Filomat 36:5 (2022), 1635–1648
https://doi.org/10.2298/FIL2205635A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ismail-May-Kantorovich Operators Preserving Affine Functions

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Dedicated to Prof. Th. M. Rassias on the occasion of his 70-th birthday

Abstract. We introduce here a modification of the Ismail-May operators, preserving affine function and estimate the order of approximation with the help of classical approach viz. the second order modulus of continuity, and the Peetre's K-functional. Further, we provide the convergence estimates for the differences of Ismail-May operators and its Kantorovich variants. In the end, the convergence of the operators have been depicted through illustrative graphics.

1. Introduction

We know that the discrete operators are not suitable for approximating integrable functions. Hence they were appropriately generalised to integral type operators and one of the technique used, is due to Kantorovich who proposed the integral modification of well known Bernstein polynomials. In the past few decades, many researchers have worked upon the approximation properties of Kantorovich modifications of different linear positive operators. For more insight into this area, one may refer to [1], [4], [8] and [20] etc. Very recently Kajla in [22], estimated convergence results of some Kantorovich type operators in terms of the modulus of continuity and Lipscitz function. Also, Bohman–Korovkin type approximation properties and order of approximation is investigated by Dogru and Gupta in [9], also the rate of convergence of certain linear positive operators was studied by Srivastava and Gupta in [25] and [26]. Generally, the integral operators reproduce constant but fail to reproduce affine functions. For $x \in (0, \infty)$, the Ismail-May operators are defined by

$$(R_n f)(x) = \sum_{j=0}^{\infty} r_{n,j}(x) f\left(\frac{j}{n}\right) \tag{1}$$

where

$$r_{n,j}(x) = e^{-(n+j)x/(1+x)} \frac{n(n+j)^{j-1}}{j!} \left(\frac{x}{1+x}\right)^j.$$

2020 *Mathematics Subject Classification*. 41A25, 41A30

Keywords. Ismail-May operators, Ismail-May-Kantorovich operators, modulus of continuity, weighted approximation, difference estimates.
Received: 19 May 2021; Accepted: 01 June 2021
Communicated by Miodrag M. Spalević
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These operators are exponential type operators as they satisfy the differential equation:

$$p(x)[(R_n f))(x)]' = \sum_{j=0}^{\infty}(j - nx)r_{n,j}(x)f\left(\frac{j}{n}\right)$$

with

$$p(x) = x(1 + x)^2.$$

These operators were proposed in [18, (3.14)] while the authors constructed several exponential type operators. The approximation properties of these operators considering different basis functions have been studied by the authors in [13] and [14].

The Kantorovich version of Ismail-May operators has the following form:

$$(R_n^k f)(x) = n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{j/n}^{(j+1)/n} f(t)dt, \quad x \in (0, \infty) \tag{2}$$

These operators have no direct link with the Ismail-May operators R_n, like other exponential type operators viz. Berstein polynomials, Baskakov operators and Szász-Mirakyan operators, which are directly connected with their Kantorovich variants. There are technical problems in finding the connection due to non-availability of proper differential equations of these basis. This may be treated as open problem for researchers.

We observe that the Ismail-May-Kantorovich operators R_n^k preserve only constant function and fail to preserve affine functions, so this motivated us to define the modified form of these operators and study its approximation properties. For a function f belonging to the class of continuous functions defined on the positive real axis, we define the following modified Kantorovich form of the classical Ismail-May operators R_n as:

$$(R_n^{mk} f)(x) = r_{n,0}f(0) + n\sum_{j=1}^{\infty} r_{n,j}(x)\int_{(2j-1)/2n}^{(2j+1)/2n} f(t)dt, \quad x \in [\frac{1}{2}, \infty) \tag{3}$$

It is seen that these modification is defined in the compact interval, rather than the positive real axis as in (2).

2. Set of Lemmas

In the sequel, we use the following basic lemmas:

Lemma 2.1. *(See[13],[14]) If* $R_n(e_i, x) = \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^i, i = 0, 1, 2, ...$ *denotes the ith-moment, with* $e_i(t) = t^i, i = 0, 1, 2, ...,$ *then we have the following recurrence relation:*

$$(R_n e_{i+1})(x) = \frac{p(x)}{n}[(R_n e_i)(x)]' + x(R_n e_i)(x),$$

and by simple computation, using the above recurrence relation, some of the moments are computed as follows:

$$(R_n e_0)(x) \;=\; 1, \;\; (R_n e_1)(x) = x, \;\; (R_n e_2)(x) = x^2 + \frac{p(x)}{n},$$

$$(R_ne_3)(x) = x^3 + \frac{3xp(x)}{n} + \frac{p(x)(x+1)(1+3x)}{n^2}$$

and

$$(R_ne_4)(x) = x^4 + \frac{6x^2p(x)}{n} + \frac{xp(x)(x+1)(7+15x)}{n^2} + \frac{p(x)(x+1)^3(1+10x+15x^2)}{n^3}.$$

