Filomat 36:5 (2022), 1663–1674
https://doi.org/10.2298/FIL2205663S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Symplectic Golden Manifolds and Lie Groupoids

Fulya Şahin[a]

[a] *Ege University, Department of Mathematics, İzmir/TURKEY*

Abstract. By considering the notion of Golden manifold and natural symplectic form on a generalized tangent bundle, we introduce generalized symplectic Golden structures on manifolds and obtain integrability conditions in terms of bivector fields, 2-forms, 1-forms and endomorphisms on manifolds and investigate isotropic subbundles. We also find certain relations between the integrability conditions of generalized symplectic Golden manifolds and Lie Groupoids which are important in mechanics as configuration space.

1. INTRODUCTION

A differentiable manifold M is called a Goden manifold if there exists a $(1,1)$-tensor field ϕ on M such that $\phi^2 = \phi + I$, where I denotes the identity map. If (M, g) is a Riemannian manifold and ϕ is a Golden structure on M such that

$$g(\phi\alpha_1, \alpha_2) = g(\alpha_1, \phi\alpha_2)$$

then (M, g, ϕ) is called a Golden Riemannian manifold. It is known that the Golden proportion has been found applications in various structures see:[7] By inspiring Golden ratio, Golden structures were introduced by Crasmareanu and Hretcanu in [12] and such structures have been studied in [6, 13, 15–17, 22, 23, 33, 34, 38–40]. The application of the golden ratio in many areas shows that the Golden Riemann manifolds will have a rich geometric structure for further research. As extension of Golden manifolds, Metallic manifolds have been introduced by Hretcanu and Crasmareanu in [23] and submanifolds of such manifolds have been studied by many authors, see:[2, 14, 19, 30]

The generalized geometry is the geometry created by using the direct sum of the tangent bundle and cotangent bundles, naming generalized tangent bundle (or big bundle), instead of the tangent bundle notion in the manifold theory. This concept was introduced by Hitchin [24] as a language that defines both complex manifolds and symplectic manifolds. Although it was seen as a virtual notion at first glance, it has been observed that the generalized geometry is a suitable language especially in explaining and defining the notions of string theory. For this reason, generalized manifolds are the most important research area of manifold theory. After Hitchin's definition of generalized complex manifolds, this topic was studied in detail by Gualtieri [21] in his doctoral thesis, see also [20]. After this stage, new generalized manifolds were studied by many authors, see: [1], [3], [4], [5], [18],[32], [36], [41].

2020 *Mathematics Subject Classification*. Primary 22A22, 53D17, 53D18

Keywords. Lie Groupoid, Lie Algebroid, Golden Manifold, Almost Generalized Golden Manifold, Generalized Symplectic Golden manifold

Received: 22 May 2021; Revised: 01 June 2021; Accepted: 02 June 2021

Communicated by Mića S. Stanković

Research supported by The Scientific and Technological Council of Turkey (TUBITAK) with number (119F065)

Email address: fulya.sahin@ege.edu.tr (Fulya Şahin)

After Crainic [10] demonstrated a one-to-one relationship between the integrability conditions of generalized complex manifolds and the notions of Lie groupoids, it has been important to examine the relationships between other generalized manifolds (generalized para-complex manifolds, Generalized contact manifolds, generalized paracontact manifolds, and so on) with Lie groupoids and Lie Algebroids. This will make it possible to examine notions in differential topology by using results in generalized manifolds.

As we mentioned above, Golden ratio has many applications in different areas and generalized geometry has also provided a useful tool for string theory, such geometry has been studied widely by physicists too, see:[27]. The aim of this paper is to combine these two notions and present a new class of generalized manifolds with natural symplectic form and relate such manifolds with Lie groupoids.

In generalized geometry, the non-degenerate symmetric bilinear form is generally considered for compatibility with structure in hand, but in this article we will consider the canonical symplectic form on the generalized manifold to ensure compatibility of the Golden structure with the bilinear form. In this direction, we first introduce generalized symplectic Golden manifolds and provide examples. We find necessary and sufficient conditions in terms of classical tensor fields for Golden structure to be integrable. We obtain isotropy conditions and show that the distributions corresponding to eigen values of Golden structures are not Dirac subbundle. Then we find orthogonality conditions for these distributions. In the last section, we show that there is a close relation between the conditions for generalized symplectic Golden structure to be integrable and Lie groupoids.

2. PRELIMINARIES

Generalized geometry is considered by taking the bundle formed by constructing the direct sum of the tangent bundle and the cotangent bundle instead of the tangent bundle of the manifold. This bundle is denoted by $TM \oplus TM^*$. The sections of this bundle is consisted of (α_1, γ) (or $\alpha_1 + \gamma$) for $\alpha_1 \in \Gamma(TM)$ and $\gamma \in \Gamma(TM^*)$. For the sections $(\alpha_1, \gamma), (\alpha_2, \delta)$ of $TM \oplus TM^* = \mathcal{T}\mathcal{M}$, a natural symplectic structure $<,>$ is defined by

$$< \alpha_1 + \gamma, \alpha_2 + \delta >= \frac{1}{2}(i_{\alpha_1}\delta - i_{\alpha_2}\gamma), \tag{1}$$

and the Courant bracket of two sections is defined by

$$[\![(\alpha_1, \gamma), (\alpha_2, \delta)]\!] = [\alpha_1, \alpha_2] + L_{\alpha_1}\delta - L_{\alpha_2}\gamma \; - \frac{1}{2}d(i_{\alpha_1}\delta - i_{\alpha_2}\gamma), \tag{2}$$

where d, L_{α_1} and i_{α_1} denote exterior derivative, Lie derivative and interior derivative with respect to α_1, respectively. The Courant bracket does not satisfy the Jacobi identity. In this paper we adapt the notions

$$\delta(\Pi^\sharp\gamma) = \Pi(\gamma, \delta) \quad \text{and} \quad \omega_\flat(\alpha_1)(\alpha_2) = \omega(\alpha_1, \alpha_2) \tag{3}$$

which are defined as $\Pi^\sharp : TM^* \to TM$, $\omega_\flat : TM \to TM^*$ for any 1-forms γ and δ, 2-form ω and bivector field Π, and vector fields α_1 and α_2. The bracket $[,]_\Pi$ on the space of 1-forms on M is defined by

$$[\gamma, \delta]_\Pi = L_{\Pi^\sharp\gamma}\delta - L_{\Pi^\sharp\delta}\gamma - d\Pi(\gamma, \delta). \tag{4}$$

The basic properties of Lie groupoids will not be given in detail in this paper. This notion is now well known in the literature, see: [31]. Generally, a Lie groupoid Γ is denoted by the set of arrows Γ_1. Being submersions, s and t ensure that s and t-fibres are manifolds. The space Γ_2 of composable arrows is a submanifold of $\Gamma_1 \times \Gamma_1$. Let Γ be a Lie groupoid on M and ω a form on Lie groupoid Γ, then ω is called multiplicative if

$$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$

where $pr_i : \Gamma \times \Gamma \to \Gamma$, $i = 1, 2$, are the canonical projections. If a Lie groupoid Γ is endowed with a form which is multiplicative, then Γ is called symplectic groupoid.

