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# Convexity and Inequalities of Some Generalized Numerical Radius Functions

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**Abstract.** In this paper, we prove that each of the following functions is convex on $\mathbb{R}$ :

$$f(t) = w_N(A^tXA^{1-t} \pm A^{1-t}XA^t),\ g(t) = w_N(A^tXA^{1-t}),\ \text{and}\ \ h(t) = w_N(A^tXA^t)$$

where $A > 0$, $X \in \mathbb{M}_n$ and $N(.)$ is a unitarily invariant norm on $\mathbb{M}_n$. Consequently, we answer positively the question concerning the convexity of the function $t \to w(A^tXA^t)$ proposed by in (2018). We provide some generalizations and extensions of $w_N(.)$ by using Kwong functions. More precisely, we prove the following

$$w_N(f(A)Xg(A) + g(A)Xf(A)) \leq w_N(AX + XA) \leq 2w_N(X)N(A),$$

which is a kind of generalization of Heinz inequality for the generalized numerical radius norm. Finally, some inequalities for the Schatten $p$-generalized numerical radius for partitioned $2 \times 2$ block matrices are established, which generalize the Hilbert-Schmidt numerical radius inequalities given by Aldalabih and Kittaneh in (2019).

## 1. Introduction and preliminaries

Based on some operator theory studies on Hilbert spaces, several generalizations for the concept of numerical radius have recently been introduced [1, 19, 21]. Abu-Omar and Kittaneh [1] introduced the so-called generalized numerical radius: If $\mathbb{M}_n$ denotes the space of all complex square matrices of size $n$, the generalized numerical radius for $A$, denoted by $w_N(A)$, is obtained via the supremum of the norm over the real parts of all rotations of $A$ i.e.

$$w_N(A) = \sup_{\theta \in \mathbb{R}} N(Re(e^{i\theta}A)).$$

Where $X = Re(X) + iIm(X)$ is the Cartesian decomposition of $X \in \mathbb{M}_n$, $Re(X) = \frac{X+X^*}{2}$ and $Im(X) = \frac{X-X^*}{2i}$, and $X^*$ denotes the adjoint of $X$. Simple computation shows that when $N$ is the usual operator norm inherited from the inner product on $H$ then $w_N(\cdot)$ coincides with the usual numerical radius norm $w(\cdot)$ which is defined as

$$w(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle| .$$

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It is well-known (see [? ]) that $w(A) = \sup_{\theta\in\mathbb{R}} \| Re(e^{i\theta}A) \|$ . We refer the reader to [1, 16, 19] for intermediate properties and inequalities of the norm $w_N(.)$.
In the present work, we restrict our attention to operator matrices $A \in \mathbb{M}_n$, where $\mathbb{M}_n$ denotes the space of all complex square matrices. We write $A > 0$ (respectively $A \geq 0$) for positive definite ( respectively for semi definite positive) matrix $A \in \mathbb{M}_n$. A norm $N(.)$ on $A \in \mathbb{M}_n$ is called unitarily invariant if $N(UAV) = N(A)$ for any $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$.

In this paper, we provide several inequalities for the matrix norm $w_N(.)$. Some results are obtained via convexity whenever $N$ is unitarily invariant norm. On the one hand, we follow up the work of Sababheh in [16] for the case of the numerical radius, to establish a new Young-type inequality for $w_N(.)$. Addressing to an open question proposed by the author in [16] about the convexity of the function $t \mapsto w(A^tXA^t)$, on $\mathbb{R}$ for $A > 0$, we provide a positive answer for the convexity of the aforementioned questioned and we prove that it is not only true for $w(.)$ but remains true for $w_N(.)$. On the other hand, motivated by the work of Bakherad [5] and Zamani [19, 21] we give some generalizations and extensions of Heinz inequality for the generalized numerical radius norm involving the so-called Kwong functions. Finally, by following the result given by Aldalabih and Kittaneh in [2] for the case of Hilbert-Schmidt numerical radius norm, we provide several Schatten $p$-generalized numerical radius inequalities. In this paper standards techniques are used to provide the results.

## 2. Convexity of some generalized numerical radius functions

Throughout this section, $N(.)$ denotes a unitarily invariant norm on $\mathbb{M}_n$. We start by proving the following basic essential lemma to demonstrate Theorem 2.1 which is the main result of this section. To provide the proof of this lemma we borrowed from [? ] the following two lemmas.

**Lemma 2.1.** *(Hölder inequality ) Let $A$, $B$ be two positive definite matrices in $\mathbb{M}_n$, $X \in \mathbb{M}_n$, $t \in [0, 1]$, and $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n$. Then*

$$N(A^tXB^t) \leq N^t(AXB)N^{1-t}(X).$$

**Lemma 2.2.** *( Heinz mean inequality) Let $A$, $B$ be two positive definite matrices in $\mathbb{M}_n$, $X \in \mathbb{M}_n$, $t \in [0, 1]$, and $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n$. Then*

$$2N(A^{\frac{1}{2}}XB^{\frac{1}{2}}) \leq N(A^tXB^{1-t} + A^{1-t}XB^t) \leq N(AX + XB).$$

**Lemma 2.3.** *Given $A > 0$, $X \in \mathbb{M}_n$, and $t \in [0, 1]$, then the following inequalities hold,*

$$w_N(A^tXA^t) \leq w_N^t(AXA)w_N^{1-t}(X), \tag{1}$$
$$2w_N(A^{\frac{1}{2}}XA^{\frac{1}{2}}) \leq w_N(A^tXA^{1-t} + A^{1-t}XA^t) \leq w_N(AX + XA). \tag{2}$$

*Proof.* For $t \in [0, 1]$, $A^t$ is a Hermitian matrix so for any $\theta \in \mathbb{R}$, we get

$$Re(e^{i\theta}A^tXA^t) = \frac{1}{2}\left(A^te^{i\theta}XA^t + A^te^{-i\theta}X^*A^t\right) = A^t\frac{e^{i\theta}X + e^{-i\theta}X^*}{2}A^t = A^tRe(e^{i\theta}X)A^t.$$

Now by using Hölder inequality-Lemma, we obtain

$$N\Big(Re(e^{i\theta}A^tXA^t)\Big) = N(A^tRe(e^{i\theta}X)A^t) \leq N^t\Big(ARe(e^{i\theta}X)A\Big)N^{1-t}\Big(Re(e^{i\theta}X)\Big).$$

Taking the supremum over all $\theta \in \mathbb{R}$, we obtain (1).