Lemma 2.2. *For each $x \geq 0$, the moments of the operators $(R_n^k f)(x)$ are given by the following:*

$$\begin{aligned}
(R_n^ke_0)(x) &= 1,\ (R_n^ke_1)(x) = x + \frac{1}{2n},\\
(R_n^ke_2)(x) &= x^2 + \frac{x(x^2+2x+2)}{n} + \frac{1}{3n^2},\\
(R_n^ke_3)(x) &= x^3 + \frac{3x^2\left(2x^2+4x+3\right)}{2n} + \frac{x\left(6x^4+20x^3+27x^2+18x+7\right)}{2n^2} + \frac{1}{4n^3},\\
(R_n^ke_4)(x) &= x^4 + \frac{2x^3\left(3x^2+6x+4\right)}{n} + \frac{x^2\left(15x^4+52x^3+72x^2+48x+15\right)}{n^2}\\
&\quad + \frac{x\left(15x^6+70x^5+137x^4+144x^3+87x^2+30x+6\right)}{n^3} + \frac{1}{5n^4}.
\end{aligned}$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned}
(R_n^ke_0)(x) &= n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{j/n}^{(j+1)/n} dt = 1,\\
(R_n^ke_1)(x) &= n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{j/n}^{(j+1)/n} tdt = \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{2j+1}{n}\right) = x + \frac{1}{2n},\\
(R_n^ke_2)(x) &= n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{j/n}^{(j+1)/n} t^2dt = \frac{1}{3n^2} + \frac{1}{n}\sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right) + \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^2\\
&= x^2 + \frac{x(x^2+2x+2)}{n} + \frac{1}{3n^2},\\
(R_n^ke_3)(x) &= n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{j/n}^{(j+1)/n} t^3dt\\
&= \frac{1}{4n^3} + \frac{1}{n^2}\sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right) + \frac{3}{2n}\sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^2 + \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^3\\
&= x^3 + \frac{3x^2\left(2x^2+4x+3\right)}{2n} + \frac{x\left(6x^4+20x^3+27x^2+18x+7\right)}{2n^2} + \frac{1}{4n^3},\\
(R_n^ke_4)(x) &= \frac{1}{5n^4} + \frac{1}{n^3}\sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right) + \frac{2}{n^2}\sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^2 + \frac{2}{n}\sum_{j=0}^{\infty} r_{n,j}(x).\left(\frac{j}{n}\right)^3 + \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^4\\
&= x^4 + \frac{2x^3\left(3x^2+6x+4\right)}{n} + \frac{x^2\left(15x^4+52x^3+72x^2+48x+15\right)}{n^2}\\
&\quad + \frac{x\left(15x^6+70x^5+137x^4+144x^3+87x^2+30x+6\right)}{n^3} + \frac{1}{5n^4}.
\end{aligned}$$

□

Lemma 2.3. *The moments of Kantorovich operators R_n^{mk}, preserving affine function are given below:*

$$
\begin{aligned}
(R_n^{mk}e_0)(x) &= 1, \ (R_n^{mk}e_1)(x) = x,\\
(R_n^{mk}e_2)(x) &= x^2 + \frac{x(x+1)^2}{n} + \frac{1}{12n^2},\\
(R_n^{mk}e_3)(x) &= x^3 + \frac{3x^2(x+1)^2}{n} + \frac{x\left(4(3x+1)(x+1)^3+1\right)}{4n^2},
\end{aligned}
$$

$$
\begin{aligned}
(R_n^{mk}e_4)(x) &= x^4 + \frac{6(x+1)^2x^3}{n} + \frac{(x+1)^3(15x+7)x^2}{n^2} + \frac{x\left((n+2)x+2\left(15x^2+10x+1\right)(x+1)^5+x^2+1\right)}{2n^3}\\
&\quad + \frac{1}{160n^4},\\
(R_n^{mk}e_5)(x) &= x^5 + \frac{10(x+1)^2x^4}{n} + \frac{5\left(6(9x+5)(x+1)^3+1\right)x^3}{6n^2} + \frac{5(x+1)^2\left(6\left(7x^2+6x+1\right)(x+1)^2+1\right)x^2}{2n^3}\\
&\quad + \frac{48x\left(105x^3+105x^2+25x+1\right)(x+1)^5+40x(3x+1)(x+1)^3+3x}{48n^4}
\end{aligned}
$$

and

$$
\begin{aligned}
(R_n^{mk}e_6)(x) &= x^6 + \frac{15(x+1)^2x^5}{n} + \frac{5\left(4(21x+13)(x+1)^3+1\right)x^4}{4n^2} + \frac{15(x+1)^2\left(4\left(14x^2+14x+3\right)(x+1)^2+1\right)x^3}{2n^3}\\
&\quad + \frac{\left(16\left(945x^3+1155x^2+385x+31\right)(x+1)^5+20(15x+7)(x+1)^3+3\right)x^2}{16n^4}\\
&\quad + \frac{(x+1)^2\left(20\left(15x^2+10x+1\right)(x+1)^2+16\left(945x^4+1260x^3+490x^2+56x+1\right)(x+1)^4+3\right)x}{16n^5}\\
&\quad + \frac{1}{448n^6}.
\end{aligned}
$$