We now recall certain notions of Lie algebroids from [29]. A Lie algebroid structure on a real vector bundle $\mathcal{L}$ on a manifold M is defined by a vector bundle map $\rho_{\mathcal{L}} : \mathcal{L} \to TM$, the anchor of $\mathcal{L}$, and an $\mathbb{R}$-Lie algebra bracket on $F(\mathcal{L})$, $[,]_{\mathcal{L}}$ satisfying the Leibniz rule

$$[\gamma, f\delta]_{\mathcal{L}} = f[\gamma, \delta]_{\mathcal{L}} + L_{\rho_{\mathcal{L}}(\gamma)}(f)\delta$$

for all $\gamma, \delta \in F(\mathcal{L})$, $f \in C^{\infty}(M)$, where $L_{\rho_{\mathcal{L}}(\gamma)}$ is the Lie derivative with respect to the vector field $\rho_{\mathcal{L}}(\gamma)$. And $F(\mathcal{L})$ denotes the set of sections in $\mathcal{L}$.

Lie algebroids and Lie groupoids are related notions. Lie groupoids are the infinitesimal equivalents of Lie algebroids. However, unlike in Lie algebras, not every Lie algebroid has a corresponding Lie groupoid. If so, a Lie algebroid is called integrable. We note that Lie algebroids have been studied widely by many authors, see: [25], [26], [35], [37] for instances.

An *IM* form (infinitesimal multiplicative form) [8] on a Lie algebroid $\mathcal{L}$ is a bundle map $u : \mathcal{L} \to TM^*$ satisfying the following properties

(i) $\langle u(\gamma), \rho(\delta)\rangle = -\langle u(\delta), \rho(\gamma)\rangle$

(ii) $u([\gamma, \delta]) = L_{\gamma}(u(\delta)) - L_{\delta}(u(\gamma)) + d\langle u(\gamma), \rho(\delta)\rangle$

for $\gamma, \delta \in F(\mathcal{L})$, where $\langle , \rangle$ denotes the usual pairing between a vector space and its dual.

Finally, we recall that a smooth manifold is a Poisson manifold [28] if $[\Pi, \Pi] = 0$, where Π is a bi-vector field and $[,]$ denotes the Schouten bracket on the space of multivector fields.

3. GENERALIZED GOLDEN MANIFOLDS

In this section we introduce generalized Golden structures on a manifold, provide examples and obtain integrability conditions for such manifolds in terms of certain geometric objects defined on such manifolds. We first present the following notion.

Definition 3.1. *An almost generalized Golden structure Φ on a smooth manifold M consists of a bundle endomorphism Φ from $\mathcal{TM}$ to itself such that*

$$\Phi^2 = \Phi + I \tag{5}$$

where I denotes the identity map. Let Φ be an almost generalized Golden structure on M endowed with $<,>$ given (5) such that

$$< \Phi X, Y > = < X, \Phi Y >, \tag{6}$$

then $(M, \Phi, <,>)$ is called a generalized symplectic Golden manifold.

We note that the notion of generalized metallical structures has been introduced in [4] and [5]. Although generalized Golden manifold given in the above definition is the special case of generalized Metallic manifold, their main properties are different. We also note that generalized metallical manifolds have been studied in [4] and [5] with the viewpoint of differential geometry, but here we will examine topologically.

(5) and (6) imply that the bundle map $\Phi : \mathcal{TM} \longrightarrow \mathcal{TM}$ is given by

$$\Phi = \begin{bmatrix} A & \Pi^{\sharp} \\ \vartheta_{\flat} & A^* \end{bmatrix}, \tag{7}$$

where Π is a bivector on M, ϑ is a 2-form on M, $A : TM \to TM$ is a bundle endomorphism, and $A^* : TM^* \to TM^*$ is dual of A.

Example 3.2. *Associated to any Golden Riemannian structure ϕ, we have a generalized Golden structure by setting*

$$\Phi = \begin{bmatrix} \phi & 0 \\ 0 & \phi^* \end{bmatrix},$$

where ϕ^* is dual of ϕ.

We give another example of generalized Golden manifolds.

Example 3.3. *Let $\mathbb{R}^2$ be the two-dimensional Euclidean space and $\{X_1, X_2\}$ a basis. Let $\{\gamma^1, \gamma^2\}$ be a dual frame. Now for some real number b and c satisfying*

$$b\sqrt{5}\gamma^2(\alpha_1)\gamma^1(\alpha_1)[\gamma(X_2) + \gamma(X_1)] = c[\gamma(X_1)\gamma^1(\alpha_1) - \gamma(X_2)\gamma^2(\alpha_1)],$$

$\forall \alpha_1 + \gamma \in \Gamma(\mathbb{R}^2 \oplus \mathbb{R}^{*2})$, *we define*

$$A = \frac{1+\sqrt{5}}{2}(X_1 \otimes \gamma^1 + X_2 \otimes \gamma^2), \vartheta = c\gamma^1 \wedge \gamma^2, \Pi = bX_1 \wedge X_2.$$

We also , as given in (7), define

$$\Phi = \begin{bmatrix} A & \Pi^\sharp \\ \vartheta_\flat & A^* \end{bmatrix}.$$

Then Φ is a generalized Golden structure on $\mathbb{R}^2$.

As usual, a generalized Golden structure is called integrable if the Nijenhuis tensor field with respect to Courant bracket is zero, i.e.,

$$[\![\Phi\xi, \Phi\zeta]\!] - \Phi([\![\Phi\xi, \zeta]\!] + [\![\xi, \Phi\zeta]\!]) + \Phi^2[\![\xi, \zeta]\!] = 0, \tag{8}$$

for all sections $\xi, \zeta \in \Gamma(\mathcal{TM})$. In the sequel we find necessary and sufficient conditions for Φ to be integrable. First note that from (5) and (7) we see that Φ is an almost generalized Golden structure on M if and only if the following expressions are satisfied

$$A^2 + \Pi^\sharp\vartheta_\flat = A + I \quad , \quad A\Pi^\sharp + \Pi^\sharp A^* = \Pi^\sharp \tag{9}$$

$$\vartheta_\flat A + A^*\vartheta_\flat = \vartheta_\flat \quad , \quad \vartheta_\flat\Pi^\sharp + A^{*2} = A^* + I. \tag{10}$$

Theorem 3.4. *A generalized Golden structure is integrable if and only if the following conditions are satisfied.*