To prove the second inequality we begin by noting that, for any $\theta \in \mathbb{R}$,

$$
\begin{aligned}
&Re\Big(e^{i\theta}(A^tXA^{1-t} + A^{1-t}XA^t)\Big)\\
&= \frac{1}{2}\Big(e^{i\theta}A^tXA^{1-t} + e^{i\theta}A^{1-t}XA^t + e^{-i\theta}A^{1-t}X^*A^t + e^{-i\theta}A^tX^*A^{1-t}\Big)\\
&= A^t\Big(\frac{e^{i\theta}X + e^{-i\theta}X^*}{2}\Big)A^{1-t} + A^{1-t}\Big(\frac{e^{i\theta}X + e^{-i\theta}X^*}{2}\Big)A^t\\
&= A^tRe(e^{i\theta}X)A^{1-t} + A^{1-t}Re(e^{i\theta}X)A^t.
\end{aligned}
$$

Then by using the well known Heinz mean inequality-Lemma and for $A = B$, we obtain

$$2N(A^{\frac{1}{2}}XA^{\frac{1}{2}}) \le N(A^tXA^{1-t} + A^{1-t}XA^t) \le N(AX + XA). \qquad (*)$$

Therefore

$$
\begin{aligned}
2N\Big(Re(e^{i\theta}A^{\frac{1}{2}}XA^{\frac{1}{2}})\Big) &= 2N\Big(A^{\frac{1}{2}}Re(e^{i\theta}X)A^{\frac{1}{2}}\Big)\\
&\le N\Big(A^tRe(e^{i\theta}X)A^{1-t} + A^{1-t}Re(e^{i\theta}X)A^t\Big) \text{ (by the left inequality of } (*))\\
&= N\left(A^t\frac{e^{i\theta}X + e^{-i\theta}X^*}{2}A^{1-t} + A^{1-t}\frac{e^{i\theta}X + e^{-i\theta}X^*}{2}A^t\right)\\
&= \frac{1}{2}N\left(e^{i\theta}A^tXA^{1-t} + e^{-i\theta}A^tX^*A^{1-t} + e^{i\theta}A^{1-t}XA^t + e^{-i\theta}A^{1-t}X^*A^t\right)\\
&= N\Big(Re(e^{i\theta}(A^tXA^{1-t} + A^{1-t}XA^t))\Big)\\
&\le N(ARe(e^{i\theta}X) + Re(e^{i\theta}X)A) \text{ (by the right inequality of } (*))\\
&= \frac{1}{2}N\left(e^{i\theta}AX + e^{-i\theta}AX^* + e^{i\theta}XA + e^{-i\theta}X^*A\right)\\
&= N\left(Re(e^{i\theta}(AX + XA))\right).
\end{aligned}
$$

Taking the supremum over all $\theta \in \mathbb{R}$, we obtain (2). □

In the following main Theorem, we generalize the result given by Sababheh [16] about the convexity of the functions $w(A^tXA^{1-t} + A^{1-t}XA^t)$ and $w(A^tXA^{1-t})$, and we answer positively the question concerning the convexity of the function $w(A^tXA^t)$.

**Theorem 2.4.** *Let $A > 0$ and $X \in \mathbb{M}_n$, then each of the following functions is convex on $\mathbb{R}$ :*

$$f(t) = w_N(A^tXA^{1-t} \pm A^{1-t}XA^t),\ g(t) = w_N(A^tXA^{1-t}),\ \textit{and}\ \ h(t) = w_N(A^tXA^t).$$

*Proof.* Replace $A$ by $A^2$ and take $t = 1$ in (2), we get

$$w_N(AXA) \le \frac{1}{2}w_N(A^2X + XA^2). \qquad (**)$$

To obtain the convexity of $f(.)$, let $t, s \in \mathbb{R}$, we have

$$\begin{aligned}
f\left(\frac{t+s}{2}\right) &= w_N(A^{\frac{t+s}{2}}XA^{1-\frac{t+s}{2}} \pm A^{1-\frac{t+s}{2}}XA^{\frac{t+s}{2}}) \\
&= w_N\Big(A^{\frac{t-s}{2}}(A^sXA^{1-t} \pm A^{1-t}XA^s)A^{\frac{t-s}{2}}\Big) \\
&\le \frac{1}{2}w_N\Big(A^{t-s}(A^sXA^{1-t} \pm A^{1-t}XA^s) + (A^sXA^{1-t} \pm A^{1-t}XA^s)A^{t-s}\Big) \quad (\text{ by } (**)) \\
&= \frac{1}{2}w_N\left(A^tXA^{1-t} \pm A^{1-s}XA^s + A^sXA^{1-s} \pm A^{1-t}XA^t\right) \\
&\le \frac{1}{2}w_N\Big(A^tXA^{1-t} \pm A^{1-t}XA^t\Big) + \frac{1}{2}w_N\Big(A^sXA^{1-s} \pm A^{1-s}XA^s\Big) \quad (\text{by the triangle inequality}) \\
&= \frac{1}{2}f(t) + \frac{1}{2}f(s).
\end{aligned}$$

The proof of the convexity of $g(t) = w_N(A^tXA^{1-t})$ on $\mathbb{R}$ follows in the same manner as the function $f(.)$. To prove the convexity of $h(t) = w_N(A^tXA^t)$, we first replace $A$ by $A^2$, $B$ by $B^2$ and $t = 1$ in the well known Heinz mean inequality-Lemma, we obtain

$$2N(AXB) \le N(A^2X + XB^2). \tag{3}$$

Now for $t, s \in \mathbb{R}$, the matrix $A^{\frac{t+s}{2}}$ is Hermitian and so that for any $\theta \in \mathbb{R}$,

$$\begin{aligned}
Re(e^{i\theta}A^{\frac{t+s}{2}}XA^{\frac{t+s}{2}}) &= \frac{1}{2}\left(A^{\frac{t+s}{2}}e^{i\theta}XA^{\frac{t+s}{2}} + A^{\frac{t+s}{2}}e^{-i\theta}X^*A^{\frac{t+s}{2}}\right) \\
&= \frac{1}{2}\left(A^{\frac{t+s}{2}}(e^{i\theta}X + e^{-i\theta}X^*)A^{\frac{t+s}{2}}\right) \\
&= A^{\frac{t+s}{2}}Re(e^{i\theta}X)A^{\frac{t+s}{2}} \\
&= A^{\frac{t-s}{2}}\Big(A^sRe(e^{i\theta}X)A^t\Big)A^{\frac{-t+s}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
h\left(\frac{t+s}{2}\right) &= w_N(A^{\frac{t+s}{2}}XA^{\frac{t+s}{2}}) \\
&= \sup_{\theta\in\mathbb{R}} N\left(Re(e^{i\theta}A^{\frac{t+s}{2}}XA^{\frac{t+s}{2}})\right) \\
&= \sup_{\theta\in\mathbb{R}} N(A^{\frac{t-s}{2}}(A^sRe(e^{i\theta}X)A^t)A^{\frac{-t+s}{2}}) \\
&\le \frac{1}{2}\sup_{\theta\in\mathbb{R}} N\Big(A^{t-s}(A^sRe(e^{i\theta}X)A^t) + (A^sRe(e^{i\theta}X)A^t)A^{-t+s}\Big) \quad (\text{by } (3)) \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} N\left(A^tRe(e^{i\theta}X)A^t + A^sRe(e^{i\theta}X)A^s\right) \\
&= \frac{1}{4}\sup_{\theta\in\mathbb{R}} N\left(A^te^{i\theta}XA^t + A^te^{-i\theta}X^*A^t + A^se^{i\theta}XA^s + A^se^{-i\theta}X^*A^s\right) \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} N\left(Re(e^{i\theta}A^tXA^t) + Re(e^{i\theta}A^sXA^s)\right) \\
&\le \frac{1}{2}\sup_{\theta\in\mathbb{R}} N\Big(Re(e^{i\theta}A^tXA^t)\Big) + \frac{1}{2}\sup_{\theta\in\mathbb{R}} N\Big(Re(e^{i\theta}A^sXA^s)\Big) \quad (\text{by the triangle inequality}) \\
&= \frac{1}{2}w_N(A^tXA^t) + \frac{1}{2}w_N(A^sXA^s) = \frac{1}{2}h(t) + \frac{1}{2}h(s).
\end{aligned}$$