Proof. Clearly, we have

$$(R_n^{mk}e_0)(x) = 1.$$

Also, by simple computation and applying Lemma 2.1, we have

$$
\begin{aligned}
(R_n^{mk}e_1)(x) &= n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{(2j-1)/2n}^{(2j+1)/2n} t\,dt = \frac{n}{2}\sum_{j=0}^{\infty} r_{n,j}(x)\left[\left(\frac{2j+1}{2n}\right)^2 - \left(\frac{2j-1}{2n}\right)^2\right]\\
&= \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right) = x,\\
(R_n^{mk}e_2)(x) &= n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{(2j-1)/2n}^{(2j+1)/2n} t^2dt = \frac{n}{3}\sum_{j=0}^{\infty} r_{n,j}(x)\left[\left(\frac{2j+1}{2n}\right)^3 - \left(\frac{2j-1}{2n}\right)^3\right]\\
&= \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^2 + \frac{1}{12n^2} = x^2 + \frac{x(x+1)^2}{n} + \frac{1}{12n^2},\\
(R_n^{mk}e_3)(x) &= n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{(2j-1)/2n}^{(2j+1)/2n} t^3dt = \frac{n}{4}\sum_{j=0}^{\infty} r_{n,j}(x)\left[\left(\frac{2j+1}{2n}\right)^4 - \left(\frac{2j-1}{2n}\right)^4\right]
\end{aligned}
$$

$$= \frac{1}{4n^2}\sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^2 + \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^3$$
$$= x^3 + \frac{3x^2(x+1)^2}{n} + \frac{x\left(4(3x+1)(x+1)^3+1\right)}{4n^2}$$
$$(R_n^{mk}e_4)(x) = n\sum_{j=0}^{\infty} r_{n,j}(x)\int_{(2j-1)/2n}^{(2j+1)/2n} t^4 dt = \frac{n}{5}\sum_{j=0}^{\infty} r_{n,j}(x)\left[\left(\frac{2j+1}{2n}\right)^5 - \left(\frac{2j-1}{2n}\right)^5\right]$$
$$= \frac{1}{160n^4} + \frac{1}{2n^2}\sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^2 + \sum_{j=0}^{\infty} r_{n,j}(x)\left(\frac{j}{n}\right)^4$$
$$= x^4 + \frac{6(x+1)^2x^3}{n} + \frac{(x+1)^3(15x+7)x^2}{n^2} + \frac{x\left((n+2)x+2\left(15x^2+10x+1\right)(x+1)^5+x^2+1\right)}{2n^3} + \frac{1}{160n^4}.$$

Proceeding similarly, we obtain the values of $(R_n^{mk}e_5)(x)$ and $(R_n^{mk}e_6)(x)$. □

From above estimate, it is clear that our modified form $(R_n^{mk}f)(x)$ preserve the affine function.

Lemma 2.4. *If we denote the i-th central moments by $\mu_i^{R_n^{mk}}(x) = (R_n^{mk}(t-x)^i)(x)$, $i \in N \cup \{0\}$, then*

$$\mu_0^{R_n^{mk}}(x) = 1,\ \mu_1^{R_n^{mk}}(x) = 0,\ \mu_2^{R_n^{mk}}(x) = \frac{x(x+1)^2}{n} + \frac{1}{12n^2},$$
$$\mu_3^{R_n^{mk}}(x) = \frac{x(x+1)^3(3x+1)}{n^2},$$
$$\mu_4^{R_n^{mk}}(x) = \frac{3x^2(x+1)^4}{n^2} + \frac{x\left(30x^5+110x^4+152x^3+96x^2+26x+3\right)(x+1)^2}{2n^3} + \frac{1}{160n^4},$$
$$\mu_5^{R_n^{mk}}(x) = \frac{5x^2(x+1)^4\left(-15x^3-4x^2+7x+2\right)}{n^3} + \frac{x\left(2\left(48\left(105x^3+105x^2+25x+1\right)(x+1)^5+40(3x+1)(x+1)^3+3\right)-3\right)}{96n^4}$$

and

$$\mu_6^{R_n^{mk}}(x) = \frac{15x^3\left(15x^3+11x^2+3x+1\right)(x+1)^4}{n^3} + \frac{x^2\left(10080x^8+67200x^7+192320x^6+307200x^5+297720x^4+177760x^3+63120x^2+12000x+917\right)}{32n^4} + \frac{x(x+1)^2\left(15120x^8+80640x^7+179200x^6+213696x^5+146700x^4+57760x^3+12240x^2+1200x+39\right)}{16n^5} + \frac{1}{448n^6}.$$

In general, $\mu_m^{R_n^{mk}}(x) = O(n^{-[\frac{(m+1)}{2}]})$, where $[\beta]$ denotes the integral part of β.