(G1) Π *satisfies the equation*

$$[\Pi^\sharp\gamma, \Pi^\sharp\delta] = \Pi^\sharp([\gamma, \delta]_\Pi), \tag{11}$$

(G2) Π *and A are related by the following formula*

$$A^*([\gamma, \delta]_\Pi) = L_{\Pi^\sharp\gamma}A^*\delta - L_{\Pi^\sharp\delta}A^*\gamma - \frac{1}{2}d\Pi(\gamma, \delta), \tag{12}$$

(G3) A, Π *are related by the following formula*

$$\begin{aligned}[A\alpha_1, \Pi^\sharp\gamma] - A[\alpha_1, \Pi^\sharp\gamma] &= \Pi^\sharp(L_{A\alpha_1}\gamma + L_{\alpha_1}A^*\gamma - L_{\alpha_1}\gamma \\ &+ \frac{1}{2}d\gamma(\alpha_1) - d\gamma(A\alpha_1)),\end{aligned} \tag{13}$$

(G4) *A, Π and $\vartheta_\flat$ are related by the following formula*

$$\begin{aligned}&L_{A\alpha_1}A^*\gamma - L_{\Pi^\sharp}\vartheta_\flat(\alpha_1) + L_{\alpha_1}\gamma - \vartheta_\flat([\alpha_1, \Pi^\sharp\gamma])\\ &-\frac{1}{2}d(\gamma(A^2\alpha_1) - \vartheta(\alpha_1, \Pi^\sharp\gamma) + \gamma(\alpha_1))\\ &+A^*(L_{\alpha_1}\gamma - \frac{1}{2}d\gamma(\alpha_1) - L_{A\alpha_1}\gamma - L_{\alpha_1}A^*\gamma + d\gamma(A\alpha_1))\end{aligned} \tag{14}$$

(G5) *N_A, $\Pi^\sharp$ and $\vartheta_\flat$ satisfy the following equation*

$$N_A(\alpha_1, \alpha_2) = \Pi^\sharp(i_{\alpha_1\wedge\alpha_2}(d\vartheta_\flat)) \tag{15}$$

(G6) *ϑ and and A are related by the following formula*

$$d\vartheta_A(\alpha_1, \alpha_2, \alpha_3) = d\vartheta(A\alpha_1, \alpha_2, \alpha_3) + d\vartheta(\alpha_1, A\alpha_2, \alpha_3) + d\vartheta(\alpha_1, \alpha_2, A\alpha_3) \tag{16}$$

for $\alpha_1, \alpha_2, \alpha_3 \in \Gamma(TM)$ and $\gamma, \delta \in \Gamma(TM^*)$.

Proof. First, for 1− forms γ and δ, from the second equation of (9) we find

$$\Pi(\gamma, \delta) = \Pi(\gamma, A^*\delta) - \Pi(\delta, A^*\gamma). \tag{17}$$

Using (2), (9), (17) and taking the vector field parts and 1−form parts of the resulting equation, we obtain (12) and (13). For vector fields α_1 and α_2, in a similar way, we get

$$\begin{aligned}&[A\alpha_1, A\alpha_1] + A[\alpha_1, \alpha_2] + [\alpha_1, \alpha_2] - A([\alpha_1, A\alpha_2] \quad + \quad [A\alpha_1, \alpha_2])\\ &\qquad\qquad -\Pi^\sharp(-L_{\alpha_2}\vartheta_\flat(\alpha_1) + L_{\alpha_1}\vartheta_\flat(\alpha_2) \quad + \quad d\vartheta(\alpha_1, \alpha_2))\end{aligned} \tag{18}$$

and

$$\begin{aligned}&L_{A\alpha_1}\vartheta_\flat(\alpha_2) - L_{A\alpha_2}\vartheta_\flat(\alpha_1) + \vartheta_\flat([\alpha_1, \alpha_2]) - \vartheta_\flat([A\alpha_1, \alpha_2])\\ &\vartheta_\flat([\alpha_1, A\alpha_2]) + d\vartheta(\alpha_1, A\alpha_2) - A^*(L_{\alpha_1}\vartheta_\flat(\alpha_2) - L_{\alpha_2}\vartheta_\flat(\alpha_1)\\ &+d\vartheta(\alpha_1, \alpha_2)\end{aligned} \tag{19}$$

Using the below formula

$$i_{\alpha_1\wedge\alpha_2}(d\vartheta) = L_{\alpha_1}(i_{\alpha_2}\vartheta) - L_{\alpha_2}(i_{\alpha_1}\vartheta) + d(i_{\alpha_1\wedge\alpha_2}\vartheta) - i_{[\alpha_1,\alpha_2]}\vartheta,$$

and the first equation of (9) in (18) we have (17). Also using the formula of exterior derivative in (19) we derive (16). Moreover, for a vector field α_1 and 1− form γ, using (2) and(9) and then taking the vector field parts and 1− form parts in resulting equation, we obtain (13) and (14). □

4. ISOTROPIC SUBBUNDLES

In this section, we are going to investigate isotropic subbundles on a generalized Golden manifold. We first find necessary conditions for the eigenbundles corresponding to ϕ and $1 - \phi$ of Φ to be isotropic. Let M be a generalized Golden manifold and Φ the bundle map given in (7). The eigenvalues of a Golden structure Φ are the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ and $1 - \phi$. We now define

$$\begin{aligned}E^{(1,0)} &= \{X + \phi\Phi X \mid X \in \Gamma(TM \oplus TM^*)\}\\ E^{(0,1)} &= \{X + (1 - \phi)\Phi X \mid X \in \Gamma(TM \oplus TM^*)\}.\end{aligned}$$

In the sequel we are going to find necessary conditions for these distributions to be isotropic. This also shows that such distributions are not isotropic in general, contrary to complex case, see [21].

