$$\begin{aligned}\beta_s(x) &= \left|\frac{(s-2)x+1}{s} - x\right| \\ &= \left|\frac{1-2x}{s}\right|.\end{aligned}$$

Finally, for all $v \in C^2[0,1]$, taking the infimum of the equation (16), we get

$$|B^*_{s,\alpha}(v;x) - v(x)| \leq 4\mathcal{K}(u, \alpha_s^2(x)) + 2\omega(v, \beta_s(x)). \tag{17}$$

As a result, using the property of Peetre's-$\mathcal{K}$ functional, we obtain

$$|B^*_{s,\alpha}(v;x) - v(x)| \leq A\omega_2(v, \alpha_s(x)) + 2\omega(v, \beta_s(x)). \tag{18}$$

Thus the proof is completed. □

4. Graphical Analysis

In this section, we show the convergence of the newly constructed operators $B^*_{s,\alpha}$ with function v. We know that the operators $B^*_{s,\alpha}$ have been given for the interval $(0,1)$. For this reason, in the following two examples we use the interval $[0+\epsilon, 1-\epsilon]$ where $\epsilon = 0.0001$.
Let the function v be

$$v(x) = x(x-1)\left(x - \frac{1}{10}\right).$$

Then for $\alpha = 0.9$, $\alpha = 0.99$ and $\alpha = 0.999$, we have plotted the convergence of the new constructed $B^*_{s,\alpha}$ Bernstein operators and B^*_s Bernstein operators [16] to the function v in Fig. 1 for $s = 25, 100, 125$.

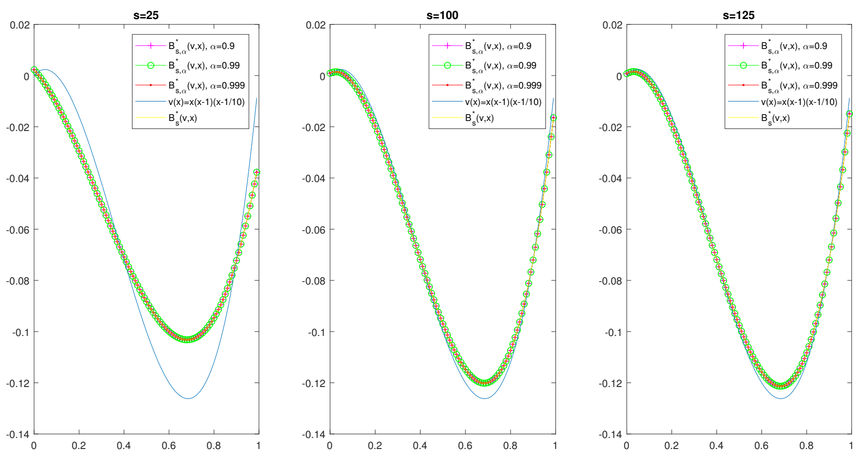

Figure 1: Convergence of $B^*_{s,\alpha}(v;x)$ for different values of α with fixed s.

The error estimation for operators $B^*_{s,\alpha}$ to the function $v(x) = x(x-1)\left(x - \frac{1}{10}\right)$ is presented in Table 1 for different values of α and s.

α	s	$\|B^*_{s,\alpha}(v) - v\|$
0.99	25	0.029026144150856
0.99	50	0.015155890904675
0.99	100	0.007741124297200
0.99	125	0.006219166275268
0.999	25	0.029017372605222
0.999	50	0.015153407208112
0.999	100	0.007740466487883
0.999	125	0.006218740529548

Table 1: Error comparison table of $B^*_{s,\alpha}$ new constructed operators

When we examine the error approximation given in Table 1, we observe that the best approximation is achieved when $\alpha = 0.999$ and $s = 125$.
As a second example, let the the function v be

$$v(x) = xe^{-5x}$$

and $x \in (0, 1)$. Then for $\alpha = 0.79$, $\alpha = 0.89$ and $\alpha = 0.99$, we have plotted the convergence of the $B^*_{s,\alpha}$ Bernstein operators to the function v in Fig. 2 for different s values.

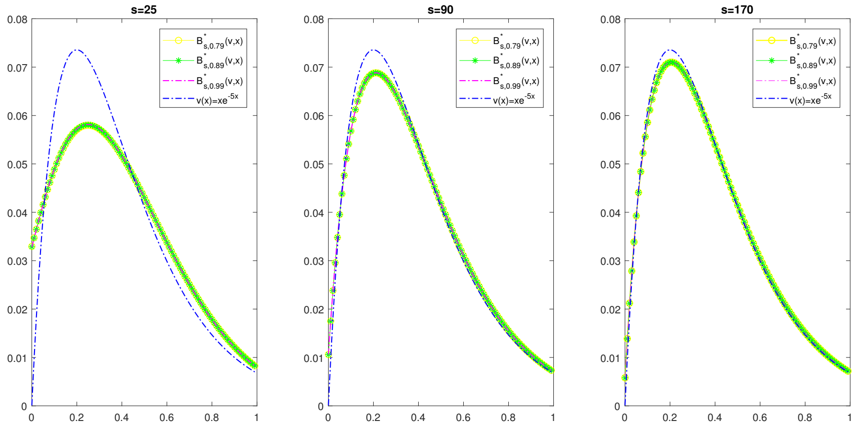

Figure 2: Convergence of $B^*_{s,\alpha}(v; x)$ for $\alpha = 0.79$, $\alpha = 0.89$ and $\alpha = 0.99$.

The error estimation for newly constructed operators $B^*_{s,\alpha}$ to the function $v(x) = xe^{-5x}$ is presented in Table 2 for different values of α and s.

α	s	$\|\|B^*_{s,\alpha}(v) - v\|\|$
0.79	25	0.032849798692534
0.79	90	0.010505004818431
0.79	170	0.005703305791541
0.89	25	0.032763043737020
0.89	90	0.010494503508916
0.89	170	0.005700131820294
0.99	25	0.032676288781506
0.99	90	0.010484002199401
0.99	170	0.005696957849047

Table 2: Error of approximation $B^*_{s,\alpha}$ for $\alpha = 0.79, 0.89, 0.99$

When we investigate the error approximation given in Table 2, we observe that the best approximation is achieved when $\alpha = 0.99$ and $s = 170$.

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