Proof. The proof of the above lemma follows by using linearity property of the operators and Lemma 2.3. □

Lemma 2.5. *For $x \in [\frac{1}{2}, \infty)$, we have*

$$(R_n^{mk}|t-x|)(x) \leq \sqrt{max\left(2, (1+x)^2 + \frac{1}{6}\right)}\sqrt{\frac{x}{n}}.$$

Proof. For $x \in [\frac{1}{2}, \infty)$, we may observe that

$$\begin{aligned}(R_n^{mk}|t-x|)(x) \leq \sqrt{(R_n^{mk}(t-x)^2)(x)} &= \sqrt{(1+x)^2 + \frac{1}{6} + \frac{1}{12n^2}} \\ &\leq \sqrt{(1+x)^2 + \frac{1}{6}}\sqrt{\frac{x}{n}}.\end{aligned}$$

For $x \in (0, \frac{1}{2n})$, we have

$$\begin{aligned}(R_n^{mk}|t-x|)(x) &= r_{n,0}(x)x + n\sum_{j=1}^{\infty} r_{n,j}(x)\int_{(2j-1)/2n}^{(2j+1)/2n} |u-x|du \\ &= r_{n,0}(x)x + (R_n^{mk}e_1)(x) - x\left((R_n^{mk}e_0)(x) - r_{n,0}(x)\right) \\ &\leq 2x \leq \sqrt{\frac{2x}{n}}.\end{aligned}$$

Thus the result. □

Lemma 2.6. *Let f be a bounded function on $[0, \infty)$, with $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$, then*

$$|(R_n^{mk}f)(x)| \leq \|f\|.$$

3. Ordinary Approximation

Let $C_b[0, \infty)$ be the space of all uniformly continuous and bounded functions defined on $[0, \infty)$ and $C_b^2[0, \infty) = \{g \in C_b^2[0, \infty) : g', g'' \in C_b[0, \infty)\}$. Then, we have the following theorems:

Theorem 3.1. *Let $f \in C_b[0, \infty)$, then for any $x \in [0, \infty)$, we have*

$$|(R_n^{mk}f)(x) - f(x)| \leq C\omega_2\left(f, \sqrt{\frac{1}{12n^2} + \frac{x(x+1)^2}{n}}\right).$$

where C is an absolute constant.

Proof. Let $g \in C_b^2[0, \infty)$ and $x, t \in [0, \infty)$. By Taylor's expansion, we have

$$(R_n^{mk}g)(x) - g(x) = g'(x)(R_n^{mk}(t-x))(x) + (R_n^{mk}(\int_x^t (t-u)g''(u)du))(x).$$

We observe that

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2\|g''\|.$$

Hence

$$\left|(R_n^{mk}g)(x) - g(x)\right| = \left|(R_n^{mk}(\int_x^t (t-u)g''(u)du))(x)\right| \leq \mu_2^{R_n^{mk}}(x)\|g''\|.$$

From Lemma 2.4, we have

$$|(R_n^{mk}g)(x) - g(x)| \leq \left(\frac{1}{12n^2} + \frac{x(x+1)^2}{n}\right)\|g''\|.$$

Making use of Lemma 2.6, we obtain

$$\begin{aligned}|(R_n^{mk}f)(x) - f(x)| &\leq |(R_n^{mk}(f-g))(x) - (f-g)(x)| + |(R_n^{mk}g)(x) - g(x)| \\ &\leq 2\|f-g\| + \left(\frac{1}{12n^2} + \frac{x(x+1)^2}{n}\right)\|g''\|.\end{aligned}$$

Now, for $g \in C_b^2[0,\infty)$ and $f \in C_b[0,\infty)$, there exists a positive constant C such that

$$K_2(f,\delta) \leq C\omega_2(f, \sqrt{\delta}), \tag{4}$$

where $\omega_2(f,.)$ denotes the usual second order modulus of continuity and the Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf\{\| f-g \| + \delta \| g'' \|\}$$

For more insight, one may refer [7, pp. 177, Theorem 2.4]. So, taking infimum on the right hand side over all $g \in C_b^2[0,\infty)$ and using the relation (4), we get the desired result.
□

Next, we use weighted modulus $\omega_\theta(f;h)$ introduced by Paltanea in [24] defined by

$$\omega_\theta(f;h) = sup\left\{|f(x) - f(y)| : x \geq 0, y \geq 0, |x-y| \leq h\theta\left(\frac{x+y}{2}\right)\right\}; h \geq 0$$

where $\theta(x) = x^{\frac{1}{2}}(1+x^m)^{-1}, x \in [0,\infty), m = 2,3,4,....$
Let $W_\theta[0,\infty)$ be the subspace of all real functions defined on $[0,\infty)$, for which the following conditions hold:
i) $\lim\limits_{h\to 0}\omega_\theta(f;h) = 0$ whenever the function $f \circ e_2$ is uniformly continuous on $[0,\infty)$.
ii) For $i = \frac{2}{2m+1}$, $f \circ e_i$ is uniformly continuous on $[1,\infty)$.

Let E be a subspace of $C[0,\infty)$ such that $C_s[0,\infty) \subset E$ with $s = max\{m+r+1, 2r+2, 2m\}$,

$$C_s[0,\infty) = \{f \in C[0,\infty) : |f(x)| \leq M(1+x^s), s \in \mathbb{N}, \text{ for all } x \geq 0, M > 0\}$$

.