Lemma 4.1. *For* $e_1 = \alpha_1 + \gamma + \phi\Phi(\alpha_1 + \gamma), e_2 = \alpha_2 + \delta + \phi\Phi(\alpha_2 + \delta) \in F(E^{(1,0)})$ *we have*

$$
\begin{aligned}
< e_1, e_2 > &= \frac{1}{2}\{-\gamma(\phi(3A + I)\alpha_2 + (A + 2I)\alpha_2) \\
&+ \delta(\phi(3A + I)\alpha_1 + (A + 2I)\alpha_1) \\
&+ \phi^2(\Pi(\gamma, \delta) + \vartheta(\alpha_2, \alpha_1)) + 2\phi(\vartheta(\alpha_2, \alpha_1))\}. \qquad (20)
\end{aligned}
$$

Proof. From (7) we have

$$e_1 = \alpha_1 + \phi A\alpha_1 + \phi\Pi^\sharp\gamma + \gamma + \phi\vartheta_\flat(\alpha_1) + \phi A^*\gamma, e_2 = \alpha_2 + \phi A\alpha_2 + \phi\Pi^\sharp\delta + \delta + \phi\vartheta_\flat(\alpha_2) + \phi A^*\delta.$$

Since $\vartheta_\flat(\alpha_2)(A\alpha_1) - \vartheta_\flat(\alpha_1)(A\alpha_2) = \vartheta_\flat(\alpha_2)(\alpha_1)$ and $A^*\delta(\Pi^\sharp\gamma) - A^*\gamma(\Pi^\sharp\delta) = \Pi(\delta, \gamma)$ we get

$$
\begin{aligned}
< e_1, e_2 > &= \frac{1}{2}\{\delta(\alpha_1) - \gamma(\alpha_2) + \phi(2\delta(A\alpha_1) + 2\delta(\Pi^\sharp\gamma) + 2\vartheta_\flat(\alpha_2)(\alpha_1) - 2\gamma(A\alpha_2)) \\
&+ \phi^2(\vartheta_\flat(\alpha_2)(\alpha_1) + \vartheta_\flat(\alpha_2)(\Pi^\sharp\gamma) + \delta(A^2\alpha_1) + \gamma(\Pi^\sharp\delta) \\
&- \vartheta_\flat(\alpha_1)(\Pi^\sharp\delta) - \gamma(A^2\alpha_2))\}. \qquad (21)
\end{aligned}
$$

On the other hand, since Π is symmetric, using the first equation of (9) we get

$$\vartheta_\flat(\alpha_1)(\Pi^\sharp\delta) = \delta(A^2\alpha_1 - A\alpha_1 - \alpha_1). \qquad (22)$$

Then putting (22) in (21) and taking into account that ϕ is Golden ratio, we obtain (20). □

The above lemma shows that $E^{(1,0)}$ is not always isotropic. Next result gives necessary conditions for $E^{(1,0)}$ to be isotropic.

Corollary 4.2. *Let* $(M, \Phi, <, >)$ *be a generalized Golden manifold. If*

$$A = -\frac{5 + \sqrt{5}}{5 + 3\sqrt{5}} I \text{ or } A = -\frac{2 + \phi}{1 + 3\phi} I$$

and

$$\Pi(\gamma, \delta) = -\sqrt{5}\vartheta(\alpha_2, \alpha_1) \text{ or } \Pi(\gamma, \delta) = (1 - 2\phi)\vartheta(\alpha_2, \alpha_1),$$

then $E^{(1,0)}$ *is isotropic.*

Similar to Lemma (4.1), we have the following.

Lemma 4.3. *For* $e_1 = \alpha_1 + \gamma + (1 - \phi)\Phi(\alpha_1 + \gamma), e_2 = \alpha_2 + \delta + (1 - \phi)\Phi(\alpha_2 + \delta) \in F(E^{(0,1)})$ *we have*

$$
\begin{aligned}
< e_1, e_2 > &= \frac{1}{2}\{\gamma(\phi(3A + I)\alpha_2 + (-4A - 3I)\alpha_2) \\
&- \delta(\phi(3A + I)\alpha_1 + (-4A - 3I)\alpha_1)\} \\
&+ (2 - \phi)\Pi(\gamma, \delta) + (3\phi - 4)\vartheta(\alpha_1, \alpha_2). \qquad (23)
\end{aligned}
$$

Hence we get the following result.

Corollary 4.4. *Let* $(M, \Phi, <, >)$ *be a generalized Golden manifold. If* $A = \frac{5-\sqrt{5}}{3\sqrt{5}-5} I$ *or* $A = \frac{3-\phi}{3\phi-4} I$ *and* $\Pi(\gamma, \delta) = \frac{2+\phi}{\phi}\vartheta(\alpha_1, \alpha_2)$ *or* $\Pi(\gamma, \delta) = \frac{5+\sqrt{5}}{1+\sqrt{5}}\vartheta(\alpha_1, \alpha_2)$ *then* $E^{(0,1)}$ *is isotropic.*

Recall that a real, maximal isotropic sub-bundle $L \subset TM \oplus TM^*$ is called an almost Dirac structure. If L is involutive, then the almost Dirac structure is said to be integrable, or simply a Dirac structure. The equivalence of generalized complex structures with the existence of transversal Dirac structures is well known, see: [21]. Next result shows that it is not possible to define a generalized Golden structure in terms of Dirac structure.

Corollary 4.5. *Let $(M, \Phi, <,>)$ be a generalized Golden manifold. Then the eigenbundles corresponding to ϕ and $1-\phi$ are not maximal isotropic.*

Proof. If $E^{(1,0)}$ and $E^{(0,1)}$ are eigenbundles corresponding to ϕ and $1-\phi$, then $E^{(1,0)} \cap E^{(0,1)} = \{0\}$. Suppose that $E^{(1,0)}$ is maximal isotropic. Let $e_0 \in F(TM \oplus TM^*)$ be a smooth section such that $< e_0, e_1 >= 0$, for any $e_1 \in F(E^{(1,0)})$. Since $E^{(1,0)}$ is maximal isotropic by assumption, it follows that $\Phi e_0 = \phi e_0$. On the other hand, for any $e \in F(TM \oplus TM^*), e_2 \in F(E^{(0,1)})$ we have

$$e = \frac{1}{2\phi - 1}\{(\phi - 1)e_1 + \phi e_2\},\ e_1 = e + \phi\Phi e, e_2 = e + (1-\phi)\Phi e.$$

Hence, using (6) we derive

$$\begin{aligned}
0 =< \Phi e_0 - \phi e_0, e > &= < e_0, \Phi e > -\phi < e_0, e > \\
&= \frac{1}{2\phi - 1}\{< e_0, e_1 - e_2 > -\phi(\phi < e_0, e_2 > \\
&\quad +(\phi - 1) < e_0, e_1 >)\} \\
&= \frac{1}{2\phi - 1}\{- < e_0, e_2 > -\phi < e_0, e_2 > - < e_0, e_2 >\} \\
&= -\frac{2+\phi}{2\phi - 1} < e_0, e_2 > \\
&= -\phi < e_0, e_2 >
\end{aligned}$$

which is a contradiction due to $<,>$ is non-degenerate.
□

Next we will check the orthogonality of the sub-bundles $E^{(1,0)}$ and $E^{(0,1)}$.