Hence, $h(.)$ is a convex function on $\mathbb{R}$ as required. □

Note that when $N(.)$ is the usual operator norm $\|.\|$ and by using the function $h(.)$ of this theorem, the question of Sababheh [16] concerning the convexity of the function $t \mapsto w(A^tXA^t)$ on $\mathbb{R}$ is answered positively .
Under the same conditions given in Theorem 2.2, we can prove the convexity of the following functions: $w_N(A^{-t}XA^{1+t}), w_N(A^tXA^{1-t}) + w_N(A^{-t}XA^{1+t})$ and $w_N(A^tXA^{1-t} + A^{-t}XA^{1+t})$.
Motivated by the work of Sababheh [16] for the numerical radius, and by the convexity of the function $w_N(A^tXA^{1-t} + A^tXA^{1-t})$ and $w_N(A^tXA^{1-t} - A^{1-t}XA^t)$ as given in Theorem 2.2, and by the Theorem 2.5 in [19] we obtain the following reversed inequalities for the generalized numerical radius norm.

**Corollary 2.5.** *Let $A > 0$ and $X \in \mathbb{M}_n$. Then,*

$$\begin{cases} w_N(A^tXA^{1-t} + A^{1-t}XA^t) \leq w_N(AX + XA) \leq 2w_N(X)N(A), & t \in [0,1] \\ w_N(A^tXA^{1-t} + A^{1-t}XA^t) \geq w_N(AX + XA) & t \notin [0,1] \end{cases}$$

Also we have the following Young-type inequality based on the convexity of the function $t \mapsto w_N(A^tXA^{1-t})$.

**Corollary 2.6.** *Let $A > 0$ and $X \in \mathbb{M}_n$. Then*

$$\begin{cases} w_N(A^tXA^{1-t}) \leq t.w_N(AX) + (1-t).w_N(XA), & t \in [0,1] \\ w_N(A^tXA^{1-t}) \geq t.w_N(AX) + (1-t).w_N(XA) & t \notin [0,1] \end{cases}$$

Motivated by the work given by Bakherad in [5] and Zamani [19], we provide some generalizations and extensions by using Kwong functions of $w_N(.)$. More precisely, in the following theorem, we prove the coming result,

$$w_N(f(A)Xg(A) + g(A)Xf(A)) \leq w_N(AX + XA) \leq 2w_N(X)N(A),$$

which is a kind of generalization of Heinz inequality for the generalized numerical radius norm. Let us first recall the definition of Kwong function: A real continuous function $f$ defined on an interval $(a, b)$ with $a \geq 0$ is called a Kwong function if the matrix

$$K_f = \left(\frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i + \lambda_j}\right)_{1\leq i,j\leq n}$$

is positive semi definite for any distinct real values $\lambda_1, \lambda_2, ..., \lambda_n$ in $(a, b)$.
Before showing the following theorem, we need the coming result provided by Najafi in [15]: For two continuous functions $f, g$ where $\frac{f(x)}{g(x)}$ is a Kwong function and $f(x).g(x) \leq x$ the following inequality holds

$$N(f(A)Xg(A) + g(A)Xf(A)) \leq N(AX + XA) \tag{4}$$

where $X \in \mathbb{M}_n$ and $A \in \mathbb{M}_n$ be a positive definite matrix.

**Theorem 2.7.** *Let $A \in \mathbb{M}_n$ be a positive definite matrix, $X \in \mathbb{M}_n$, and $f, g$ be two positive continuous functions on $(0,\infty)$ such that $f(x).g(x) \leq x$, for all $x \in (0,\infty)$, and $\frac{f(x)}{g(x)}$ is a Kwong function, then*

$$w_N\left(f(A)Xg(A) + g(A)Xf(A)\right) \leq w_N(AX + XA) \leq 2w_N(X)N(A).$$

*Proof.* We have $w_N(f(A)Xg(A) + g(A)Xf(A)) = \sup_{\theta\in\mathbb{R}} N\Big(Re\Big(e^{i\theta}(f(A)Xg(A) + g(A)Xf(A))\Big)\Big)$

$$= \sup_{\theta\in\mathbb{R}} N\left(\frac{e^{i\theta}f(A)Xg(A) + e^{i\theta}g(A)Xf(A) + e^{-i\theta}g(A)X^*f(A) + e^{-i\theta}f(A)X^*g(A)}{2}\right)$$
$$= \sup_{\theta\in\mathbb{R}} N\Big(f(A)Re(e^{i\theta}X)g(A) + g(A)Re(e^{i\theta}X)f(A)\Big)$$
$$\leq \sup_{\theta\in\mathbb{R}} N\left(ARe(e^{i\theta}X) + Re(e^{i\theta}X)A\right) \quad (using \;\; (4))$$

$$= \sup_{\theta\in\mathbb{R}} N\left(A\left(\frac{e^{i\theta}X + e^{-i\theta}X^*}{2}\right) + \left(\frac{e^{i\theta}X + e^{-i\theta}X^*}{2}\right)A\right)$$
$$= \sup_{\theta\in\mathbb{R}} N(Re(e^{i\theta}AX) + Re(e^{i\theta}XA))$$
$$= \sup_{\theta\in\mathbb{R}} N\left(Re\left(e^{i\theta}(AX + XA)\right)\right)$$
$$= w_N(AX + XA) \le 2w_N(X)N(A) \text{ by Theorem 2.5 in [19].} \quad \square$$

Notice that by choosing $f(x) = x^t$, $g(x) = x^{1-t}$ where $0 \le t \le 1, x > 0$, we have $\frac{f(x)}{g(x)}$ is a Kwong function and $f(x).g(x) = x$, for all $x \in (0,\infty)$. Therefore, the following Heinz inequality for the generalized numerical radius norm,

$$w_N(A^tXA^{1-t} + A^{1-t}XA^t) \le w_N(AX + XA)$$

is obtained.