Theorem 3.2. *If $f \in C_s[0,\infty)$ and $f'' \in W_\theta[0,\infty)$, the for $x \in (0,\infty)$, we have:*

$$\left|(R_n^{mk}f)(x) - f(x) - \frac{1}{2}f''(x)\left(\frac{x(x+1)^2}{n} + \frac{1}{12n^2}\right)\right|$$
$$\leq \frac{1}{2}\left[\frac{x(x+1)^2}{n} + \frac{1}{12n^2} + \sqrt{2}\sqrt{\left(R_n^{mk}\left[1+\left(x+\frac{|t-x|}{2}\right)^m\right]^2\right)(x)}\right]\omega_\theta\left(f''; \left(\frac{\mu_6^{R_n^{mk}}(x)}{x}\right)^{\frac{1}{2}}\right),$$

where $\mu_6^{R_n^{mk}}(x)$ is the 6th central moment as indicated in Lemma 2.4.

Theorem 3.3. *If $f \in C_s[0,\infty)$ and $f'' \in W_\theta[0,\infty)$, the for $x \in (0,\infty)$, we have:*

$$\left|(R_n^{mk} f)(x) - f(x) - \frac{1}{2} f''(x)\mu_2^{R_n^{mk}}(x)\right| \leq \frac{1}{2}\left[\mu_2^{R_n^{mk}}(x) + \sqrt{\frac{2}{x}}\mu_2^{R_n^{mk}}(x) C_{n,m}(x)\right]\omega_\theta\left(f''; \sqrt{\frac{\mu_4^{R_n^{mk}}(x)}{\mu_2^{R_n^{mk}}(x)}}\right)$$

where

$$C_{n,m}(x) = 1 + \frac{1}{M_{n,3}(x)}\sum_{k=0}^{m}\binom{m}{k} x^{m-k}\frac{M_{n,k+3}(x)}{2^k},$$

$M_{n,k}(x)$ are the absolute moments of order k and for the operators R_n^{mk}, $\frac{M_{n,k}}{M_{n,3}}(4 \leq k \leq m)$ is a bounded ratio for fixed k and m, when $n \to \infty$. Also, $\mu_2^{R_n^{mk}}(x)$ and $\mu_4^{R_n^{mk}}(x)$ are the 2nd and 4th central moments respectively as indicated in Lemma 2.4.

The proofs of the above two theorems follow along the lines of [17], we omit the details.

4. Difference of Operators

The differences of operators have been investigated by many researchers for the past few years and the recent work may be studied in [3], [10] and [15] etc. Here, in this section, we provide the quantitative estimate for the differences of Ismail-May operators $(R_n f)(x)$ with its Kantorovich version $(R_n^k f)(x)$, Ismail-May-Kantorovich operators $(R_n^k f)(x)$ with Szász-Mirakyan-Kantorovich operators $(S_n^k f)(x)$ and Ismail-May-Kantorovich operators $(R_n^k f)(x)$ with Baskakov- Kantorovich operators $(V_n^k f)(x)$. We consider the weighted modulus of continuity $\Omega(g,\delta)$ see ([2]), defined as $\Omega(g,\delta) = \sup\limits_{|h|<\delta, x\geq 0} \dfrac{|g(x) - g(x+h)|}{(1+h^2)(1+x^2)}$.

If we have two positive linear operators, say

$$(U_n f)(x) = \sum_{j=0}^{\infty} u_{n,j}(x) F_{n,j}(f)$$

and

$$(V_n f)(x) = \sum_{j=0}^{\infty} u_{n,j}(x) G_{n,j}(f),$$

then the result on difference of operators (with same basis) in weighted space is given below (as provided by Aral et al. [5], also see Gupta et al. [15]):

Theorem A. For $f \in C_b[0,\infty)$, then we have

$$|((U_n - V_n)f)(x)| \leq \frac{1}{2}\|f''\| A(x) + 8\Omega(f'',\alpha_1)(1 + A(x)) + 16\Omega(f,\alpha_2)(B(x)+1),$$

where

$$\begin{aligned} A(x) &= \sum_{j=0}^{\infty} u_{n,j}(x)\Big[(1+(b^{F_{n,j}})^2)\mu_2^{F_{n,j}} + (1+(b^{G_{n,j}})^2)\mu_2^{G_{n,j}}\Big], \\ B(x) &= \sum_{j=0}^{\infty} u_{n,j}(x)\Big[(1+(b^{(FG)_{n,j}})^2)\Big], \\ \alpha_1^4 &= \sum_{j=0}^{\infty} u_{n,j}(x)\Big[(1+(b^{F_{n,j}})^2)\mu_6^{F_{n,j}} + (1+(b^{G_{n,j}})^2)\mu_6^{G_{n,j}}\Big], \\ \alpha_2^4 &= \sum_{j=0}^{\infty} u_{n,j}(x)(1+(b^{(FG)_{n,j}})^2)(b^{F_{n,j}} - b^{G_{n,j}})^4, \end{aligned}$$

with

$$b^{(FG)_{n,j}} = min(b^{F_{n,j}}, b^{G_{n,j}}), b^{H_{n,j}} = H_{n,j}(e_1),$$

and

$$\mu_r^{H_{n,j}} = \sum_{i=0}^{r} \binom{r}{i}(-1)^i H_{n,j}(e_{r-i})[H_{n,j}(e_1)]^i.$$

Also, for the difference of two linear positive operators with different basis, we apply the following result as provided by Gupta [10] and Gupta-Acu [11].