Lemma 4.6. *Let $(M, \Phi, <,>)$ be a generalized Golden manifold. The eigenbundles corresponding to ϕ and $1-\phi$ are orthogonal to each other.*

Proof. For $X = \alpha_1 + \phi(A\alpha_1 + \Pi^\sharp\gamma) + \gamma + \phi(\vartheta_\flat(\alpha_1) + A^*\gamma) \in F(E^{(1,0)})$ and $Y = \alpha_2 + (1-\phi)(A\alpha_2 + \Pi^\sharp\delta) + \delta + (1-\phi)(\vartheta_\flat(\alpha_2) + A^*\delta) \in F(E^{(0,1)}$, by direct computations, we have

$$\begin{aligned}
&2 < X, Y >= \delta(\alpha_1) + \phi\delta(A\alpha_1) + \phi\Pi(\gamma, \delta) + (1-\phi)\vartheta(\alpha_2, \alpha_1) \\
&-\vartheta(\alpha_2, A\alpha_1) - \vartheta_\flat(\alpha_2)\Pi^\sharp(\gamma) + (1-\phi)\delta(A\alpha_1) - \delta(A^2\alpha_1) \\
&-\Pi(\gamma, A^*\delta) - \gamma(\alpha_2) - (1-\phi)\gamma(A\alpha_2) - (1-\phi)\Pi(\delta, \gamma) \\
&-\phi\vartheta(\alpha_1, \alpha_2) + \vartheta(\alpha_1, A\alpha_2) + \vartheta_\flat(\alpha_1)\Pi^\sharp(\delta) - \phi\gamma(A\alpha_2) \\
&+\gamma(A^2\alpha_2) + A^*\gamma(\Pi^\sharp\delta).
\end{aligned}$$

Since Π and ϑ are skew symmetric and $\vartheta(\alpha_2, A\alpha_1) - \vartheta(\alpha_1, A\alpha_2) = \vartheta(\alpha_2, \alpha_1)$, we get

$$\begin{aligned}
&2 < X, Y >= \delta(\alpha_1) + \delta(A\alpha_1) + \Pi(\gamma, \delta) \\
&-\vartheta_\flat(\alpha_2)\Pi^\sharp(\gamma) - \delta(A^2\alpha_1) - \Pi(\gamma, A^*\delta) - \gamma(\alpha_2) - \gamma(A\alpha_2) \\
&+\vartheta_\flat(\alpha_1)\Pi^\sharp(\delta) + \gamma(A^2\alpha_2) + \Pi(\delta, A^*\gamma).
\end{aligned}$$

Using the first equation of (9) we arrive at

$$2 < X, Y >= \Pi(\gamma, \delta) - \Pi(\gamma, A^*\delta) + \Pi(\delta, A^*\gamma).$$

Then $\Pi(\gamma, A^*\delta) - \Pi(\delta, A^*\gamma) = \Pi(\delta, \gamma)$ implies that

$$2 < X, Y >= \Pi(\gamma, \delta) + \Pi(\gamma, \delta).$$

Using again skew symmetric Π, we obtain

$$< X, Y >= 0.$$

□

In this section, finally we investigate the integrability conditions for $E^{(1,0)}$.

Theorem 4.7. *Let $(M, \Phi, <,>)$ be a generalized Golden manifold. Then $E^{(1,0)}$ is integrable if and only if the following conditions are satisfied*

$$\begin{aligned}(\phi+1)[\Pi^\sharp\gamma, \Pi^\sharp\delta] &= \phi A[\Pi^\sharp\gamma, \Pi^\sharp\delta] + \Pi^\sharp\{\gamma, \delta\}_\Pi + \phi\Pi^\sharp(L_{\Pi^\sharp\gamma}A^*\delta - L_{\Pi^\sharp\delta}A^*\gamma \\ &\quad -d\Pi(\gamma, A^*\delta)),\end{aligned} \tag{24}$$

$$\begin{aligned}\phi(\{\gamma, \delta\}_\Pi - \vartheta_\flat([\Pi^\sharp\gamma, \Pi^\sharp\delta]) &= (\phi + 1 + \phi A^*)(L_{\Pi^\sharp\gamma}A^*\delta \\ &- L_{\Pi^\sharp\delta}A^*\gamma - d\Pi(\gamma, A^*\delta))\end{aligned} \tag{25}$$

$$\begin{aligned}&\phi([\alpha_1, A\alpha_2] + [A\alpha_1, \alpha_2] + [A\alpha_1, A\alpha_2] + A[\alpha_1, \alpha_2] \\ &-A[A\alpha_1, A\alpha_2] = -N_A(\alpha_1, \alpha_2) + \Pi^\sharp(i_{\alpha_1\wedge\alpha_2}(d\vartheta) + \phi(L_{A\alpha_1}\vartheta_\flat(\alpha_2) \\ &-L_{A\alpha_2}\vartheta_\flat(\alpha_1) - d\vartheta(\alpha_2, A\alpha_1)),\end{aligned} \tag{26}$$

$$\begin{aligned}&(\phi I - A^*)(L_{\alpha_1}\vartheta_\flat(\alpha_2) - L_{\alpha_2}\vartheta_\flat(\alpha_1) + d\vartheta(\alpha_1, \alpha_2) + (\phi(I - A^*) + I) \\ &(L_{A\alpha_1}\vartheta_\flat(\alpha_2) - L_{A\alpha_2}\vartheta_\flat(\alpha_1) - d\vartheta(\alpha_2, A\alpha_1) = \vartheta_\flat((\phi - 1)[\alpha_1, \alpha_2] \\ &[\alpha_1, A\alpha_2] + [A\alpha_1, \alpha_2] + \phi[A\alpha_1, A\alpha_2]),\end{aligned} \tag{27}$$

$$\begin{aligned}\phi\{[\alpha_1, \Pi^\sharp\gamma] &+ [A\alpha_1, \Pi^\sharp\gamma] - A[A\alpha_1, \Pi^\sharp\gamma] - \Pi^\sharp(L_{\alpha_1}\gamma + L_{A\alpha_1}A^*\gamma \\ &\qquad -L_{\Pi^\sharp\gamma}\vartheta_\flat(\alpha_1) - d(\gamma(\alpha_1) + \frac{1}{2}\gamma(A\alpha_1))\} \\ &= -[A\alpha_1, \Pi^\sharp\gamma] + A[\alpha_1, \Pi^\sharp\gamma] + \Pi^\sharp(-L_{\alpha_1}\gamma + L_{\alpha_1}A^*\gamma \\ &\qquad +L_{A\alpha_1}\gamma - d(\gamma(A\alpha_1) - \frac{1}{2}\gamma(\alpha_1)))\end{aligned} \tag{28}$$

and

$$\begin{aligned}&L_{\alpha_1}\gamma + \phi L_{\alpha_1}A^*\gamma + \phi L_{A\alpha_1}\gamma + (\phi + 1)L_{A\alpha_1}A^*\gamma - (\phi + 1)L_{\Pi^\sharp\gamma}\vartheta_\flat(\alpha_1) \\ &-\frac{1}{2}d((2\phi + 1)\gamma(\alpha_1) + (3\phi + 1)\gamma(A\alpha_1)) = \vartheta_\flat([\alpha_1, \Pi^\sharp\gamma] + \phi[A\alpha_1, \Pi^\sharp\gamma]) \\ &+A^*((\phi - 1)L_{\alpha_1}\gamma + L_{\alpha_1}A^*\gamma + L_{A\alpha_1}\gamma + \phi L_{A\alpha_1}A^*\gamma - \phi L_{\phi^\sharp\gamma}\vartheta\flat(\alpha_1) \\ &-\frac{1}{2}d((2\phi - 1)\gamma(\alpha_1) + (\phi + 2)\gamma(AX)))\end{aligned} \tag{29}$$

for $\alpha_1, \alpha_2 \in F(TM)$ and $\gamma, \delta \in F(TM^*)$.