By using the following two Lemmas (see [24] for proofs), we can find more inequalities for the generalized numerical radius.

**Lemma 2.8.** *Let $A, B, X \in \mathbb{M}_n$ such that $A, B$ are positive definite, and $f, g$ are two positive continuous functions on $(0,\infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is a Kwong function. Then,*

$$N\left(A^{\frac{1}{2}}\left(f(A)Xg(B) + g(A)Xf(B)\right)B^{\frac{1}{2}}\right) \le \frac{k}{2}N(A^2X + 2AXB + XB^2) \tag{5}$$

*holds for $k = \max_{\lambda\in\sigma(A)\cup\sigma(B)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$ where $\sigma(A)$ represents the spectrum of $A$.*

**Lemma 2.9.** *Let $A, B, X \in \mathbb{M}_n$ such that $A, B$ are positive definite. And for any two positive continuous functions on $(0,\infty)$ with $h(x) = \frac{f(x)}{g(x)}$ is kwong, then*

$$N\left(f(A)Xg(B) + g(A)Xf(B)\right) \le \frac{k'}{2}N(A^2X + 2AXB + XB^2) \tag{6}$$

*holds for $k' = \max_{\lambda\in\sigma(A)\cup\sigma(B)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda^2}\right\}$.*

**Theorem 2.10.** *Let $A \in \mathbb{M}_n$ be a positive definite matrix, and $f, g$ be two positive continuous functions on $(0,\infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is a Kwong function. Then,*

$$w_N(A^{\frac{1}{2}}H_{f,g}(A)A^{\frac{1}{2}}) \le \frac{k}{2}w_N(A^2X + 2AXA + XA^2),$$

*where $H_{f,g}(A) = f(A)Xg(A) + g(A)Xf(A)$ and $k = \max_{\lambda\in\sigma(A)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$.*

*Proof.* By using the inequality (3) and applying the same technique as the proof in Theorem 2.5 the required result is obtained. □

**Theorem 2.11.** *Let $A \in \mathbb{M}_n$ be a positive definite matrix, and $f, g$ be two positive real continuous functions on $(0,\infty)$ such that $h(x) = \frac{f(x)}{g(x)}$ is a Kwong function. Then*

$$w_N(f(A)Xg(A) + g(A)Xf(A)) \le \frac{k'}{2}w_N(A^2X + 2AXB + XB^2)$$

*holds for $k' = \max_{\lambda\in\sigma(A)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda^2}\right\}$.*

*Proof.* By using the inequality (4) and applying a similar proof as in of Theorem 2.5, the required result is obtained. □

## 3. Inequalities for $w_p(.)$

In this section, $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ denote the singular values of a matrix $A \in \mathbb{M}_n$ i.e. the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$. For $1 \leq p < \infty$, the Schatten $p$-norm of $A$ is denoted and defined by

$$\| A \|_p = \Big( \sum_{j=1}^{n} s_j^p(A) \Big)^{\frac{1}{p}}.$$

Abu-Omar and Kittaneh in [1] give some properties for the Hilbert-Schmidt numerical radius norm $w_2(.)$ as a concrete example of $w_N(.)$ when $N(.) = \|.\|_2$. The aim of this section is to provide some inequalities for the Schatten $p$-generalized numerical radius $w_p(.) = w_N(.)$ with $N(.) = \|.\|_p$. In the following theorem we provide an upper bound for the Schatten p-generalized numerical radius $w_p(.)$.

**Theorem 3.1.** *For $2 \leq p < \infty$, and $A, B, X, Y \in \mathbb{M}_n$, we have the following inequality*

$$w_p(AXB \pm BYA) \leq 2^{\frac{5}{2}-\frac{2}{p}} \max(\| XB \|_p, \| BY \|_p)\bigg( w_p^p(A) - n(2^{1-p} - 2^{2-\frac{3p}{2}}) \,|\, s_n(A) \,|^p - \frac{1}{2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big| \bigg)^{\frac{1}{p}}.$$

Before we provide the proof of this theorem, we need the following two lemmas. The first one is borrowed from ([4] Theorem 4.1 ).

**Lemma 3.2.** *For $2 \leq p < \infty$ and $A, B \in \mathbb{M}_n$, we have*

$$\| A + B \|_p^p + \| A - B \|_p^p \geq 2^{2-\frac{p}{2}} \Big( \| A \|_p^p + \| B \|_p^p \Big) + n 2^{2-\frac{p}{2}} c_p\Big(s_n(A), s_n(B)\Big), \tag{7}$$

*where $c_p(s, t) = (2^{\frac{p}{2}} - 2) \min(| s |^p, | t |^p)$ and $s, t \in \mathbb{C}$.*

**Lemma 3.3.** *For $2 \leq p < \infty$ and $A \in \mathbb{M}_n$, we have*

$$\| A \|_p^p + \| A^* \|_p^p \leq 2^{\frac{3p}{2}-1} w_p^p(A) - n(2^{\frac{p}{2}} - 2) \,|\, s_n(A) \,|^p - 2^{\frac{3p}{2}-2} \Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|. \tag{8}$$

*Proof.* By using the fact that $w_p(A) = \sup_{\theta \in \mathbb{R}} \| Re(e^{i\theta} A) \|_p = \sup_{\theta \in \mathbb{R}} \| Im(e^{i\theta} A) \|_p$,
then
$w_p(A) \geq \max\Big( \| Re(A) \|_p, \| Im(A) \|_p \Big)$. So

$$\begin{aligned} 2^p w_p^p(A) \geq \max\Big( \| A + A^* \|_p^p, \| A - A^* \|_p^p \Big) &= \frac{1}{2}\Big( \| A + A^* \|_p^p + \| A - A^* \|_p^p \Big) \\ &+ \frac{1}{2}\Big| \| A + A^* \|_p^p - \| A - A^* \|_p^p \Big| \\ &\geq 2^{1-\frac{p}{2}}\Big( \| A \|_p^p + \| A^* \|_p^p \Big) + n 2^{1-\frac{p}{2}} c_p(s_n(A), s_n(A^*)) \\ &+ \frac{1}{2}\Big| \| A + A^* \|_p^p - \| A - A^* \|_p^p \Big|. \text{ (by (6) for } B = A^* \text{)} \end{aligned}$$

Hence,

$$\begin{aligned} \| A \|_p^p + \| A^* \|_p^p \leq\ & 2^{\frac{3p}{2}-1} w_p^p(A) - n c_p(s_n(A), s_n(A^*)) \\ & - 2^{\frac{3p}{2}-2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|. \end{aligned}$$