Theorem B. For the two operators $S_n := \sum_{j=0}^{\infty} s_{n,j}(x)L_{n,j}(f)$ and $T_n := \sum_{j=0}^{\infty} t_{n,j}(x)M_{n,j}(f)$, we have

$$|((S_n - T_n)f)(x)| \leq \frac{C(x)}{2}\|f''\| + 2\omega(f,\gamma_1) + 2\omega(f,\gamma_2), n \in N,$$

where $C(x) = \sum_{j=0}^{\infty} s_{n,j}(x)\mu_2^{L_{n,j}} + \sum_{j=0}^{\infty} t_{n,j}(x)\mu_2^{M_{n,j}}$,

$$\gamma_1^2 = \sum_{j=0}^{\infty} s_{n,j}(x)\left[L_{n,j}(e_1) - x\right]^2$$

and

$$\gamma_2^2 = \sum_{j=0}^{\infty} t_{n,j}(x)\left[M_{n,j}(e_1) - x\right]^2$$

with $f^{(i)} \in C_b[0,\infty), i \in \{0,1,2\}, x \in [0,\infty)$ and $\|.\| = \sup_{x\in[0,\infty)} |f(x)| < \infty$,

4.1. Error Estimation: Ismail-May operators and Ismail-May-Kantorovich operators

Here, we obtain the quantitative estimate for the difference between the operators $(R_n f)(x)$ and $(R_n^k f)(x)$. As the application of Theorem A, we have the following:

Theorem 4.1. *Let $f \in C_b[0,\infty)$. Then for Ismail-May operators defined by*

$(R_n f)(x) = \sum_{j=0}^{\infty} r_{n,j}(x)F_{n,j}(f)$, *where* $F_{n,j}(f) = f\left(\frac{j}{n}\right)$

and Ismail-May-Kantorovich operators defined by

$$(R_n^k f)(x) = \sum_{j=0}^{\infty} r_{n,j}(x)J_{n,j}(f), \textit{where } J_{n,j}(f) = n\int_{j/n}^{(j+1)/n} f(t)dt, \tag{5}$$

we immediately have

$$|((R_n - R_n^k)f)(x)| \leq \frac{1}{2}\|f''\|A(x) + 8\Omega\,(f'',\alpha_1)\,(1 + A(x)) + 16\Omega\,(f,\alpha_2)\,(1 + B(x)),$$

where the values of $A(x), B(x), \alpha_1$ and α_2 are provided in the proof below.

Proof. We have

$$b^{J_{n,j}} = J_{n,j}(e_1) = \frac{2j+1}{2n}$$

Also,

$$\begin{aligned}\mu_2^{J_{n,j}} &= J_{n,j}(e_2) - 2[J_{n,j}(e_1)]^2 + [J_{n,j}(e_1)]^2 \\ &= \frac{n}{3}\left(\left(\frac{j+1}{n}\right)^3 - \left(\frac{j}{n}\right)^3\right) - \left(\frac{2j+1}{2n}\right)^2 = \frac{1}{12n^2}\end{aligned}$$

and

$$\begin{aligned}\mu_6^{J_{n,j}} &= J_{n,j}(e_6) - 6J_{n,j}(e_5)J_{n,j}(e_1) + 15J_{n,j}(e_4)[J_{n,j}(e_1)]^2 - 20J_{n,j}(e_3)[J_{n,j}(e_1)]^3 + 15J_{n,j}(e_2)[J_{n,j}(e_1)]^4 - 5[J_{n,j}(e_1)]^6 \\ &= \frac{1}{448n^6}.\end{aligned}$$

Next, we have

$$b^{F_{n,j}} = F_{n,j}(e_1) = \frac{j}{n},\ \ \mu_r^{F_{n,j}} = 0, r \in N.$$

From Lemma 2.1 and the values obtained above, we get

$$\begin{aligned}A(x) &= \sum_{j=0}^{\infty} r_{n,j}(x)\Big[(1 + (b^{F_{n,j}})^2)\mu_2^{F_{n,j}} + (1 + (b^{J_{n,j}})^2)\mu_2^{J_{n,j}}\Big] = \sum_{j=0}^{\infty} r_{n,j}(x)\left[\left(1 + \left(\frac{2j+1}{2n}\right)^2\right)\frac{1}{12n^2}\right] \\ &= \frac{1 + 4nx(2 + 2x + x^2) + 4n^2(1 + x^2)}{48n^4}.\end{aligned}$$

Further, using Lemma 2.1, we have

$$B(x) = \sum_{j=0}^{\infty} r_{n,j}(x)\Big[(1 + (b^{(FJ)_{n,j}})^2)\Big] = \sum_{j=0}^{\infty} r_{n,j}(x)\left[\left(1 + \frac{j^2}{n^2}\right)\right] = \frac{n + nx^2 + x(x+1)^2}{n}.$$

Next proceeding in the same manner, we get

$$\begin{aligned}\alpha_1^4 &= \sum_{j=0}^{\infty} r_{n,j}(x)\Big[(1 + (b^{F_{n,j}})^2)\mu_6^{F_{n,j}} + (1 + (b^{J_{n,j}})^2)\mu_6^{J_{n,j}}\Big] = \sum_{j=0}^{\infty} r_{n,j}(x)\left[\left(1 + \left(\frac{2j+1}{2n}\right)^2\right)\frac{1}{448n^6}\right] \\ &= \frac{1 + 4nx(2 + 2x + x^2) + n^2(2 + 4x^2)}{896n^8}.\end{aligned}$$