Proof. We will prove only the first two conditions. The rest can be obtain in a similar way. For $e_\gamma = \gamma + \phi\Phi\gamma$, we have $e_\gamma = \phi\Pi^\sharp\gamma + \gamma + \phi A^*\gamma$. Now using (2) and taking into account that ϕ is the Golden ratio, we obtain

$$\begin{aligned} [\![e_\gamma, e_\delta]\!] &= (\phi+1)[\Pi^\sharp, \Pi^\sharp] + \phi\{\gamma,\delta\}_\Pi + (\phi+1)(L_{\Pi^\sharp\gamma}A^*\delta - L_{\Pi^\sharp\delta}A^*\gamma \\ &\quad -d\Pi(\gamma, A^*\delta)). \end{aligned} \tag{30}$$

Now, for any e_γ and e_δ elements of $E^{(1,0)}$, the bracket $[\![e_\gamma, e_\delta]\!]$ is belong to $E^{(1,0)}$ if and only if

$$\Phi[\![e_\gamma, e_\delta]\!] = \phi[\![e_\gamma, e_\delta]\!].$$

This condition is equivalent to the condition

$$[\![e_\gamma, e_\delta]\!] = (\phi - 1)\Phi[\![e_\gamma, e_\delta]\!]. \tag{31}$$

Using (30), (3) and (4) in (31), and then taking the vector field parts and 1− form parts we obtain (24) and (25). □

5. RELATIONS WITH LIE GROUPOIDS

In this section we relate generalized Golden structures with Lie groupoids in terms of classical tensor fields ϑ, Π and A. We first recall some basic information for symplectic manifolds and Lie groupoids, for details on Lie groupoids and its integrability see:[11].

The relation between the condition $(G1)$ and 2− form ω follows from [10]. Since $\Pi^\sharp$ and $[,]_\Pi$ define a Lie algebroid structure on TM^*, we have the following result.

Theorem 5.1. *The integrability condition (G1) defines a symplectic groupoid (Ξ, ω) on the generalized Golden manifold M.*

Proof. If M is a generalized Golden manifold, we have (G1). Then assertion follows from Theorem 3.2 of [10]. □

We note that complex version of (G2) is equivalent to closed ω_A when ω is a symplectic form, [10]. But this is not valid for Golden generalized manifold.

Lemma 5.2. *Given a non-degenerate bivector Π and a symplectic form ω as inverse of Π on a manifold M, then Π satisfies (G2) and if and only if*

$$\begin{aligned} d\omega_A(\alpha_1,\alpha_2,\alpha_3) &= \frac{1}{2}\alpha_3\omega(\alpha_1,\alpha_2) - \alpha_1\omega(\alpha_2,\alpha_3) - 2\alpha_1\omega(A\alpha_2,\alpha_3) \\ &\quad +2\alpha_3\omega(\alpha_1, A\alpha_2) - \omega([\alpha_1,\alpha_2],\alpha_3) + 2\omega(A[\alpha_1,\alpha_2],\alpha_3) \end{aligned}$$

Proof. For vector fields α_1, α_2 and α_3, taking $\gamma = i_{\alpha_1}\omega$ and $\delta = i_{\alpha_2}\omega$ in (12), we have

$$\begin{aligned} A^*(i_{\alpha_1\wedge\alpha_2}(d\omega) \quad + \quad i_{[\alpha_1,\alpha_2]}\omega))(\alpha_3) &= \alpha_1\omega(\alpha_2, A\alpha_3) - \omega(\alpha_2, A[\alpha_1,\alpha_3]) \\ &\quad -\alpha_2\omega(\alpha_1, A\alpha_3) + \omega(\alpha_1, A[\alpha_2,\alpha_3]) - \frac{1}{2}\alpha_3\omega(\alpha_2,\alpha_1). \end{aligned}$$

Then closed ω and the formula of exterior derivative gives Lemma.
□

The following result presents the existence of the Hitchin pair in the generalized Golden manifold. see [10], for the definition of the Hitchin pair.

Theorem 5.3. *Let M be a generalized Golden manifold such that $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$. Let Π be an integrable Poisson structure on M, and (ϑ, ω) the symplectic groupoid over M. Then there is multiplicative (1,1)-tensors J on ϑ with the property that (J, ω) is a Hitchin pair.*

Proof. For $2-$ form ω, taking $\gamma = i_{\alpha_1}\omega$ and $\delta = i_{\alpha_2}\omega$ in $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$, we find

$$\omega(A\alpha_1, \alpha_2) = \omega(\alpha_1, A\alpha_2),$$

that is ω and A commute. Also considering the assumption $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$ and (17), we obtain

$$\Pi(\delta, A^*\gamma) = -\Pi(\gamma, A^*\delta).$$

Using this in (G2), it turns out the following form

$$A^*([\gamma,\delta]_\Pi) = L_{\Pi^\sharp\gamma}A^*\delta - L_{\Pi^\sharp\delta}A^*\gamma - d\Pi(A^*\gamma,\delta).$$

Then by following proof of Lemma 2.8 of [10], we conclude that ω_A is closed. As a result we show that there is a $1-1$ correspondence between Hitchin pairs (ω, A) and the second equation of(10) and (G2) with the condition $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$. Moreover, by taking $u = A^*$, $\rho = \Pi^\sharp$ and $\mathcal{L} = TM^*$, we have

$$< A^*\gamma, \Pi^\sharp\delta >= \Pi(\delta, A^*\gamma) = -\Pi(\gamma, A^*\delta) = - < A^*\delta, \Pi^\sharp\gamma >$$

which is the first condition for $IM-$ form. Furthermore, the assumption $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$ and (G2) imply the second condition of $IM-$ form. Thus we obtain that (G2) with $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$. Then it follows from Theorem 5.1 that there is a $1-1$ correspondence with closed multiplicative forms on ϑ. In a similar way, one can see that ω_J is multiplicative.
□