But, $c_p(s_n(A), s_n(A^*)) = (2^{\frac{p}{2}} - 2)\min(s_n(A), s_n(A^*))$. Then,

$$\| A \|_p^p + \| A^* \|_p^p \leq 2^{\frac{3p}{2}-1} w_p^p(A) - n(2^{\frac{p}{2}} - 2)\min(s_n(A), s_n(A^*)) - 2^{\frac{3p}{2}-2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|.$$

And the required inequality holds by letting $s_n(A) = s_n(A^*)$. □

*Proof.* For the proof of Theorem 3.1, we distinct two cases: First case we let $X, Y, A \in \mathbb{M}_n$ such that $\| X \|_p \leq 1, \| Y \|_p \leq 1$, and $w_p(A) \leq 1$, then

$$\begin{aligned} w_p(AX \pm YA) &\leq \| AX \pm YA \|_p \\ &\leq \| AX \|_p + \| YA \|_p \quad \text{(by triangle inequality)} \\ &\leq \| A \|_p + \| A^* \|_p \quad (\| X \|_p \leq 1, \| Y \|_p \leq 1, \| A \|_p = \| A^* \|_p) \\ &\leq 2^{1-\frac{1}{p}}\Big( \| A \|_p^p + \| A^* \|_p^p \Big)^{\frac{1}{p}}. \quad \text{(by concavity of } t^{\frac{1}{p}}) \end{aligned}$$

Then by using the inequality (7) we get

$$\begin{aligned} w_p(AX \pm YA) &\leq 2^{1-\frac{1}{p}}\Big(2^{\frac{3p}{2}-1} w_p^p(A) - n(2^{\frac{p}{2}} - 2) \mid s_n(A) \mid^p - 2^{\frac{3p}{2}-2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|\Big)^{\frac{1}{p}} \\ &= 2^{1-\frac{1}{p}} 2^{\frac{3}{2}-\frac{1}{p}}\Big(w_p^p(A) - n2^{1-\frac{3p}{2}}(2^{\frac{p}{2}} - 2) \mid s_n(A) \mid^p - \frac{1}{2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|\Big)^{\frac{1}{p}} \\ &\leq 2^{\frac{5}{2}-\frac{2}{p}}\Big(w_p^p(A) - 2^{1-\frac{3p}{2}} n c_p(s_n(A), s_n(A^*)) - \frac{1}{2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|\Big)^{\frac{1}{p}} \\ &= 2^{\frac{5}{2}-\frac{2}{p}}\Big(1 - 2^{1-\frac{3p}{2}} n c_p(s_n(A), s_n(A^*)) - \frac{1}{2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|\Big)^{\frac{1}{p}}. \quad (w_p(A) \leq 1) \end{aligned}$$

For the general case we replace, $X$ by $\frac{X}{\max\left(\|X\|_p, \|Y\|_p\right)}$, $Y$ by $\frac{Y}{\max\left(\|X\|_p, \|Y\|_p\right)}$ and $A$ by $\frac{A}{w_p(A)}$ respectively, we get

$$w_p\Big(\frac{A}{w_p(A)} \frac{X}{\max\big( \| X \|_p, \| Y \|_p \big)} \pm \frac{Y}{\max\big( \| X \|_p, \| Y \|_p \big)} \frac{A}{w_p(A)}\Big)$$

$$\leq 2^{\frac{5}{2}-\frac{2}{p}}\Big(1 - 2^{1-\frac{3p}{2}} n c_p\Big(s_n\Big(\frac{A}{w_p(A)}\Big), s_n\Big(\frac{A^*}{w_p(A)}\Big)\Big) - \frac{1}{2}\Big| \| Re\Big(\frac{A}{w_p(A)}\Big) \|_p^p - \| Im\Big(\frac{A}{w_p(A)}\Big) \|_p^p \Big|\Big)^{\frac{1}{p}}.$$

But $c_p(s_n(A), s_n(A^*)) = (2^{\frac{p}{2}} - 2)\min\big( \mid s_n(A) \mid^p, \mid s_n(A^*) \mid^p \big) = (2^{\frac{p}{2}} - 2) \mid s_n(A) \mid^p$ then

$$w_p\,(AX \pm YA) \leq 2^{\frac{5}{2}-\frac{2}{p}} w_p(A) \max(\| X \|_p, \| Y \|_p) \times$$

$$\Big(1 - n2^{1-\frac{3p}{2}}(2^{\frac{p}{2}} - 2) \mid s_n(\frac{A}{w_p(A)}) \mid^p - \frac{1}{2}\Big| \| Re\Big(\frac{A}{w_p(A)}\Big) \|_p^p - \| Im\Big(\frac{A}{w_p(A)}\Big) \|_p^p \Big|\Big)^{\frac{1}{p}}.$$

Therefore, $w_p(AX \pm YA) \leq 2^{\frac{5}{2}-\frac{2}{p}} \max\big( \| X \|_p, \| Y \|_p \big) \times$

$$\Big(w_p^p(A) - 2^{1-\frac{3p}{2}} n(2^{\frac{p}{2}} - 2) \mid s_n(A) \mid^p - \frac{1}{2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|\Big)^{\frac{1}{p}}.$$

Now by replacing $X$ by $XB$ and $Y$ by $BY$ in the last inequality, we find,

$$w_p(AXB \pm BYA) \leq 2^{\frac{5}{2}-\frac{2}{p}} \max\big( \| XB \|_p, \| BY \|_p \big)\Big(w_p^p(A) - n(2^{1-p} - 2^{2-\frac{3p}{2}}) \mid s_n(A) \mid^p - \frac{1}{2}\Big| \| Re(A) \|_p^p - \| Im(A) \|_p^p \Big|\Big)^{\frac{1}{p}}.$$

□

An application of Theorem 3.1 is the following corollary, which can be seen as a kind of generalization of the inequalities given by Hirzallah and Kittaneh in [13].

**Corollary 3.4.** *Let $A, B \in \mathbb{M}_n$. Then*

$$w_2(AB \pm BA) \leq 2\sqrt{2} \parallel B \parallel_2 \sqrt{w_2^2(A) - \frac{1}{2}\Big| \parallel Re(A) \parallel_2^2 - \parallel Im(A) \parallel_2^2 \Big|},$$

*and*

$$w_2(A^2) \leq \sqrt{2} \parallel A \parallel_2 \sqrt{w_2^2(A) - \frac{1}{2}\Big| \parallel Re(A) \parallel_2^2 - \parallel Im(A) \parallel_2^2 \Big|}.$$

The following theorem provides an estimation of the Schatten $p$-generalized numerical radius $w_p(.)$ of $2 \times 2$ block matrix entries. To start, we recall the below lemma [7].