Now $b^{(FJ)_{n,j}} = \frac{j}{n}$, hence on applying Lemma 2.1, we obtain

$$\begin{aligned}\alpha_2^4 &= \sum_{j=0}^{\infty} r_{n,j}(x)(1 + (b^{(FJ)_{n,j}})^2)(b^{F_{n,j}} - b^{J_{n,j}})^4 = \sum_{j=0}^{\infty} r_{n,j}(x)\left(1 + \frac{j^2}{n^2}\right)\left(\frac{j}{n} - \frac{2j+1}{2n}\right)^4 \\ &= \frac{n(1+x)^2 + x(x+1)^2}{16n^5}.\end{aligned}$$

Hence the result follows.

□

4.2. Error Estimation: Ismail-May-Kantorovich operators and Szász-Mirakyan-Kantorovich operators

The Szász-Mirakyan operators are defined as $(S_n f)(x) = \sum_{j=0}^{\infty} s_{n,j}(x)F_{n,j}(f)$,

where $s_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}$ and $F_{n,j}(f) = F(\frac{j}{n})$.
Their generalization was introduced by Jain in [21] but the Durrmeyer type modification of the generalized Szász-Mirakyan operators has not been discussed much in the last four decades due to complications in finding moments. However, this difficulty was overcome by Gupta and Greubel in [16].

Lemma 4.2. *[12] If* $(S_n e_i)(x) = \displaystyle\sum_{j=0}^{\infty} s_{n,j}(x)\left(\frac{j}{n}\right)^i$, $i \in N \cup \{0\}$ *denotes the ith-moment, then we have the following recurrence relation:*

$$(S_n e_{i+1})(x) = \frac{x}{n}[((S_n e_i)(x)]' + x(S_n e_i)(x).$$

and the few moments are:

$$\begin{aligned}(S_n e_0)(x) &= 1, \ (S_n e_1)(x) = x, \ (S_n e_2)(x) = x^2 + \frac{x}{n} \\ &\text{and} \\ (S_n e_3)(x) &= x^3 + \frac{3x^2}{n} + \frac{x}{n^2}.\end{aligned}$$

The Szász-Mirakyan-Kantorovich operators are defined as

$(S_n^k f)(x) = \sum_{j=0}^{\infty} s_{n,j}(x)J_{n,j}(f)$, where $J_{n,j}(f) = n\int_{j/n}^{(j+1)/n} f(t)dt$

Here, we obtain the quantitative estimate for the difference between the operators $(R_n^k f)(x)$ and $(S_n^k f)(x)$. As the application of Theorem B, we have the following:

Theorem 4.3. *Let $f \in C_b[0,\infty)$. Then for Ismail-May-Kantorovich operators defined by*

$(R_n^k f)(x) = \sum_{j=0}^{\infty} r_{n,j}(x)J_{n,j}(f)$

and for Szász-Mirakyan-Kantorovich operators defined by

$(S_n^k f)(x) = \sum_{j=0}^{\infty} s_{n,j}(x)J_{n,j}(f)$, *we have*

$$|((R_n^k - S_n^k)f)(x)| \leq \frac{1}{12n^2}\|f''\| + 2\omega\left(f, \sqrt{\frac{1}{4n^2} + \frac{x(x+1)^2}{n}}\right) + 2\omega\left(f, \sqrt{\frac{1}{4n^2} + \frac{x}{n}}\right).$$

Proof. Here $\mu_2^{J_{n,j}} = \frac{1}{12n^2}$.
Thus

$$C(x) = \sum_{j=0}^{\infty} r_{n,j}(x)\mu_2^{J_{n,j}} + \sum_{j=0}^{\infty} s_{n,j}(x)\mu_2^{J_{n,j}} = \frac{1}{6n^2}$$

Now, by simple analysis, using Lemma 2.1 and Lemma 4.2, we obtain

$$\gamma_1^2 = \sum_{j=0}^{\infty} r_{n,j}(x)\left[J_{n,j}(e_1) - x\right]^2 = \frac{1}{4n^2} + \frac{x(x+1)^2}{n}$$

and

$$\gamma_2^2 = \sum_{j=0}^{\infty} s_{n,j}(x)\left[J_{n,j}(e_1) - x\right]^2 = \frac{1}{4n^2} + \frac{x}{n}$$

Hence, we have the required result. □

4.3. Error Estimation: Ismail-May-Kantorovich operators and Baskakov-Kantorovich operators

The Baskakov operators are defined as $(V_n f)(x) = \sum_{j=0}^{\infty} v_{n,j}(x)F_{n,j}(f)$,

where $v_{n,j}(x) = \binom{n+j-1}{j}\frac{t^j}{(1+t)^{n+j}}$ and $F_{n,j}(f) = F(\frac{j}{n})$