Lemma 5.4. *If Π is a non-degenerate bivector on a generalized Golden manifold M such that $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$, ω is the inverse 2-form (defined by $\omega_\flat = (\Pi^\sharp)^{-1}$) and Π satisfies the first equation of (9) then $\vartheta = \omega_A + \omega - A^*\omega$, where A is a bundle morphism on M.*

Proof. For $\alpha_1 \in \chi(M)$, applying $\omega_\flat$ to (9), we have

$$\omega_\flat(A^2\alpha_1) + \vartheta_\flat(\alpha_1) = \omega_\flat(A\alpha_1) + \omega_\flat(\alpha_1).$$

Now for $\alpha_2 \in \chi(M)$, the assumption $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$ ensures that ω and A commute, hence we obtain

$$\omega(A\alpha_1, A\alpha_2) + \vartheta(\alpha_1,\alpha_2) = \omega(A\alpha_1,\alpha_2) + \omega(\alpha_1,\alpha_2).$$

Thus we get

$$A^*\omega(\alpha_1,\alpha_2) + \vartheta(\alpha_1,\alpha_2) = \omega(A\alpha_1,\alpha_2) + \omega(\alpha_1,\alpha_2) \tag{5.1}$$

which gives the assertion. □

Theorem 5.5. *Let M be a generalized Golden manifold and (ϑ, ω, J) the induced symplectic groupoid over M with the induced multiplicative (1,1)-tensor such that $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$. Assume that (Π, J) satisfy (G1), (G2) with integrable Π. Then for a 2-form on M, the following assertions are equivalent.*

(i) (G5) is satisfied and $A^2 + \Pi^\sharp\vartheta_\flat = A + I$,
(ii) $\omega_J + \omega - J^*\omega = t^*\vartheta - s^*\vartheta$,

Proof. By assumption $\Pi(\gamma,\delta) = 2\Pi(\gamma, A^*\delta)$, thus it follows that ω and J commute and $\omega_J = 0$. Since ω and ω_J are closed, we get $i_{\alpha_1\wedge\alpha_2}(d(J^*\omega)) = -i_{N_J(\alpha_1,\alpha_2)}\omega$. Putting $\widetilde{\vartheta} = \omega_J + \omega - J^*\omega$, closed ω and ω_J give

$$i_{\alpha_1\wedge\alpha_2}(d\widetilde{\vartheta}) = i_{N_J(\alpha_1,\alpha_2)}\omega. \tag{5.2}$$

Since $d\phi = 0 \Leftrightarrow d\phi(\alpha_1,\alpha_2,\gamma) = 0$, we have

$$d\phi(\alpha_1,\alpha_2,\gamma) = 0 \Leftrightarrow d\widetilde{\vartheta}(\alpha_1,\alpha_2,\gamma) - d(t^*\vartheta)(\alpha_1,\alpha_2,\gamma) + d(s^*\vartheta)(\alpha_1,\alpha_2,\gamma) = 0.$$

On the other hand, we obtain

$$d(t^*\vartheta)(\alpha_1,\alpha_2,\gamma) = d\vartheta(dt(\alpha_1),dt(\alpha_2),dt(\gamma)). \tag{5.3}$$

If we take $dt = \rho$ in (5.3) for A, we get

$$d(t^*\vartheta)(\alpha_1,\alpha_2,\gamma) = d\vartheta(dt(\alpha_1),dt(\alpha_2),\rho(\gamma)). \tag{5.4}$$

Moreover from [8] we know that

$$Id_\vartheta = m \circ (t, Id_\vartheta). \tag{5.5}$$

Using (5.5) in (5.4), we get

$$d(t^*\vartheta)(\alpha_1,\alpha_2,\gamma) = d\vartheta(\alpha_1,\alpha_2,\rho(\gamma)).$$

On the other hand, by direct computation

$$d(s^*\vartheta)(\alpha_1,\alpha_2,\gamma) = d\vartheta(ds(\alpha_1),ds(\alpha_2),ds(\gamma)).$$

Since $\gamma \in kerds$, then $ds(\gamma) = 0$. Hence we have $d(s^*\vartheta) = 0$. Thus we get

$$d\widetilde{\vartheta}(\alpha_1,\alpha_2,\gamma) = d\vartheta(\alpha_1,\alpha_2,\rho(\gamma)). \tag{5.6}$$

Using (5.2) in (5.6), we derive

$$\omega(N_J(\alpha_1,\alpha_2),\gamma) = d\vartheta(\alpha_1,\alpha_2,\rho(\gamma)). \tag{5.7}$$

On the other hand, it is clear that $\phi = 0 \Leftrightarrow \widetilde{\vartheta} - t^*\vartheta + s^*\vartheta = 0$. Thus we obtain

$$\widetilde{\vartheta}(\alpha_1,\gamma) = \vartheta(\alpha_1,\rho(\gamma)).$$

Since $\widetilde{\vartheta} = \omega_J + \omega - J^*\omega$, we get

$$\omega(J\alpha_1,\gamma) + \omega(\alpha_1,\gamma) - \omega(J\alpha_1,J\gamma) = \vartheta(\alpha_1,\rho(\gamma)). \tag{5.8}$$

Since Poisson bivector Π is integrable, it defines a Lie algebroid whose anchor map is $\rho = \Pi^\sharp$. Let us use $\Pi^\sharp$ instead of ρ in (5.7) and (5.8). Then we get

$$\omega(N_J(\alpha_1,\alpha_2),\gamma) = d\vartheta(\alpha_1,\alpha_2,\Pi^\sharp(\gamma)), \tag{5.9}$$

$$\omega(J\alpha_1,\gamma) + \omega(\alpha_1,\gamma) - \omega(J\alpha_1,J\gamma) = \vartheta(\alpha_1,\Pi^\sharp(\gamma)).$$

Since $\omega(\gamma,\alpha_1) = \gamma(\alpha_1)$, $\omega_J(\gamma,\alpha_1) = \gamma(J\alpha_1)$, from (5.9) we have

$$\begin{aligned} -\gamma(N_J(\alpha_1,\alpha_2)) &= d\vartheta(\alpha_1,\alpha_2,\Pi^\sharp(\gamma)) \\ &= \Pi(\gamma, i_{\alpha_1\wedge\alpha_2}d\vartheta) \\ &= -\gamma(\Pi^\sharp(i_{\alpha_1\wedge\alpha_2}d\vartheta)). \end{aligned}$$

Hence we get

$$N_J(\alpha_1,\alpha_2) = \Pi^\sharp(i_{\alpha_1\wedge\alpha_2}d\vartheta). \tag{5.10}$$

On the other hand, from (5.8) we obtain

$$-\gamma(A\alpha_1) - \gamma(\alpha_1) + \gamma(A^2\alpha_1) = \Pi(\gamma, i_{\alpha_1}\vartheta) = -\gamma(\Pi^\sharp\vartheta_\sharp\alpha_1).$$