**Lemma 3.5.** *Let $T = [T_{i,j}]$, $T_{i,j} \in \mathbb{M}_n$ for $1 \leq i, j \leq 2$, be a block matrix. Then:*
*For $p \in [2, \infty[$,*

$$\parallel T \parallel_p \leq \frac{1}{2^{\frac{2}{p}-1}} \Big( \sum_{i,j} \parallel T_{i,j} \parallel_p^p \Big)^{\frac{1}{p}}. \tag{9}$$

*For $p \in [1, 2]$,*

$$\parallel T \parallel_p \leq \Big( \sum_{i,j} \parallel T_{i,j} \parallel_p^p \Big)^{\frac{1}{p}}. \tag{10}$$

**Theorem 3.6.** *Let $A, B, C, D \in \mathbb{M}_n$ and $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then:*
*For $p \in [2, \infty[$,*

$$w_p(T) \leq \frac{1}{2^{\frac{2}{p}-1}} \Big( w_p^p(A) + w_p^p(D) + \frac{1}{2^{p-1}} \Big( \parallel B \parallel_p + \parallel C \parallel_p \Big)^p \Big)^{\frac{1}{p}}. \tag{11}$$

*For $p \in [1, 2]$,*

$$w_p(T) \leq \Big( w_p^p(A) + w_p^p(D) + \frac{1}{2^{p-1}} \Big( \parallel B \parallel_p + \parallel C \parallel_p \Big)^p \Big)^{\frac{1}{p}}. \tag{12}$$

*Proof.* Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{M}_{2n}$, then for $\theta \in \mathbb{R}$ we have

$$Re(e^{i\theta}T) = Re\left(e^{i\theta}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} e^{i\theta}A & e^{i\theta}B \\ e^{i\theta}C & e^{i\theta}D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} e^{-i\theta}A^* & e^{-i\theta}C^* \\ e^{-i\theta}B^* & e^{-i\theta}D^* \end{pmatrix}$$
$$= \begin{pmatrix} Re(e^{i\theta}A) & \frac{1}{2}(e^{i\theta}B + e^{-i\theta}C^*) \\ \frac{1}{2}(e^{i\theta}B + e^{-i\theta}C^*)^* & Re(e^{i\theta}D) \end{pmatrix} = \begin{pmatrix} Re(e^{i\theta}A) & F \\ F^* & Re(e^{i\theta}D) \end{pmatrix},$$

where $F = \frac{1}{2}(e^{i\theta}B + e^{-i\theta}C^*)$ and as,

$$\begin{aligned} \parallel F \parallel_p^p &= \frac{1}{2^p} \parallel e^{i\theta}B + e^{-i\theta}C^* \parallel_p^p \\ &\leq \frac{1}{2^p} \Big( \parallel e^{i\theta}B \parallel_p + \parallel e^{-i\theta}C^* \parallel_p \Big)^p \quad \text{(using triangle inequality)} \\ &= \frac{1}{2^p} \Big( \parallel B \parallel_p + \parallel C \parallel_p \Big)^p. \qquad ( \parallel C \parallel_p = \parallel C^* \parallel_p ) \end{aligned}$$

Then for $p \in [2, \infty[$,

$$\| Re(e^{i\theta}T) \|_p \leq \frac{1}{2^{\frac{2}{p}-1}}\Big( \| Re(e^{i\theta}A) \|_p^p + \| Re(e^{i\theta}D) \|_p^p + 2 \| F \|_p^p \Big)^{\frac{1}{p}} \qquad (\text{ by } (8))$$

$$\leq \frac{1}{2^{\frac{2}{p}-1}}\left(\| Re(e^{i\theta}A) \|_p^p + \| Re(e^{i\theta}D) \|_p^p + \frac{1}{2^{p-1}}(\| B \|_p + \| C \|_p)^p\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{2^{\frac{2}{p}-1}}\left(w_p^p(A) + w_p^p(D) + \frac{1}{2^{p-1}}(\| B \|_p + \| C \|_p)^p\right)^{\frac{1}{p}}.$$

By taking the supremum over $\theta \in \mathbb{R}$ the demanded inequality (10) is reached.
For $p \in [1, 2]$, we have

$$\| Re(e^{i\theta}T) \|_p \leq \left(\| Re(e^{i\theta}A) \|_p^p ++ \| Re(e^{i\theta}D) \|_p^p + 2 \| F \|_p^p\right)^{\frac{1}{p}} \qquad (\text{ by } (9))$$

$$\leq \left(\| Re(e^{i\theta}A) \|_p^p + \| Re(e^{i\theta}D) \|_p^p + \frac{1}{2^{p-1}}(\| B \|_p + \| C \|_p)^p\right)^{\frac{1}{p}}$$

$$\leq \left(w_p^p(A) + w_p^p(D) + \frac{1}{2^{p-1}}(\| B \|_p + \| C \|_p)^p\right)^{\frac{1}{p}}.$$

By taking the supremum over $\theta \in \mathbb{R}$ the inequality (11) is satisfied. □

We point out that a lower bound for the Schatten $p$-generalized numerical radius has already been established by Bottazi and Conde in [10]. Indeed, Using a Clarkson inequality obtained by Hirzallah and Kittaneh in [12] it follows directly that inequality (10) is bounded below by $\frac{1}{2^{p-1}} \| T \|_p^p$ and (11) is bounded below by $\frac{1}{2} \| T \|_p^p$.
An application of Theorem 3.6 is the following.

**Corollary 3.7.** *Let $A, B, D \in \mathbb{M}_n$. Then,*
*For $p \in [2, \infty[$ the following inequalities hold:*

1. $w_p\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \leq \frac{1}{2^{\frac{2}{p}-1}}\left(w_p^p(A) + w_p^p(D)\right)^{\frac{1}{p}}$,
2. $w_p\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \leq \frac{1}{2^{\frac{2}{p}}-1}\left(w_p^p(A) + \frac{1}{2^{p-1}} \| B \|_p^p\right)^{\frac{1}{p}}$,
3. $w_p\begin{pmatrix} A & B \\ B & A \end{pmatrix} \leq \frac{1}{2^{\frac{2}{p}}-1}(w_p^p(A+B) + w_p^p(A-B))^{\frac{1}{p}}$.

*And for $p \in [1, 2]$, the following hold:*

1. $w_p\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \leq \left(w_p^p(A) + w_p^p(D)\right)^{\frac{1}{p}}$,
2. $w_p\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \leq \left(w_p^p(A) + \frac{1}{2^{p-1}} \| B \|_p^p\right)^{\frac{1}{p}}$,
3. $w_p\begin{pmatrix} A & B \\ B & A \end{pmatrix} \leq \left(w_p^p(A+B) + w_p^p(A-B)\right)^{\frac{1}{p}}$.