Lemma 4.4. *[15] If* $(V_n e_i)(x) = \sum_{j=0}^{\infty} v_{n,j}(x)\left(\frac{j}{n}\right)^i$, $i \in N \cup \{0\}$ *denotes the ith-moment, then we have the following recurrence relation:*

$$n(V_n e_{i+1})(x) = x(x+1)[(V_n e_i)(x)]' + nx(V_n e_i)(x).$$

and the few moments are:

$$(V_n e_0)(x) \quad = \quad 1,\ (V_n e_1)(x) = x,\ (V_n e_2)(x) = \frac{x^2(n+1)+x}{n}$$

and

$$(V_n e_3)(x) \quad = \quad \frac{x^3(n+1)(n+2) + 3x^2(n+1) + x}{n^2}.$$

The Baskakov-Kantorovich operators are defined as

$$(V_n^k f)(x) = \sum_{j=0}^{\infty} v_{n,j}(x)J_{n,j}(f),$$

where $J_{n,j}(f)$ is given in (5). Here, we obtain the quantitative estimate for the difference between the operators $(R_n^k f)(x)$ and $(V_n^k f)(x)$. As the application of Theorem B, we have the following:

Theorem 4.5. *Let* $f \in C_b[0,\infty)$. *Then for Ismail-May-Kantorovich operators defined by*

$(R_n^k f)(x) = \sum_{j=0}^{\infty} r_{n,j}(x)J_{n,j}(f)$,

and Baskakov-Kantorovich operators defined by

$(V_n^k f)(x) = \sum_{j=0}^{\infty} v_{n,j}(x)J_{n,j}(f)$, *we have*

$$|((R_n^k - V_n^k)f)(x)| \quad \le \quad \frac{1}{12n^2}\|f''\| + 2\omega\left(f, \sqrt{\frac{1}{4n^2} + \frac{x(x+1)^2}{n}}\right) + 2\omega\left(f, \sqrt{\frac{1}{4n^2} + \frac{x(x+1)}{n}}\right),$$

here $J_{n,j}(f)$ *is given in (5).*

Proof. Here

$$C(x) = \sum_{j=0}^{\infty} r_{n,j}(x)\mu_2^{J_{n,j}} + \sum_{j=0}^{\infty} v_{n,j}(x)\mu_2^{J_{n,j}} = \frac{1}{6n^2}$$

Now, using Lemma 2.1 and Lemma 4.4, we are led to

$$\gamma_1^2 = \sum_{j=0}^{\infty} r_{n,j}(x)\left[J_{n,j}(e_1) - x\right]^2 = \sum_{j=0}^{\infty} r_{n,j}(x)\left[\frac{2j+1}{2n} - x\right]^2 = \frac{1}{4n^2} + \frac{x(x+1)^2}{n}$$

and

$$\gamma_2^2 = \sum_{j=0}^{\infty} v_{n,j}(x)\left[J_{n,j}(e_1) - x\right]^2 = \sum_{j=0}^{\infty} v_{n,j}(x)\left[\frac{2j+1}{2n} - x\right]^2 = \frac{1}{4n^2} + \frac{x(x+1)}{n}.$$

Combining the above estimates, we have the required result. □

5. Graphical Representation

The convergence of modified operators $R_n^{mk} f$ to the function f, where $f(x) = x^2 - x + 1$ for different values of n is represented in the following graph:

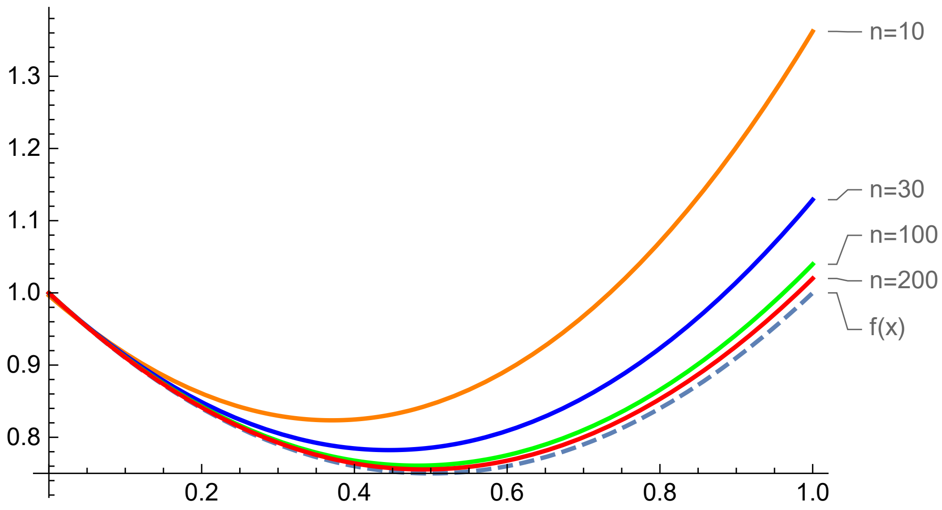

Figure 1: Convergence of $(R_n^{mk} f)(x)$ to the function $f(x) = x^2 - x + 1$ for $n = 10$, $n = 30$, $n = 100$ and $n = 200$.

It may be concluded that the operators R_n^{mk} converge to the function more rapidly as the value of n increases.

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