Thus we get

$$A^2 + \Pi^\sharp\vartheta_\sharp = A + I. \tag{5.11}$$

Then (i)⇔(ii) follows from (5.10) and (5.11). □

References

[1] M. Aldi, D. Grandini, Generalized almost product structures and generalized CRF-structures. J. Geom. Phys. 121, (2017), 93-107.
[2] M. A. Akyol, Remarks on metallic maps between metallic Riemannian manifolds and constancy of certain maps. Honam Math. J. 41 (2019), no. 2, 343–356.
[3] M. A. Blaga, A. Nannicini, Generalized quasi-statistical structures. Bull. Belg. Math. Soc. Simon Stevin 27(5), (2020), 731–754.
[4] M. A. Blaga, A. Nannicini, Generalized metallic structures. Rev. Un. Mat. Argentina 61(1), (2020), 73–86.
[5] M. A. Blaga, A. Nannicini, On the geometry of metallic pseudo-Riemannian structures. Riv. Math. Univ. Parma (N.S.) 11(1), (2020), 69-87.
[6] A. M. Blaga, C.E. Hretcanu, Golden warped product Riemannian manifolds, Lib. Math. (N.S.) 37(2), (2017), 39–50.
[7] M. Crasmareanu, Dirac structures from Lie integrability. Int. J. Geom. Methods Mod. Phys. Vol. 9, No. 4, (2012), 1220005 (7 pages).
[8] H. Bursztyn, M. Crainic, A. Weinstein, C. Zhu, Integration of twisted Dirac brackets, Duke Math. J., 123, 549-607, (2004).
[9] A. Cannas da Silva, Symplectic geometry, Handbook of differential geometry, Elsevier/North-Holland, Amsterdam, Vol. II, (2006), 79–188.
[10] M. Crainic, Generalized complex structures and Lie brackets, Bull. Braz. Math. Soc., New Series 42(4), 559-578 (2011).
[11] M. Crainic, R.L. Fernandes, Lectures on integrability of Lie brackets, Lectures on Poisson geometry, 1-107, Geom. Topol. Monogr., 17, Geom. Topol. Publ., (2011).
[12] M. Crasmareanu, C.E. Hretcanu, Golden differential geometry, Chaos Solitons Fractals 38(5), (2008), 1229-1238.
[13] F. E. Erdoğan , C. Yıldırım, On a study of the totally umbilical semi-invariant submanifolds of Golden Riemannian manifolds, Journal of Polytechnic, 21,(2018), 967-970.
[14] F. E. Erdoğan, Transversal lightlike submanifolds of metallic semi-Riemannian manifolds. Turkish J. Math. 42 (2018), no. 6, 3133-3148.
[15] F. Etayo, R. Santamaría, A.Upadhyay, On the geometry of almost Golden Riemannian manifolds, Mediterr. J. Math., 14:187, (2017).
[16] A. Gezer, N. Cengiz, A. Salimov, On integrability of Golden Riemannian structures, Turkish J. Math., 37(4), (2013), 693-703.
[17] B.Gherici, s-golden manifolds, Mediterr. J. Math., 16:56, (2019).
[18] R. R. Gomez, J. Talvacchia, On products of generalized geometries. Geom. Dedicata 175, (2015), 211-218.
[19] S. Gönül, I. K. Erken, A. Yazla, C. Murathan, A neutral relation between metallic structure and almost quadratic ϕ-structure. Turkish J. Math. 43 (2019), no. 1, 268-278.
[20] M. Gualtieri, Generalized complex geometry. Ann. of Math. (2) 174(1), (2011), 75-123.
[21] M. Gualtieri, Generalized complex geometry, Ph.D. thesis, Univ. Oxford, arXiv:ma.th.DG/0401221, (2003).
[22] C.E. Hretcanu, M. Crasmareanu, Applications of the Golden ratio on Riemannian manifolds, Turkish J. Math. 33(2), (2009), 179-191.
[23] C.E. Hretcanu, M. Crasmareanu, Metallic structures on Riemannian manifolds, Rev. Un. Mat. Argentina, 54(2), (2013), 15-27.
[24] N. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Math., 54, (2003), 281-308.
[25] C. Ida, P. Popescu, On almost complex Lie algebroids. Mediterr. J. Math. 13 (2016), no. 2, 803–824.
[26] A. Ionescu, Vanishing theorems on holomorphic Lie algebroids. Mediterr. J. Math. 15 (2018), no. 6, Art. 213, 14 pp.
[27] P. Koerber, Lectures on generalized complex geometry for physicists, arXiv:1006.1536v2.
[28] P. Libermann, C. M. Marle, Symplectic Geometry and Analytical Mechanics, Springer, (1987).
[29] K. Mackenzie, Lie groupoids and Lie algebroids in differential geometry, Cambridge, Cambridge University Press, London Mathematical Society Lecture Note Series, vol. 124, (1987).
[30] A. Manea, Some remarks on metallic Riemannian structures. An. Stiin. Univ. Al. I. Cuza Iasi. Mat. (N.S.) 65 (2019), no. 1, 37-46.
[31] I. Moerdijk , J. Mrcun, Introduction to Foliations and Lie Groupoids, Cambridge University Press, (2003).
[32] L. Ornea, R. Pantilie, On holomorphic maps and generalized complex geometry. J. Geom. Phys. 61(8), (2011), 1502-1515.
[33] C. Ozgur, N. Y. Ozgur, Classification of metallic shaped hypersurfaces in real space forms, Turkish J. Math., 39(5), (2015), 784–794.
[34] M. Ozkan, Prolongations of golden structures to tangent bundles, Differ. Geom. Dyn. Syst., 16 (2014), 227–238.
[35] V. Pirhadi, A. Razavi, Integrability of transitive Jacobi manifolds,Iran. J. Sci. Technol. Trans. A Sci. 43 (2019), no. 4, 1657–1664.
[36] Y. S. Poon, A. Wade, Generalized contact structures, J. Lond. Math. Soc. 83(2), (2011), 333-352.
[37] L. Popescu, Mechanical structures on Lie algebroids. Mediterr. J. Math. 15 (2018), no. 4, Art. 154, 24 pp.
[38] N. Poyraz, E. Yasar, Lightlike submanifolds of golden semi-Riemannian manifolds, J. Geom. Phys., 141, (2019), 92–104.
[39] N. O. Poyraz, E. Yasar, Lightlike hypersurfaces of a golden semi-Riemannian manifold, Mediterr. J. Math., 14:204 (2017).
[40] B. Sahin, M. A. Akyol, Golden maps between golden Riemannian manifolds and constancy of certain maps, Math. Commun. 19(2), (2014), 333–342.
[41] A. Wade, Local structure of generalized contact manifolds, Differential Geom. Appl. 30(1), (2012), 124-135.