We first cite the following facts, which will be needed in the next propositions and theorems: For $p \in [1, \infty[$,

$$\left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\|_p = \left\|\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right\|_p, \tag{13}$$

$$\left\| \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \right\|_p = \left\| \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \right\|_p, \tag{14}$$

$$\left\| \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right\|_p = \left( \| A \|_p^p + \| B \|_p^p \right)^{\frac{1}{p}}. \tag{15}$$

A. Al-Natoor and W. Audeh [3] recently provided the following refinement of the triangle inequality for the of the Schatten $p$-norm

$$\| A + B \|_p \leq 2^{1-\frac{1}{p}} w_p \begin{pmatrix} 0 & A \\ B^* & 0 \end{pmatrix} \leq \| A \|_p + \| B \|_p,$$

when $A, B \in \mathbb{M}_n$. By using the following lemma (see [18] for a proof), we can find more inequalities concerning $w_p(.)$.

**Lemma 3.8.** *Let $X \geq mI \geq 0$ for some positive real number $m$, $Y \in \mathbb{M}_n$, and $N(.)$ is a unitarily invariant norm. Then*

$$mN(Y) \leq \frac{1}{2} N(YX + XY). \tag{16}$$

We have the following proposition.

**Proposition 3.9.** *Let $A, B \in \mathbb{M}_n$ be Hermitian matrices and $0 \leq mI \leq X$ for some positive real number $m$. Then*

$$m \| A - B \|_p \leq w_p(AX - XB) \leq \| AX - XB \|_p. \tag{17}$$

*Proof.* Let $T = AX - XB$, then $T + T^* = (A - B)X + X(A - B)$. It follows, that

$$\begin{aligned} m \| A - B \|_p &\leq \frac{1}{2} \| (A - B)X + X(A - B) \|_p \quad \text{(Y by A-B in (17))} \\ &= \frac{1}{2} \| T + T^* \|_p \\ &= \| Re(T) \|_p \\ &\leq w_p(T) = w_p(AX - XB). \end{aligned}$$

The right inequality follows from the fact that for all $A \in \mathbb{M}_n$ $w_p(A) \leq \| A \|_p$. □

Also we have the following two theorems.

**Theorem 3.10.** *Let $A, B \in \mathbb{M}_n$, $p \in [1, \infty[$ and $0 \leq mI \leq X$ for some positive real number $m$. Then*

$$m \| A - B \|_p \leq 2^{\frac{-1}{p}} w_p \begin{pmatrix} 0 & AX - XB \\ A^*X - XB^* & 0 \end{pmatrix} \leq \frac{1}{2} \left( \| AX - XB \|_p + \| A^*X - XB^* \|_p \right)$$

*Proof.* Let $\widetilde{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$, $\widetilde{B} = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$ and $\widetilde{X} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$. We have,

$$\begin{aligned} \| \widetilde{A} - \widetilde{B} \|_p &= \left\| \begin{pmatrix} 0 & A - B \\ A^* - B^* & 0 \end{pmatrix} \right\|_p \\ &= \left\| \begin{pmatrix} A - B & 0 \\ 0 & A^* - B^* \end{pmatrix} \right\|_p \quad \text{(by (12) )} \\ &= \left\| \begin{pmatrix} A - B & 0 \\ 0 & A - B \end{pmatrix} \right\|_p \quad \text{(by (13))} \\ &= 2^{\frac{1}{p}} \| A - B \|_p. \quad \text{(for A = B in (14))} \end{aligned}$$

So,

$$\begin{aligned} 2^{\frac{1}{p}} m \parallel A-B \parallel_p &= m \parallel \widetilde{A}-\widetilde{B} \parallel_p \leq w_p\left(\widetilde{AX}-\widetilde{XB}\right) \text{ (By the left hand side of (17))} \\ &= w_p\begin{pmatrix} 0 & AX-XB \\ A^*X-XB^* & 0 \end{pmatrix} \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \parallel e^{i\theta}(AX-XB)+e^{-i\theta}(A^*X-XB^*)^* \parallel_p \quad \text{(by (15))} \\ &\leq 2^{\frac{1}{p}-1}\Big( \sup_{\theta\in\mathbb{R}} \parallel e^{i\theta}(AX-XB) \parallel_p + \sup_{\theta\in\mathbb{R}} \parallel e^{-i\theta}(A^*X-XB^*) \parallel_p \Big) \\ &= 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} | e^{i\theta} | \parallel AX-XB \parallel_p + 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} | e^{-i\theta} | \parallel A^*X-XB^* \parallel_p \\ &= 2^{\frac{1}{p}-1}\left(\parallel AX-XB \parallel_p + \parallel A^*X-XB^* \parallel_p\right). \end{aligned}$$

Thus,

$$m \parallel A-B \parallel_p \leq \frac{1}{2^{\frac{1}{p}}} w_p\begin{pmatrix} 0 & AX-XB \\ A^*X-XB^* & 0 \end{pmatrix} \leq \frac{1}{2}\Big( \parallel AX-XB \parallel_p + \parallel A^*X-XB^* \parallel_p \Big)$$

as required. □

**Theorem 3.11.** *Let $X, Y \in \mathbb{M}_n$ and $p \in [1, \infty[$. Then,*

$$w_p^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \geq \frac{1}{4}\Big( \parallel X^*X+YY^* \parallel_p^p + \parallel XX^*+Y^*Y \parallel_p^p \Big)^{\frac{1}{p}}. \tag{18}$$

*Proof.* Let $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, $H_\theta = Re(e^{i\theta}T)$ and $K_\theta = Im(e^{i\theta}T)$. Then

$$H_\theta^2 = \frac{1}{4}\begin{pmatrix} XX^*+Y^*Y+2Re(e^{2i\theta}XY) & 0 \\ 0 & X^*X+YY^*+2Re(e^{2i\theta}YX) \end{pmatrix},$$

$$K_\theta^2 = \frac{1}{4}\begin{pmatrix} XX^*+Y^*Y-2Re(e^{2i\theta}XY) & 0 \\ 0 & X^*X+YY^*-2Re(e^{2i\theta}YX) \end{pmatrix}.$$

And so, $H_\theta^2 + k_\theta^2 = \frac{1}{2}\begin{pmatrix} XX^*+Y^*Y & 0 \\ 0 & X^*X+YY^* \end{pmatrix}$.
If $M = XX^*+Y^*Y$ and $N = X^*X+YY^*$ then,

$$\begin{aligned} \frac{1}{2} \parallel \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \parallel_p &= \parallel H_\theta^2 + K_\theta^2 \parallel_p \\ &\leq \parallel H_\theta^2 \parallel_p + \parallel K_\theta^2 \parallel_p \\ &\leq \parallel H_\theta \parallel_p^2 + \parallel K_\theta \parallel_p^2 \leq w_p^2(T) + w_p^2(T) = 2w_p^2(T). \end{aligned}$$

Therefore,

$$\begin{aligned} w_p^2(T) &\geq \frac{1}{4} \parallel \begin{pmatrix} XX^*+Y^*Y & 0 \\ 0 & X^*X+YY^* \end{pmatrix} \parallel_p \\ &= \frac{1}{4} \parallel \begin{pmatrix} 0 & XX^*+Y^*Y \\ X^*X+YY^* & 0 \end{pmatrix} \parallel_p \quad \text{(by (12))} \\ &= \frac{1}{4}\left(\parallel XX^*+Y^*Y \parallel_p^p + \parallel X^*X+YY^* \parallel_p^p\right)^{\frac{1}{p}} \quad \text{(by (14))} \end{aligned}$$

as required. □

As an application of this theorem is the following.

**Corollary 3.12.** *For* $X \in \mathbb{M}_n$ *then* $w_p^2(X) \geq \frac{1}{2^{2+\frac{1}{p}}} \parallel X^*X + XX^* \parallel_p$ .

*Proof. We have*

$$w_p^2\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \geq \frac{1}{4}\left(\parallel X^*X + XX^* \parallel_p^p + \parallel XX^* + X^*X \parallel_p^p\right)^{\frac{1}{p}} \quad (\textit{for } X = Y \textit{ in (18)})$$
$$= 2^{\frac{1}{p}-2} \parallel XX^* + X^*X \parallel_p .$$

*But,*

$$w_p\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} = 2^{\frac{1}{p}-1} \sup_{\theta\in\mathbb{R}} \parallel e^{i\theta}X + e^{-i\theta}X^* \parallel_p \quad (\textit{for } A{=}B{=}X \textit{ in (15)})$$
$$= 2^{\frac{1}{p}} \sup_{\theta\in\mathbb{R}} \parallel Re(e^{i\theta}X) \parallel_p = 2^{\frac{1}{p}} w_p(X).$$

*So,* $2^{\frac{2}{p}} w_p^2(X) = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \geq 2^{\frac{1}{p}-2} \parallel X^*X + XX^* \parallel_p$ . *And therefore,*

$$w_p^2(X) \geq \frac{1}{2^{2+\frac{1}{p}}} \parallel X^*X + XX^* \parallel_p$$

*as required.* □

**Remark** First note that the majority of the inequalities and results provided in this paper, can be extended to $\mathbb{B}(H)$, the space of all bounded linear operators on a complex separable Hilbert space $H$. Second a further investigation could be done to find an upper bound for the Schatten $p-$generalized numerical radius for partitioned $2 \times 2$ block matrices by using some ideas for [6, 22].

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## References

[1] A. Abu-Omar, F. Kittaneh, A generalization of the numerical radius, Linear Algebra Appl.569 (2019), 323-334.
[2] A. Aldalabih, F. Kittaneh, Hilbert-Schmidt numerical radius inequalities for operator matrices, Linear Algebra Appl.581 (2019), 72-84.
[3] A. Al-Natoor, W. Audeh, Refinement of triangle inequality for the Schatten $p-$norm, Adv. Oper. Theory 5 (2020), 1635-1645.
[4] F. Alrimawi, O. Hirzallah, and F. Kittaneh, Norm inequalities related to Clarkson inequalities, Electronic Journal of Linear Algebra, ISSN 1081-3810, Volume 34, ( 2018), 163-169.
[5] M. Bakherad, Some generalized numerical radius inequalities involving Kwong function, Hacet. J. Math. Stat. Vol. 48(4)(2019), 951-958.
[6] W. Bani-Domi, F. Kittaneh, Norm and numerical radius inequalities for Hilbert space operators, Linear and Multilinear Algebra. 69 (5)(2021), 934-945.
[7] R. Bhatia, F. Kittaneh, Norm inequalities for partitioned operators and an application, Math. Ann. 287(1990), 719-726.
[8] R. Bhatia, Matrix Analysis, Grad. Texts in Math., Spring-Verlag, New York, 1997.
[9] P. Bhunia, K. Paul, Development of inequalities and characterization of equality conditions for the numerical radius. Linear Algebra Appl. 630 (2021), 306-315.
[10] T. Bottazzi, C. Conde, Generalized numerical radius and related inequalities, arXiv: 1909.09243(2019).
[11] K. Feki, Generalized numerical radius inequalities of operators in Hilbert spaces. Adv. Oper. Theory 6 (2021), no. 1, Paper No. 6, 19 pp.

[12] O. Hirzallah, F. Kittaneh, Non-commutative Clarkson inequalities for unitarily invariant norms, Pac. J. Math.202(2002), 363-369.
[13] O. Hirzallah, F. Kittaneh, Numerical radius inequalities for several operators, Math, Scand.114(2014), 110-119.
[14] F. Kittaneh, M.Sal Moslehian and T. Yamazaki, Chartesian decomposition and numerical radius inequalities arXiv: 1511.020904v1 [math.FA] 6 Nov 2015.
[15] H. Najafi, Some results on Kwong functions and related inequalities, Linear Algebra Appl.439 (2013), no.9, 2634-2461.
[16] M. Sababheh, Numerical radius inequalities via convexity, Linear Algebra Appl. 549 (2018), 67-78.
[17] D. Sain, P. Bhunia, A. Bhanja,K. Paul, On a new norm on $\mathbb{B}(H)$ and its applications to numerical radius inequalities. Ann. Funct. Anal. 12 (2021), no. 4, Paper No.51, 25 pp.
[18] J.L. Von Neumann and T. Ando, An inequality for trace ideals, Comm. Math. phys. 76 (1980), no. 2, 143-148.
[19] A. Zamani, M. S. Moslehian, Q. Xu, C. Fu, Numerical radius inequalities concerning with algebra norms. Mediterr. J. Math. 18 (2021), no. 2, Paper No. 38, 13 pp. MR4203694
[20] A. Zamani, $A$-numerical radius and product of semi-Hilbertian operators. Bull. Iranian Math. Soc. 47 (2021), no. 2, 371–377.
[21] A. Zamani, P. Wójcik, Another generalization of the numerical radius for Hilbert space operators, Linear Algebra Appl. 609 (2021), 114-128.
[22] Q.Xu, Z. Ye, A. Zamani, Some upper bounds for the $A-$numerical radius of $2 \times 2$block matrices. Adv. Oper. Theory 6 (2021), no. 1, 1-13.
[23] T. Yamazaki, On upper and lower bounds for the numerical radius and an equality condition, Studia Math. 178 (2007), no.1, 83-89.
[24] H. Zuo, Y. Seo, M. Fujii, Further refinements of Zhan's inequality for unitarily invariant norms, Ann. Funct. Anal. 6(2015), no. 2, 234-241.